A Module View of Integral Closures

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The Problem

Let

$$P := \mathbf{F}[x_n, \ldots, x_1]$$

and

$$Q := P[y_{m-1}, \ldots, y_1]/I$$

be an *integrally closed* extension of P (with $Q = P$ corresponding to the case $m = 1$).

$$f(T) \in Q[T]$$

a monic, irreducible polynomial of some degree d, defining a *simple integral extension*

$$S := Q[z]/\langle f(z) \rangle,$$

which may or may not be integrally closed (in its field of fractions).

Find a *canonical form* for the *integral closure* $ic(S)$.

Module Bases

Suppose that $\Delta \in P$ satisfies

$$S \subseteq ic(S) \subseteq \frac{1}{\Delta} S.$$

With $y_0 := 1$,

$$S = P \left\langle z^i y_j : 0 \le j < m,\ 0 \le i < d \right\rangle$$

$$\frac{1}{\Delta} S = P \left\langle \frac{z^i y_j}{\Delta} : 0 \le j < m,\ 0 \le i < d \right\rangle .$$

So it is *obvious* to conjecture that

$$ic(S) = P \left\langle \frac{f_{i,j}}{\Delta} : 0 \le i < d, 0 \le j < m \right\rangle$$

with $LM(f_{i,j}) = \underline{x}^{\alpha_{i,j}} z^i y_j$ for some $\underline{x}^{\alpha_{i,j}} \in P$.

Example

$$P := \mathbf{F}_2[x_2, x_1]$$

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$$f_1(T) := T^3 + T^1 x_2 x_1 + T^0 \left(x_2^3 x_1^2 + x_2^2 x_1^3\right)$$

$$Q := P[y]/\langle f_1(y)\rangle.$$

$$\Delta_1 := x_2^3 x_1^2 + x_2^2 x_1^3$$

$$Q = P\langle 1, y, y^2\rangle \subseteq ic(Q) \subseteq P\left\langle \frac{1}{\Delta_1}, \frac{y}{\Delta_1}, \frac{y^2}{\Delta_1}\right\rangle = \frac{1}{\Delta_1} Q$$

$$ic(Q) = P\left\langle 1,\ y,\ \frac{y^2}{x_2 x_1}\right\rangle$$

$$= P[y_2, y_1]/I_1$$

$$I_1 := \left\langle y_1^2 + y_2 x_2 x_1,\ y_2 y_1 + y_1 + (x_2^2 x_1 + x_2 x_1^2),\ y_2^2 + y_2 + y_1(x_2 + x_1)\right\rangle$$

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$$f_2(T) := T^3 + T^1 (y_2 + y_1)(x_2 + 1)x_1 + T^0 (y_2 + y_1)^2 x_2^2 x_1$$

$$S := ic(Q)[z]/\langle f_2(z)\rangle$$

$$\Delta_2 := x_2^8 x_1^5 + x_2^6 x_1^7 + x_2^8 x_1^3 + x_2^4 x_1^7 + x_2^6 x_1^3 + x_2^4 x_1^5$$

$$S = P\langle z^0 y_0, z^0 y_1, z^0 y_2, z^1 y_0, z^1 y_1, z^1 y_2, z^2 y_0, z^2 y_1, z^2 y_2\rangle$$

$$P\langle z^i y_j : 0 \leq i, j < 3\rangle \subseteq ic(S) \subseteq \frac{1}{\Delta_2} P\langle z^i y_j : 0 \leq i, j < 3\rangle$$

$$ic(S) = \frac{1}{\delta_2} P\langle f_{i,j} : 0 \leq i, j < 3\rangle,\ LM(f_{i,j}) := \underline{x}^{\alpha_{i,j}} z^i y_j$$

$$ic(S) = P[z_8, \ldots, z_1; x_2, x_1]/I_2,\ z_{3i+j} := f_{i,j}$$

$$I_2 = \langle z_i z_j - NF(z_i z_j)\rangle$$

Truth

A P-module basis for S is actually

$$z_0 := y_0;\ z_1 := y_1;\ z_2 := y_2;\ z_3 := zy_0;\ z_4 := zy_1;$$

$$z_5 := \frac{zy_2 + zy_1x_2 + zy_0}{x_1};\ z_6 := z^2y_0;\ z_7 := \frac{z^2y_1}{x_1};$$

$$z_8 := \frac{z^2y_2 + z^2y_1(x_2x_1 + x_2 + x_1) + z^2y_0(x_2^3x_1^2 + x_2^3x_1 + x_2x_1^2 + 1)}{x_2(x_2+1)x_1^2(x_1+1)};$$

$$z_9 := (z^2y_2(x_1+1) + z^2y_1(x_2^2x_1 + x_2x_1^2 + x_2x_1 + x_1^2 + x_2 + x_1)$$
$$+z^2y_0(x_2^3x_1 + x_2^2x_1^2 + x_1 + 1))/x_2(x_2+1)(x_2+x_1)x_1^2$$

The integral closure is then of the form

$$ic(S) = P[z_9, z_8, z_7, z_6, z_5, z_4, z_3, z_2, z_1; x_2, x_1]/\overline{I}$$

with $\overline{I}$ the ideal of induced relations most of the form $z_iz_j - NF(z_iz_j)$ except for one of the form $SP(z_9, z_8) - NF(SP(z_9, z_8))$.

Weighted Truth

A P-module basis for S with the induced weight function

$$W := \begin{pmatrix} 19 & 12 & 15 & 9 & 9 \\ 12 & 9 & 9 & 9 & 0 \end{pmatrix}$$

is however

$$w_0 := y_0;\ w_1 := y_1;\ w_2 := y_2;\ w_3 := zy_0;\ w_4 := zy_1;$$

$$w_5 := \frac{zy_2 + zy_1x_2 + zy_0}{x_1};$$

$$w_6 := zy_2;$$

$$\delta w_7 := z^2y_0(x_2^3x_1^2 + x_2^2x_1^3 + x_2^3x_1 + x_2x_1^3 + x_2x_1 + 1)$$
$$+z^2y_2(x_2x_1 + 1) + z^2y_1(x_2 + x_1);$$

$$\delta w_8 := z^2y_1(x_2^2x_1 + x_2x_1^2 + x_2x_1 + x_1^2 + x_2 + x_1)$$
$$+z^2y_2(x_2^2x_1 + x_2x_1 + x_1 + 1) + z^2y_0(x_2^3x_1 + x_2^2x_1^2 + x_2^2x_1 + x_2x_1 + x_1 + 1);$$

$$\delta w_9 := z^2y_2(x_2x_1^2 + x_1) + z^2y_1(x_2x_1 + x_1^2)$$
$$+z^2y_0(x_2^2x_1^2 + x_2x_1^3 + x_2x_1^2 + x_1)$$

for $\delta := x_1^2(x_1 + 1)x_2(x_2 + 1)(x_2 + x_1)$.

The integral closure is then of the form

$$ic(S) = P[w_9, w_8, w_7, w_6, w_5, w_4, w_3, w_2, w_1; x_2, x_1]/\overline{I}$$

with $\overline{I}$ the ideal of induced relations most of the form $w_iw_j - NF(w_iw_j)$ except for $w_4x_2 + w_5x_1 + w_6 + w_3$, of the form $SP(w_4, w_5) - NF(SP(w_4, w_5))$.

Black Box Algorithm Ingredients

Any *good* algorithm for computing the integral closure $ic(S)$ as described above should produce, as output,

1. a $\delta \in P$ minimal such that $ic(S)\delta \subseteq S$;
2. a ring $R := \mathbf{F}[z_{s-1}, \ldots, z_1; x_n, \ldots, x_1]$ with a monomial order that properly differentiates between dependent and independent variables;
3. an ideal $\overline{I}$ of induced relations of predictable form, so that $ic(S) = R/\overline{I}$;
4. an embedding $\phi : S \to ic(S)$;
5. an embedding $\psi : ic(S)\delta \to S$.

(The implementation *IntegralClosure* in MAGMA works only for $n = 1$ independent variable, and produces a P-module basis as a subset of S/δ, meaning δ is implicit, the R, $\overline{I}$, and ϕ are not given, and ψ is unnecessary. And the implementation *normal* in SINGULAR does not necessarily produce δ restricted to P, does give R, $\overline{I}$ (called *norid*), ϕ (called *normap*), but no ψ, making it extremely difficult to figure out what the new variables are in terms of S, especially since the ordering used on R makes the new variables less important than even the independent variables of S.)

Module Basis for the Weighted Example

$$M_1 := \delta \cdot (1, 0, 0, 0, 0, 0, 0, 0, 0)$$
$$M_2 := \delta \cdot (0, 1, 0, 0, 0, 0, 0, 0, 0)$$
$$M_3 := \delta \cdot (0, 0, 1, 0, 0, 0, 0, 0, 0)$$
$$M_4 := \delta \cdot (0, 0, 0, 1, 0, 0, 0, 0, 0)$$
$$M_5 := \delta \cdot (0, 0, 0, 0, 1, 0, 0, 0, 0)$$
$$M_6 := \delta / x_1 \cdot (0, 0, 0, 1, x_2, 1, 0, 0, 0)$$
$$M_7 := \delta \cdot (0, 0, 0, 0, 0, 1, 0, 0, 0)$$
$$M_8 := (0, 0, 0, 0, 0, 0, x_2^3 x_1^2 + x_2^2 x_1^3 + x_2^3 x_1 + x_2 x_1^3 + x_2 x_1 + 1, x_2 + x_1, x_2 x_1 + 1)$$
$$M_9 := (0, 0, 0, 0, 0, 0,$$
$$x_2^3 x_1 + x_2^2 x_1^2 + x_2^2 x_1 + x_2 x_1 + x_1 + 1, x_2^2 x_1 + x_2 x_1^2 + x_2 x_1 + x_1^2 + x_2 + x_1, x_2^2 x_1 + x_2 x_1 + x_1 + 1)$$
$$M_{10} := x_1 (0, 0, 0, 0, 0, 0, x_2^2 x_1 + x_2 x_1^2 + x_2 x_1 + 1, x_2 + x_1, x_2 x_1 + 1)$$

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$$\delta = x_2^3 x_1^3 + x_2^2 x_1^4 + x_2^3 x_1^2 + x_2 x_1^4 + x_2^2 x_1^2 + x_2 x_1^3$$

Minimization

It may be necessary to switch to a new set in *independent* variables in order to *minimize* the description of the integral closure.

$$Q := P := \mathbf{F}_2[x_2, x_1],\ WEIGHT(Q) := \begin{pmatrix} | & 1 & 1 \\ | & 1 & 0 \end{pmatrix}$$

$$f(T) := T^4 + T^2 x_2 x_1 + T^0 x_2^3 x_1^2$$

$$S := Q[z]/\langle f(z)\rangle,\ WEIGHT(S) = \begin{pmatrix} 5 & | & 4 & 4 \\ 3 & | & 4 & 0 \end{pmatrix}$$

$$\Delta := x_2^3 x_1^2$$

$$ic(S) = \mathbf{F}_2[z_3, z_2, z_1; x_2, x_1]/I,\ WEIGHT(ic(S)) = \begin{pmatrix} 3 & 2 & 5 & | & 4 & 4 \\ 1 & 2 & 3 & | & 4 & 0 \end{pmatrix}$$

$$I = \left\langle z_1^2 + z_2 x_2 x_1, z_2 z_1 + z_3 x_2 + z_1, z_2^2 + z_2 + x_2, z_3 z_1 + x_2 x_1, z_3 z_2 + z_1, z_3^2 + z_2 x_1 + x_1 \right\rangle$$

$$\delta := x_2^2 x_1,\ z_3\delta := z^3 + z x_2 x_1,\ z_2\delta := z^2 x_2,\ z_1\delta := z\delta$$

$$weight(z_2) = (2,2) < weight(x_2) = (4,4),\ x_2 = z_2^2 + z_2,\ z_1 = z_3 z_2$$

$$ic(S) = \mathbf{F}_2[\overline{z_1}; \overline{x}_2, \overline{x}_1]/\overline{I},\ WEIGHT(ic(S)) = \begin{pmatrix} 3 & | & 2 & 4 \\ 1 & | & 2 & 0 \end{pmatrix}$$

$$\overline{I} = \left\langle \overline{z}_1^2 + \overline{x}_2 \overline{x}_1 + \overline{x}_1 \right\rangle$$

$$\delta := x_2^2 x_1,\ \overline{z}_1\delta := z^3 + z x_2 x_1,\ \overline{x}_2\delta := z^2 x_2,\ \overline{x}_1 := x_1\delta$$

$$\psi((\overline{z}_1,\ \overline{x}_2,\ \overline{x}_1)\delta) = (z^3 + z x_2 x_1,\ z^2 x_2,\ x_1\delta)$$

$$\phi(z,\ x_2,\ x_1) = (\overline{z}_1 \overline{x}_2,\ \overline{x}_2^2 + \overline{x}_2,\ \overline{x}_1)$$

MAGMA isn't set up to do examples with two or more *independent* variables, but SINGULAR is. Compare the ideal returned by SINGULAR (edited to fit the page) immediately below to the suggested *predictable* (in the sense described above) form following it. The latter has $y_1 = m_2$, $y_2 = m_3$, $y_3 = z$ and the reverse map given above by the module basis page.) Note that the grevlex $T(1) = z \succ T(2) = y_2 \succ T(3) = y_1 \succ T(4) = x_2 \succ T(5) = x_1 \succ T(6) \succ T(7) \succ T(8) \succ T(9)$ order employed by SINGULAR's normal function completely ignores any original ordering, puts the *independent* variables in the middle, gives poor information about what the new variables are, and gives basis functions that, while correct, have no *meaning*.

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 A Computer Algebra System for Polynomial Computations   /   version 2-0-4
                                                       0<
     by: G.-M. Greuel, G. Pfister, H. Schoenemann        \   April 2003
FB Mathematik der Universitaet, D-67653 Kaiserslautern    \
> LIB "normal.lib";
> ring r1=2,(z,y2,y1,x2,x1),(dp(3),dp(2));
> poly f1=y1^2+y2*x2*x1;
> poly f2=(y2+1)*y1+x2*x1*(x2+x1);
> poly f3=(y2+1)*y2+y1*(x2+x1);
> poly f4=z^3+z*(y2+y1)*(x2+1)*x1+(y2+y1)^2*x2^2*x1;
> ideal i1=f1,f2,f3,f4;
> list nor=normal(i1);
> def r2=nor[1];
> setring r2;
> ideal i2=interred(norid);i2;
i2[1]=T(5)*T(6)+T(3)*T(7)

i2[2]=T(2)^2+T(3)*T(4)+T(3)*T(5)+T(2)

i2[3]=T(1)^2+T(2)*T(7)+T(3)*T(7)

i2[4]=T(4)^2*T(7)+T(4)*T(5)*T(7)+T(2)*T(6)+T(6)

i2[5]=T(2)*T(4)*T(7)+T(3)*T(6)

i2[6]=T(2)*T(3)*T(7)+T(3)^2*T(7)+T(3)*T(5)*T(7)+T(3)*T(6)*T(7)+T(4)*T(6)*T(7)
+T(1)*T(3)*T(8)+T(1)*T(4)*T(8)+T(1)*T(5)*T(8)+T(1)*T(2)*T(9)+T(1)*T(3)*T(9)
+T(1)*T(5)*T(9)+T(2)*T(5)*T(9)+T(2)*T(6)*T(9)+T(4)*T(6)*T(9)+T(2)*T(7)*T(9)
+T(3)*T(7)*T(9)+T(2)*T(9)^2+T(3)*T(9)^2+T(2)*T(6)+T(3)*T(6)+T(1)*T(8)
+T(1)*T(9)+T(3)*T(9)+T(6)*T(9)+T(6)
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i2[7]=T(1)*T(2)*T(7)+T(1)*T(3)*T(7)+T(3)^2*T(7)+T(3)*T(4)*T(7)+T(1)*T(5)*T(7)
+T(2)*T(5)*T(7)+T(3)*T(5)*T(7)+T(4)*T(5)*T(7)+T(1)*T(6)*T(7)+T(2)*T(6)*T(7)
+T(3)*T(6)*T(7)+T(1)*T(7)^2+T(2)*T(7)^2+T(3)*T(7)^2+T(1)*T(2)*T(8)
+T(1)*T(4)*T(8)+T(4)*T(5)*T(8)+T(5)*T(7)*T(8)+T(1)*T(2)*T(9)+T(1)*T(3)*T(9)
+T(3)*T(4)*T(9)+T(1)*T(5)*T(9)+T(2)*T(5)*T(9)+T(2)*T(6)*T(9)+T(3)*T(6)*T(9)
+T(4)*T(6)*T(9)+T(2)*T(9)^2+T(3)*T(9)^2+T(1)*T(2)+T(2)*T(3)+T(3)^2+T(2)*T(5)
+T(3)*T(5)+T(2)*T(6)+T(3)*T(6)+T(4)*T(6)+T(1)*T(7)+T(2)*T(7)+T(5)*T(7)
+T(6)*T(7)+T(7)^2+T(3)*T(8)+T(7)*T(8)+T(2)*T(9)+T(3)*T(9)+T(6)*T(9)+T(7)*T(9)
+T(1)

i2[8]=T(3)*T(6)^2+T(2)*T(6)*T(7)+T(4)*T(6)*T(7)+T(6)^2*T(7)+T(6)*T(7)^2
+T(1)*T(7)*T(8)+T(6)*T(7)

i2[9]=T(2)*T(4)*T(6)+T(2)*T(6)+T(4)*T(6)+T(6)^2+T(6)*T(7)+T(1)*T(8)+T(6)

i2[10]=T(1)*T(2)*T(6)+T(2)*T(6)^2+T(3)*T(4)*T(7)+T(3)*T(5)*T(7)+T(4)*T(6)*T(7)
+T(3)*T(7)^2+T(2)*T(3)*T(8)+T(1)*T(4)*T(8)+T(3)*T(4)*T(8)+T(2)*T(5)*T(8)
+T(3)*T(5)*T(8)+T(4)*T(5)*T(8)+T(5)^2*T(8)+T(2)*T(7)*T(8)+T(4)*T(7)*T(8)
+T(5)*T(7)*T(8)+T(1)*T(2)*T(9)+T(1)*T(3)*T(9)+T(1)*T(4)*T(9)+T(1)*T(5)*T(9)
+T(2)*T(5)*T(9)+T(1)*T(6)*T(9)+T(2)*T(6)*T(9)+T(3)*T(6)*T(9)+T(4)*T(6)*T(9)
+T(1)*T(7)*T(9)+T(3)*T(7)*T(9)+T(4)*T(7)*T(9)+T(5)*T(7)*T(9)+T(5)*T(8)*T(9)
+T(2)*T(9)^2+T(3)*T(9)^2+T(1)*T(2)+T(1)*T(3)+T(1)*T(5)+T(2)*T(6)+T(3)*T(6)
+T(6)^2+T(1)*T(7)+T(3)*T(7)+T(1)*T(8)+T(3)*T(8)+T(5)*T(8)+T(3)*T(9)+T(6)*T(9)
+T(1)+T(6)

i2[11]=T(4)^2*T(5)+T(4)*T(5)^2+T(2)*T(3)+T(3)

i2[12]=T(3)*T(4)*T(5)+T(4)*T(5)^2+T(3)^2+T(2)*T(5)+T(3)*T(5)+T(4)*T(5)+T(5)^2
+T(2)*T(7)+T(3)*T(7)+T(5)*T(7)+T(5)*T(9)+T(5)+T(7)

i2[13]=T(2)*T(4)*T(5)+T(3)^2

i2[14]=T(3)^2*T(5)+T(5)^3+T(1)*T(6)^2+T(1)*T(3)*T(7)+T(1)*T(4)*T(7)
+T(3)*T(5)*T(7)+T(4)*T(5)*T(7)+T(1)*T(6)*T(7)+T(2)*T(6)*T(7)+T(3)*T(6)*T(7)
+T(4)*T(6)*T(7)+T(4)*T(7)^2+T(5)*T(7)^2+T(1)*T(3)*T(8)+T(3)^2*T(8)
+T(1)*T(4)*T(8)+T(3)*T(4)*T(8)+T(1)*T(5)*T(8)+T(3)*T(5)*T(8)+T(5)^2*T(8)
+T(3)*T(7)*T(8)+T(4)*T(7)*T(8)+T(5)*T(7)*T(8)+T(1)*T(2)*T(9)+T(1)*T(3)*T(9)
+T(2)*T(3)*T(9)+T(1)*T(4)*T(9)+T(1)*T(5)*T(9)+T(3)*T(5)*T(9)+T(4)*T(5)*T(9)
+T(5)^2*T(9)+T(1)*T(6)*T(9)+T(3)*T(6)*T(9)+T(4)*T(6)*T(9)+T(1)*T(7)*T(9)
+T(2)*T(7)*T(9)+T(4)*T(7)*T(9)+T(5)*T(7)*T(9)+T(5)*T(8)*T(9)+T(2)*T(9)^2
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+T(1)*T(3)+T(3)^2+T(1)*T(5)+T(3)*T(5)+T(1)*T(6)+T(2)*T(6)+T(4)*T(6)+T(6)^2
+T(1)*T(7)+T(3)*T(7)+T(4)*T(7)+T(5)*T(7)+T(6)*T(7)+T(7)^2+T(3)*T(8)+T(5)*T(8)
+T(7)*T(8)+T(1)*T(9)+T(2)*T(9)+T(5)*T(9)+T(3)+T(5)+T(7)

i2[15]=T(3)^2*T(4)+T(3)*T(4)^2+T(2)*T(5)+T(3)*T(6)+T(4)*T(6)+T(2)*T(7)
+T(2)*T(9)+T(3)*T(9)+T(3)+T(6)

i2[16]=T(2)*T(3)*T(4)+T(3)*T(4)^2+T(4)*T(5)^2+T(2)*T(3)+T(3)*T(4)+T(4)*T(5)
+T(5)^2+T(2)*T(6)+T(4)*T(6)+T(5)*T(7)+T(2)*T(9)+T(5)*T(9)+T(5)+T(7)

i2[17]=T(1)*T(3)*T(4)+T(1)*T(4)*T(5)+T(4)*T(6)^2+T(2)*T(6)*T(7)+T(3)*T(6)*T(7)
+T(4)*T(6)*T(7)+T(6)^2*T(7)+T(3)*T(7)^2+T(6)*T(7)^2+T(1)*T(2)*T(8)
+T(1)*T(3)*T(8)+T(1)*T(5)*T(8)+T(1)*T(7)*T(8)+T(3)*T(4)*T(9)+T(2)*T(6)*T(9)
+T(1)*T(3)+T(2)*T(3)+T(3)^2+T(1)*T(5)+T(2)*T(5)+T(3)*T(5)+T(1)*T(6)+T(3)*T(6)
+T(4)*T(6)+T(6)^2+T(1)*T(7)+T(2)*T(7)+T(1)*T(8)+T(2)*T(9)+T(6)

i2[18]=T(1)*T(2)*T(4)+T(1)*T(4)*T(6)+T(1)*T(4)*T(7)+T(3)*T(4)*T(7)
+T(4)*T(5)*T(7)+T(4)*T(5)*T(8)+T(1)*T(4)+T(3)*T(7)+T(5)*T(7)+T(6)*T(7)
+T(7)^2+T(3)*T(8)+T(7)*T(9)

i2[19]=T(3)*T(4)*T(7)*T(9)+T(2)*T(5)^2+T(3)*T(5)^2+T(4)*T(5)^2+T(1)*T(3)*T(6)
+T(1)*T(4)*T(6)+T(1)*T(6)^2+T(2)*T(6)^2+T(4)*T(6)^2+T(1)*T(3)*T(7)
+T(1)*T(4)*T(7)+T(3)*T(4)*T(7)+T(2)*T(5)*T(7)+T(3)*T(5)*T(7)+T(4)*T(5)*T(7)
+T(5)^2*T(7)+T(1)*T(6)*T(7)+T(4)*T(6)*T(7)+T(6)^2*T(7)+T(4)*T(7)^2+T(5)*T(7)^2
+T(6)*T(7)^2+T(1)*T(3)*T(8)+T(2)*T(3)*T(8)+T(3)^2*T(8)+T(1)*T(4)*T(8)
+T(1)*T(5)*T(8)+T(2)*T(5)*T(8)+T(4)*T(5)*T(8)+T(2)*T(6)*T(8)+T(3)*T(6)*T(8)
+T(2)*T(7)*T(8)+T(1)*T(3)*T(9)+T(2)*T(3)*T(9)+T(3)*T(4)*T(9)+T(3)*T(6)*T(9)
+T(4)*T(6)*T(9)+T(1)*T(7)*T(9)+T(2)*T(7)*T(9)+T(3)*T(7)*T(9)+T(4)*T(7)*T(9)
+T(5)*T(7)*T(9)+T(6)*T(7)*T(9)+T(3)*T(8)*T(9)+T(2)*T(9)^2+T(3)*T(9)^2
+T(1)*T(2)+T(1)*T(3)+T(2)*T(3)+T(3)^2+T(1)*T(5)+T(4)*T(5)+T(5)^2+T(1)*T(6)
+T(2)*T(6)+T(3)*T(6)+T(4)*T(6)+T(6)^2+T(1)*T(7)+T(5)*T(7)+T(6)*T(8)+T(7)*T(8)
+T(2)*T(9)+T(3)*T(9)+T(5)*T(9)+T(6)*T(9)+T(7)*T(9)+T(1)+T(5)+T(7)

i2[20]=T(2)*T(6)*T(7)*T(8)+T(1)*T(2)*T(8)^2+T(1)*T(3)*T(8)^2+T(3)*T(4)^2*T(9)
+T(1)*T(6)^2*T(9)+T(2)*T(6)^2*T(9)+T(1)*T(3)*T(7)*T(9)+T(3)^2*T(7)*T(9)
+T(1)*T(5)*T(7)*T(9)+T(2)*T(5)*T(7)*T(9)+T(3)*T(5)*T(7)*T(9)
+T(4)*T(5)*T(7)*T(9)+T(5)^2*T(7)*T(9)+T(2)*T(6)*T(7)*T(9)+T(6)^2*T(7)*T(9)
+T(1)*T(7)^2*T(9)+T(4)*T(7)^2*T(9)+T(5)*T(7)^2*T(9)+T(6)*T(7)^2*T(9)
+T(1)*T(3)*T(8)*T(9)+T(3)^2*T(8)*T(9)+T(3)*T(4)*T(8)*T(9)+T(2)*T(5)*T(8)*T(9)
+T(4)*T(5)*T(8)*T(9)+T(5)^2*T(8)*T(9)+T(2)*T(6)*T(8)*T(9)+T(1)*T(7)*T(8)*T(9)
+T(3)*T(7)*T(8)*T(9)+T(5)*T(7)*T(8)*T(9)+T(3)*T(6)*T(9)^2+T(1)*T(7)*T(9)^2
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+T(3)*T(7)*T(9)^2+T(3)*T(8)*T(9)^2+T(5)*T(8)*T(9)^2+T(2)*T(3)*T(5)+T(3)*T(5)^2
+T(4)*T(5)^2+T(1)*T(3)*T(6)+T(4)^2*T(6)+T(1)*T(6)^2+T(2)*T(6)^2+T(4)*T(6)^2
+T(3)^2*T(7)+T(1)*T(4)*T(7)+T(3)*T(4)*T(7)+T(3)*T(5)*T(7)+T(4)*T(5)*T(7)
+T(5)^2*T(7)+T(1)*T(6)*T(7)+T(2)*T(6)*T(7)+T(3)*T(6)*T(7)+T(4)*T(7)^2
+T(5)*T(7)^2+T(1)*T(2)*T(8)+T(1)*T(3)*T(8)+T(2)*T(3)*T(8)+T(3)^2*T(8)
+T(3)*T(5)*T(8)+T(6)^2*T(8)+T(2)*T(7)*T(8)+T(3)*T(7)*T(8)+T(3)^2*T(9)
+T(2)*T(4)*T(9)+T(3)*T(4)*T(9)+T(1)*T(5)*T(9)+T(2)*T(5)*T(9)+T(3)*T(5)*T(9)
+T(2)*T(6)*T(9)+T(4)*T(6)*T(9)+T(6)^2*T(9)+T(3)*T(7)*T(9)+T(6)*T(7)*T(9)
+T(5)*T(8)*T(9)+T(6)*T(8)*T(9)+T(7)*T(8)*T(9)+T(2)*T(9)^2+T(3)*T(9)^2
+T(7)*T(9)^2+T(3)*T(4)+T(4)*T(5)+T(5)^2+T(3)*T(6)+T(6)^2+T(5)*T(7)+T(6)*T(7)
+T(3)*T(8)+T(1)*T(9)+T(2)*T(9)+T(5)*T(9)+T(6)*T(9)+T(7)*T(9)+T(3)+T(5)+T(6)
+T(7)

i2[21]=T(2)*T(4)^2*T(8)+T(3)*T(4)^2*T(8)+T(3)^2*T(7)*T(9)+T(3)*T(5)*T(7)*T(9)
+T(5)^2*T(7)*T(9)+T(2)*T(6)*T(7)*T(9)+T(4)*T(6)*T(7)*T(9)+T(3)*T(7)^2*T(9)
+T(4)*T(7)^2*T(9)+T(5)*T(7)^2*T(9)+T(1)*T(3)*T(8)*T(9)+T(1)*T(4)*T(8)*T(9)
+T(1)*T(2)*T(9)^2+T(1)*T(3)*T(9)^2+T(3)*T(4)*T(9)^2+T(1)*T(5)*T(9)^2
+T(2)*T(5)*T(9)^2+T(2)*T(6)*T(9)^2+T(4)*T(6)*T(9)^2+T(3)*T(7)*T(9)^2
+T(4)*T(7)*T(9)^2+T(5)*T(7)*T(9)^2+T(2)*T(9)^3+T(3)*T(9)^3+T(1)*T(6)^2
+T(3)^2*T(7)+T(1)*T(4)*T(7)+T(3)*T(4)*T(7)+T(1)*T(5)*T(7)+T(2)*T(5)*T(7)
+T(3)*T(5)*T(7)+T(5)^2*T(7)+T(3)*T(6)*T(7)+T(4)*T(6)*T(7)+T(6)^2*T(7)
+T(1)*T(7)^2+T(2)*T(7)^2+T(4)*T(7)^2+T(5)*T(7)^2+T(6)*T(7)^2+T(1)*T(2)*T(8)
+T(1)*T(3)*T(8)+T(3)^2*T(8)+T(4)^2*T(8)+T(3)*T(5)*T(8)+T(5)^2*T(8)
+T(5)*T(7)*T(8)+T(1)*T(3)*T(9)+T(2)*T(3)*T(9)+T(3)^2*T(9)+T(2)*T(4)*T(9)
+T(1)*T(5)*T(9)+T(2)*T(5)*T(9)+T(3)*T(5)*T(9)+T(4)*T(5)*T(9)+T(1)*T(6)*T(9)
+T(2)*T(6)*T(9)+T(3)*T(6)*T(9)+T(1)*T(7)*T(9)+T(2)*T(7)*T(9)+T(3)*T(7)*T(9)
+T(4)*T(7)*T(9)+T(6)*T(7)*T(9)+T(7)^2*T(9)+T(5)*T(8)*T(9)+T(1)*T(9)^2
+T(2)*T(9)^2+T(3)*T(9)^2+T(6)*T(9)^2+T(7)*T(9)^2+T(1)*T(2)+T(2)*T(3)+T(3)^2
+T(2)*T(5)+T(3)*T(5)+T(4)*T(6)+T(2)*T(7)+T(3)*T(7)+T(5)*T(7)+T(7)^2+T(1)*T(8)
+T(3)*T(8)+T(5)*T(8)+T(1)*T(9)+T(3)*T(9)+T(4)*T(9)+T(5)*T(9)+T(6)*T(9)
+T(7)*T(9)+T(9)^2+T(1)+T(6)+T(9)

i2[22]=T(4)*T(5)^3+T(2)*T(3)*T(5)+T(1)*T(4)*T(5)+T(4)*T(5)^2+T(3)*T(4)*T(7)
+T(4)*T(5)*T(7)+T(4)*T(5)*T(9)+T(2)*T(3)+T(1)*T(5)+T(2)*T(5)+T(5)^2+T(3)*T(6)
+T(1)*T(7)+T(2)*T(7)+T(3)*T(7)+T(4)*T(7)+T(5)*T(7)+T(5)*T(9)+T(3)+T(5)+T(7)

i2[23]=T(1)*T(5)^3+T(1)*T(6)*T(7)^2+T(1)*T(7)^3+T(1)*T(2)*T(5)*T(8)
+T(2)*T(5)^2*T(8)+T(5)^3*T(8)+T(1)*T(3)*T(6)*T(8)+T(1)*T(6)^2*T(8)
+T(3)*T(4)*T(7)*T(8)+T(1)*T(5)*T(7)*T(8)+T(2)*T(5)*T(7)*T(8)
+T(3)*T(5)*T(7)*T(8)+T(5)^2*T(7)*T(8)+T(4)*T(6)*T(7)*T(8)+T(1)*T(7)^2*T(8)
+T(2)*T(7)^2*T(8)+T(3)*T(7)^2*T(8)+T(4)*T(7)^2*T(8)+T(1)*T(2)*T(8)^2
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+T(1)*T(3)*T(8)^2+T(3)^2*T(8)^2+T(3)*T(4)*T(8)^2+T(1)*T(5)*T(8)^2
+T(3)*T(5)*T(8)^2+T(5)^2*T(8)^2+T(3)*T(7)*T(8)^2+T(4)*T(7)*T(8)^2
+T(3)*T(4)^2*T(9)+T(1)*T(2)*T(5)*T(9)+T(1)*T(3)*T(5)*T(9)+T(2)*T(5)^2*T(9)
+T(3)*T(5)^2*T(9)+T(1)*T(3)*T(6)*T(9)+T(1)*T(4)*T(6)*T(9)+T(1)*T(3)*T(7)*T(9)
+T(3)^2*T(7)*T(9)+T(2)*T(5)*T(7)*T(9)+T(3)*T(5)*T(7)*T(9)+T(1)*T(6)*T(7)*T(9)
+T(3)*T(6)*T(7)*T(9)+T(4)*T(6)*T(7)*T(9)+T(6)^2*T(7)*T(9)+T(1)*T(7)^2*T(9)
+T(6)*T(7)^2*T(9)+T(1)*T(3)*T(8)*T(9)+T(1)*T(4)*T(8)*T(9)+T(2)*T(5)*T(8)*T(9)
+T(4)*T(5)*T(8)*T(9)+T(1)*T(6)*T(8)*T(9)+T(2)*T(6)*T(8)*T(9)
+T(5)*T(7)*T(8)*T(9)+T(5)*T(8)^2*T(9)+T(1)*T(3)*T(9)^2+T(2)*T(3)*T(9)^2
+T(1)*T(4)*T(9)^2+T(3)*T(4)*T(9)^2+T(2)*T(5)*T(9)^2+T(1)*T(6)*T(9)^2
+T(3)*T(6)*T(9)^2+T(1)*T(7)*T(9)^2+T(2)*T(7)*T(9)^2+T(4)*T(7)*T(9)^2
+T(5)*T(7)*T(9)^2+T(3)*T(8)*T(9)^2+T(5)*T(8)*T(9)^2+T(1)*T(2)*T(3)
+T(1)*T(2)*T(5)+T(1)*T(3)*T(5)+T(2)*T(3)*T(5)+T(1)*T(5)^2+T(4)*T(5)^2+T(5)^3
+T(1)*T(6)^2+T(2)*T(6)^2+T(4)*T(6)^2+T(6)^3+T(3)^2*T(7)+T(1)*T(4)*T(7)
+T(3)*T(4)*T(7)+T(1)*T(5)*T(7)+T(2)*T(5)*T(7)+T(5)^2*T(7)+T(1)*T(6)*T(7)
+T(3)*T(6)*T(7)+T(4)*T(6)*T(7)+T(2)*T(7)^2+T(3)*T(7)^2+T(6)*T(7)^2
+T(1)*T(2)*T(8)+T(2)*T(3)*T(8)+T(3)^2*T(8)+T(1)*T(4)*T(8)+T(2)*T(5)*T(8)
+T(3)*T(5)*T(8)+T(1)*T(6)*T(8)+T(3)*T(6)*T(8)+T(6)^2*T(8)+T(1)*T(7)*T(8)
+T(2)*T(7)*T(8)+T(3)*T(7)*T(8)+T(4)*T(7)*T(8)+T(5)*T(7)*T(8)+T(6)*T(7)*T(8)
+T(3)*T(8)^2+T(5)*T(8)^2+T(2)*T(3)*T(9)+T(3)*T(4)*T(9)+T(2)*T(5)*T(9)
+T(4)*T(5)*T(9)+T(5)^2*T(9)+T(2)*T(6)*T(9)+T(1)*T(7)*T(9)+T(2)*T(7)*T(9)
+T(3)*T(7)*T(9)+T(5)*T(7)*T(9)+T(6)*T(7)*T(9)+T(6)*T(8)*T(9)+T(7)*T(8)*T(9)
+T(1)*T(3)+T(2)*T(3)+T(1)*T(5)+T(3)*T(5)+T(4)*T(5)+T(5)^2+T(6)^2+T(3)*T(7)
+T(4)*T(7)+T(6)*T(7)+T(7)^2+T(1)*T(8)+T(3)*T(8)+T(5)*T(8)+T(7)*T(8)+T(6)*T(9)
+T(3)

i2[24]=T(4)^4+T(4)*T(5)*T(8)^2+T(4)^3*T(9)+T(3)^2*T(7)*T(9)
+T(3)*T(5)*T(7)*T(9)+T(5)^2*T(7)*T(9)+T(3)*T(6)*T(7)*T(9)+T(4)*T(6)*T(7)*T(9)
+T(6)^2*T(7)*T(9)+T(4)*T(7)^2*T(9)+T(5)*T(7)^2*T(9)+T(6)*T(7)^2*T(9)
+T(1)*T(2)*T(8)*T(9)+T(1)*T(4)*T(8)*T(9)+T(1)*T(5)*T(8)*T(9)
+T(1)*T(7)*T(8)*T(9)+T(1)*T(2)*T(9)^2+T(1)*T(3)*T(9)^2+T(1)*T(5)*T(9)^2
+T(2)*T(5)*T(9)^2+T(4)*T(6)*T(9)^2+T(3)*T(7)*T(9)^2+T(4)*T(7)*T(9)^2
+T(5)*T(7)*T(9)^2+T(2)*T(9)^3+T(3)*T(9)^3+T(1)*T(4)^2+T(2)*T(4)^2+T(3)*T(5)^2
+T(4)*T(5)^2+T(5)^3+T(1)*T(3)*T(6)+T(1)*T(6)^2+T(4)*T(6)^2+T(6)^3
+T(1)*T(5)*T(7)+T(2)*T(6)*T(7)+T(3)*T(6)*T(7)+T(1)*T(7)^2+T(4)*T(7)^2
+T(5)*T(7)^2+T(6)*T(7)^2+T(1)*T(3)*T(8)+T(2)*T(3)*T(8)+T(1)*T(4)*T(8)
+T(2)*T(4)*T(8)+T(1)*T(5)*T(8)+T(2)*T(5)*T(8)+T(3)*T(5)*T(8)+T(4)*T(5)*T(8)
+T(2)*T(6)*T(8)+T(4)*T(6)*T(8)+T(1)*T(7)*T(8)+T(2)*T(7)*T(8)+T(4)*T(7)*T(8)
+T(2)*T(8)^2+T(5)*T(8)^2+T(1)*T(3)*T(9)+T(2)*T(3)*T(9)+T(2)*T(4)*T(9)
+T(4)^2*T(9)+T(1)*T(5)*T(9)+T(3)*T(5)*T(9)+T(2)*T(6)*T(9)+T(3)*T(6)*T(9)
+T(1)*T(7)*T(9)+T(2)*T(7)*T(9)+T(3)*T(7)*T(9)+T(7)^2*T(9)+T(1)*T(8)*T(9)
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+T(3)*T(8)*T(9)+T(7)*T(8)*T(9)+T(5)*T(9)^2+T(6)*T(9)^2+T(7)*T(9)^2+T(1)*T(2)
+T(3)*T(4)+T(4)^2+T(1)*T(5)+T(2)*T(5)+T(3)*T(5)+T(4)*T(5)+T(5)^2+T(3)*T(6)
+T(6)^2+T(1)*T(7)+T(2)*T(7)+T(3)*T(7)+T(6)*T(7)+T(7)^2+T(3)*T(8)+T(5)*T(8)
+T(6)*T(8)+T(8)^2+T(2)*T(9)+T(5)*T(9)+T(7)*T(9)+T(1)+T(3)+T(6)

i2[25]=T(3)*T(4)^3+T(3)*T(4)^2+T(1)*T(5)^2+T(1)*T(4)*T(6)+T(4)*T(6)^2
+T(1)*T(5)*T(7)+T(2)*T(5)*T(7)+T(4)*T(5)*T(7)+T(5)^2*T(7)+T(3)*T(6)*T(7)
+T(6)^2*T(7)+T(2)*T(7)^2+T(4)*T(7)^2+T(5)*T(7)^2+T(6)*T(7)^2+T(2)*T(3)*T(8)
+T(3)*T(4)*T(8)+T(2)*T(5)*T(8)+T(3)*T(5)*T(8)+T(5)^2*T(8)+T(1)*T(7)*T(8)
+T(2)*T(7)*T(8)+T(4)*T(7)*T(8)+T(5)*T(7)*T(8)+T(1)*T(2)*T(9)+T(1)*T(4)*T(9)
+T(1)*T(5)*T(9)+T(1)*T(6)*T(9)+T(2)*T(6)*T(9)+T(3)*T(6)*T(9)+T(1)*T(7)*T(9)
+T(3)*T(7)*T(9)+T(4)*T(7)*T(9)+T(5)*T(7)*T(9)+T(5)*T(8)*T(9)+T(1)*T(2)
+T(1)*T(3)+T(1)*T(6)+T(3)*T(6)+T(6)^2+T(7)*T(8)+T(1)*T(9)+T(7)*T(9)+T(1)

i2[26]=T(2)*T(4)^3+T(3)*T(4)^2+T(4)*T(6)^2+T(3)^2*T(7)+T(3)*T(4)*T(7)
+T(3)*T(5)*T(7)+T(5)^2*T(7)+T(3)*T(6)*T(7)+T(6)^2*T(7)+T(4)*T(7)^2+T(5)*T(7)^2
+T(6)*T(7)^2+T(1)*T(2)*T(8)+T(1)*T(4)*T(8)+T(1)*T(5)*T(8)+T(4)*T(5)*T(8)
+T(1)*T(7)*T(8)+T(1)*T(2)*T(9)+T(1)*T(3)*T(9)+T(1)*T(4)*T(9)+T(1)*T(5)*T(9)
+T(2)*T(5)*T(9)+T(4)*T(6)*T(9)+T(3)*T(7)*T(9)+T(4)*T(7)*T(9)+T(5)*T(7)*T(9)
+T(2)*T(9)^2+T(3)*T(9)^2+T(1)*T(2)+T(1)*T(4)+T(1)*T(5)+T(1)*T(6)+T(2)*T(6)
+T(3)*T(6)+T(6)^2+T(1)*T(7)+T(5)*T(7)+T(7)^2+T(1)*T(8)+T(2)*T(8)+T(5)*T(8)
+T(3)*T(9)+T(6)*T(9)+T(1)+T(6)

i2[27]=T(1)*T(4)^3+T(4)*T(6)^2+T(3)^2*T(7)+T(3)*T(5)*T(7)+T(5)^2*T(7)
+T(3)*T(6)*T(7)+T(6)^2*T(7)+T(4)*T(7)^2+T(5)*T(7)^2+T(6)*T(7)^2+T(1)*T(2)*T(8)
+T(1)*T(4)*T(8)+T(1)*T(5)*T(8)+T(1)*T(7)*T(8)+T(1)*T(2)*T(9)+T(1)*T(3)*T(9)
+T(1)*T(5)*T(9)+T(2)*T(5)*T(9)+T(4)*T(6)*T(9)+T(3)*T(7)*T(9)+T(4)*T(7)*T(9)
+T(5)*T(7)*T(9)+T(2)*T(9)^2+T(3)*T(9)^2+T(2)*T(6)+T(3)*T(6)+T(6)^2+T(3)*T(7)
+T(5)*T(7)+T(6)*T(7)+T(7)^2+T(1)*T(8)+T(1)*T(9)+T(3)*T(9)+T(6)*T(9)+T(6)

i2[28]=T(4)*T(5)*T(8)^2*T(9)+T(4)^3*T(9)^2+T(2)*T(6)*T(7)*T(9)^2
+T(3)*T(6)*T(7)*T(9)^2+T(6)^2*T(7)*T(9)^2+T(3)*T(7)^2*T(9)^2
+T(6)*T(7)^2*T(9)^2+T(1)*T(2)*T(8)*T(9)^2+T(1)*T(3)*T(8)*T(9)^2
+T(1)*T(5)*T(8)*T(9)^2+T(1)*T(7)*T(8)*T(9)^2+T(2)*T(6)*T(9)^3
+T(3)*T(6)*T(7)*T(8)+T(6)^2*T(7)*T(8)+T(4)*T(7)^2*T(8)+T(6)*T(7)^2*T(8)
+T(1)*T(2)*T(8)^2+T(2)*T(4)*T(8)^2+T(3)*T(4)*T(8)^2+T(1)*T(7)*T(8)^2
+T(4)*T(7)*T(8)^2+T(3)*T(5)^2*T(9)+T(5)^3*T(9)+T(1)*T(3)*T(7)*T(9)
+T(1)*T(4)*T(7)*T(9)+T(3)*T(5)*T(7)*T(9)+T(5)^2*T(7)*T(9)+T(3)*T(6)*T(7)*T(9)
+T(4)*T(6)*T(7)*T(9)+T(3)*T(7)^2*T(9)+T(5)*T(7)^2*T(9)+T(2)*T(3)*T(8)*T(9)
+T(1)*T(4)*T(8)*T(9)+T(2)*T(4)*T(8)*T(9)+T(4)^2*T(8)*T(9)+T(4)*T(5)*T(8)*T(9)
+T(1)*T(6)*T(8)*T(9)+T(3)*T(6)*T(8)*T(9)+T(1)*T(7)*T(8)*T(9)
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+T(2)*T(7)*T(8)*T(9)+T(4)*T(7)*T(8)*T(9)+T(2)*T(8)^2*T(9)+T(1)*T(2)*T(9)^2
+T(1)*T(3)*T(9)^2+T(2)*T(3)*T(9)^2+T(2)*T(4)*T(9)^2+T(3)*T(4)*T(9)^2
+T(4)^2*T(9)^2+T(3)*T(5)*T(9)^2+T(1)*T(6)*T(9)^2+T(3)*T(6)*T(9)^2
+T(4)*T(6)*T(9)^2+T(6)^2*T(9)^2+T(3)*T(7)*T(9)^2+T(4)*T(7)*T(9)^2
+T(6)*T(7)*T(9)^2+T(1)*T(8)*T(9)^2+T(2)*T(8)*T(9)^2+T(3)*T(8)*T(9)^2
+T(5)*T(8)*T(9)^2+T(7)*T(8)*T(9)^2+T(3)*T(9)^3+T(5)*T(9)^3+T(2)*T(4)^2+T(4)^3
+T(3)*T(5)^2+T(4)*T(5)^2+T(5)^3+T(1)*T(3)*T(6)+T(4)*T(6)^2+T(1)*T(3)*T(7)
+T(1)*T(4)*T(7)+T(3)*T(4)*T(7)+T(1)*T(5)*T(7)+T(4)*T(5)*T(7)+T(1)*T(6)*T(7)
+T(3)*T(6)*T(7)+T(4)*T(6)*T(7)+T(6)^2*T(7)+T(1)*T(7)^2+T(3)*T(7)^2+T(4)*T(7)^2
+T(5)*T(7)^2+T(6)*T(7)^2+T(1)*T(2)*T(8)+T(1)*T(3)*T(8)+T(2)*T(3)*T(8)
+T(1)*T(4)*T(8)+T(3)*T(4)*T(8)+T(5)^2*T(8)+T(3)*T(6)*T(8)+T(4)*T(6)*T(8)
+T(6)^2*T(8)+T(1)*T(7)*T(8)+T(3)*T(7)*T(8)+T(7)^2*T(8)+T(1)*T(8)^2+T(2)*T(8)^2
+T(4)*T(8)^2+T(5)*T(8)^2+T(6)*T(8)^2+T(7)*T(8)^2+T(1)*T(2)*T(9)+T(2)*T(3)*T(9)
+T(1)*T(5)*T(9)+T(4)*T(5)*T(9)+T(5)^2*T(9)+T(2)*T(6)*T(9)+T(4)*T(6)*T(9)
+T(3)*T(7)*T(9)+T(1)*T(8)*T(9)+T(2)*T(8)*T(9)+T(3)*T(8)*T(9)+T(5)*T(8)*T(9)
+T(6)*T(8)*T(9)+T(8)^2*T(9)+T(2)*T(9)^2+T(3)*T(9)^2+T(4)*T(9)^2+T(5)*T(9)^2
+T(6)*T(9)^2+T(9)^3+T(1)*T(3)+T(2)*T(3)+T(1)*T(4)+T(4)^2+T(2)*T(5)+T(3)*T(5)
+T(4)*T(5)+T(2)*T(6)+T(2)*T(7)+T(4)*T(7)+T(5)*T(7)+T(6)*T(7)+T(7)^2+T(1)*T(8)
+T(2)*T(8)+T(3)*T(8)+T(5)*T(8)+T(7)*T(8)+T(8)^2+T(7)*T(9)+T(8)*T(9)+T(9)^2
+T(3)+T(5)+T(6)+T(7)
```

(This was produced by the author's own program, written in MAGMA but generally untested, employing the author's q-th power algorithm, and may contain transcription errors.)

$$m2 * m2$$
$$+m3 * (x_2x_1);$$
$$m3 * m2$$
$$+m2$$
$$+m1 * (x_2^2x_1 + x_2x_1^2);$$
$$m3 * m3$$
$$+m3$$
$$+m2 * (x_2 + x_1);$$
$$m4 * m2$$
$$+m5;$$
$$m4 * m3$$
$$+m7;$$
$$m4 * m4$$
$$+m10$$
$$+m8 * x_1;$$
$$m5 * m2$$
$$+m7 * (x_2x_1);$$
$$m5 * m3$$
$$+m5$$
$$+m4 * (x_2^2x_1 + x_2x_1^2);$$
$$m5 * m4$$
$$+m10 * (x_2^2x_1 + x_2x_1 + x_1)$$
$$+m9 * (x_2x_1^2 + x_1)$$
$$+m8 * x_1;$$
$$m5 * m5$$
$$+m10 * (x_2^3x_1^2 + x_2^2x_1^3 + x_2^2x_1^2 + x_2x_1^3 + x_2x_1)$$
$$+m9 * (x_2^2x_1^2 + x_2x_1^3)$$
$$+m8 * (x_2^2x_1^2 + x_2x_1^3 + x_2x_1^2);$$

$m6 * m2$
$$+m7 * x_2^2$$
$$+m4 * (x_2^2 + x_2x_1);$$
$m6 * m3$
$$+m5$$
$$+m4 * (x_2^3 + x_2^2x_1);$$
$m6 * m4$
$$+m10 * (x_2^3 + x_2x_1 + x_1)$$
$$+m9 * (x_2^2x_1 + x_1)$$
$$+m8 * x_1;$$
$m6 * m5$
$$+m10 * (x_2^4x_1 + x_2^3x_1^2 + x_2^3x_1 + x_2^2x_1^2 + x_2x_1)$$
$$+m9 * (x_2^3x_1 + x_2^2x_1^2)$$
$$+m8 * (x_2^3x_1 + x_2^2x_1^2 + x_2x_1^2);$$
$m6 * m6$
$$+m10 * (x_2^5 + x_2^4x_1 + x_2^4 + x_2^3x_1 + x_2^2)$$
$$+m9 * (x_2^4 + x_2^3x_1 + x_2^2 + x_2x_1)$$
$$+m8 * (x_2^4 + x_2^3x_1 + x_2^3);$$
$m7 * m2$
$$+m5$$
$$+m4 * (x_2^2x_1 + x_2x_1^2);$$
$m7 * m3$
$$+m7$$
$$+m5 * (x_2 + x_1);$$
$m7 * m4$
$$+m10 * (x_2^2x_1 + x_2x_1^2 + x_2x_1 + x_1^2 + 1)$$
$$+m9 * (x_2x_1 + x_1^2)$$
$$+m8 * (x_2x_1 + x_1^2 + x_1);$$
$m7 * m5$
$$+m10 * (x_2x_1^2 + x_2x_1 + x_1);$$
$$+m9 * (x_2x_1^2 + x_1)$$
$$+m8 * (x_2^2x_1^2 + x_2x_1^3 + x_1);$$
$m7 * m6$
$$+m10 * (x_2^3 + x_2x_1 + x_1)$$
$$+m9 * (x_2x_1^2 + x_1)$$
$$+m8 * (x_2^3x_1 + x_2^2x_1^2 + x_1);$$
$m7 * m7$
$$+m10 * (x_2^3x_1 + x_2^2x_1^2 + 1)$$
$$+m9 * (x_2^2x_1^2 + x_2x_1^3)$$
$$+m8 * x_1;$$

$m8 * m2$

$+m10 * (x_2^2 + x_2)$
$+m9 * (x_2x_1);$

$m8 * m3$

$+m10 * (x_2^2 + x_2x_1 + x_2 + x_1 + 1)$
$+m9 * (x_2 + x_1 + 1)$
$+m8 * (x_2 + x_1 + 1);$

$m8 * m4$

$+m7 * (x_2 + 1)$
$+m5 * (x_2 + 1)$
$+m4 * (x_2 + 1)$
$+m3 * (x_2^3x_1)$
$+m2 * (x_2^3 + x_2^2x_1 + x_2^2);$

$m8 * m5$

$+m7 * (x_2^2x_1 + x_2x_1)$
$+m4 * (x_2^3x_1 + x_2^3x_1^2 + x_2^2x_1 + x_2x_1^2)$
$+m3 * (x_2^4x_1 + x_2^3x_1^2 + x_2^3x_1)$
$+m2 * (x_2^3x_1)$
$+m1 * (x_2^5x_1^2 + x_2^4x_1^3);$

$m8 * m6$

$+m7 * (x_2^3 + x_2^2)$
$+m6 * (x_2 + 1)$
$+m5 * (x_2 + 1)$
$+m4 * (x_2^4 + x_2^3x_1 + x_2^2 + x_2x_1)$
$+m3 * (x_2^5 + x_2^4x_1 + x_2^4)$
$+m2 * (x_2^3x_1)$
$+m1 * (x_2^6x_1 + x_2^5x_1^2 + x_2^5 + x_2^3x_1^2 + x_2^4 + x_2^3x_1);$

$m8 * m7$

$+m5 * (x_2^2 + x_2x_1 + x_1 + 1)$
$+m4 * (x_2^3x_1 + x_2^2x_1^2 + x_2^2x_1 + x_2x_1^2)$
$+m3 * (x_2^3x_1)$
$+m2 * (x_2^4x_1 + x_2^3x_1^2 + x_2^3 + x_2^2x_1 + x_2^2)$
$+m1 * (x_2^5x_1 + x_2^3x_1^3 + x_2^4x_1 + x_2^3x_1^2);$

$m8 * m8$

$+m10 * (x_2^2x_1 + x_2 + x_1 + 1)$
$+m9 * (x_2^2x_1 + x_2^2 + x_1 + 1)$
$+m8 * (x_2^2 + x_2x_1 + x_2 + x_1)$
$+m7 * x_2^3$
$+m6 * x_2^2$
$+m5 * x_2^2;$

$m9 * m2$
$+m10 * (x_2^2 + x_2x_1 + x_1)$
$+m9 * x_1$
$+m8 * x_1;$
$m9 * m3$
$+m10 * (x_2 + 1)$
$+m9 * x_2$
$+m8 * (x_2 + x1);$
$m9 * m4$
$+m7 * (x_2 + 1)$
$+m4 * (x_2x_1 + x_2 + x_1 + 1)$
$+m3 * (x_2^2x_1)$
$+m2 * x_2^3$
$+m1 * (x_2^4x_1 + x_2^3x_1^2);$
$m9 * m5$
$+m5 * (x_2x_1 + x_1)$
$+m4 * (x_2^3x_1 + x_2^2x_1^2 + x_2^2x_1 + x_2x_1^2)$
$+m3 * (x_2^4x_1)$
$+m2 * (x_2^4x_1 + x_2^3x_1^2 + x_2^2x_1)$
$+m1 * (x_2^4x_1^2 + x_2^3x_1^3);$
$m9 * m6$
$+m7 * (x_2 + 1)$
$+m6 * (x_2 + 1)$
$+m5 * (x_2^2 + 1)$
$+m4 * (x_2^3 + x_2^2x_1 + x_2 + 1)$
$+m3 * (x_2^5 + x_2^4 + x_2^3x_1)$
$+m2 * (x_2^5 + x_2^4x_1 + x_2^2x_1)$
$+m1 * (x_2^5x_1 + x_2^4x_1^2 + x_2^5 + x_2^4x_1 + x_2^4 + x_2^3x_1);$
$m9 * m7$
$+m7 * (x_2x_1 + x_1)$
$+m5 * (x_2^2 + x_2x_1 + x_2 + x_1)$
$+m3 * (x_2^4x_1 + x_2^3x_1^2 + x_2^2x_1)$
$+m2 * (x_2^3x_1 + x_2^2x_1^2 + x_2^3)$
$+m1 * (x_2^5x_1 + x_2^4x_1^2);$

$m9 * m8$
$$+m10 * (x_2^3 + x_2^2x_1 + x_1 + 1)$$
$$+m9 * (x_2^2 + x_2x_1 + x_1 + 1)$$
$$+m8 * (x_2^2 + x_2)$$
$$+m7 * x_2^2$$
$$+m6 * x_2^2$$
$$+m4 * (x_2^4 + x_2^3x_1 + x_2^2);$$

$m9 * m9$
$$+m10 * (x_2^2 + 1)$$
$$+m9 * (x_2^2 + x_2x_1 + x_1 + 1)$$
$$+m8 * (x_2^2 + x_2x_1 + x_2 + x_1)$$
$$+m6 * x_2^2$$
$$+m5 * x_2^2$$
$$+m4 * (x_2^2x_1);$$

$m10 * m2$
$$+m10 * x_1$$
$$+m9 * x_1$$
$$+m8 * x_1;$$

$m10 * m3$
$$+m10 * (x_1 + 1)$$
$$+m9 * x_1;$$

$m10 * m4$
$$+m4 * (x_2x_1 + x_1)$$
$$+m3 * (x_2^2x_1)$$
$$+m2 * (x_2^2x_1);$$

$m10 * m5$
$$+m5 * (x_2x_1 + x_1)$$
$$+m3 * (x_2^3x_1^2)$$
$$+m2 * (x_2^2x_1)$$
$$+m1 * (x_2^4x_1^2 + x_2^3x_1^3);$$

$m10 * m6$
$$+m7 * (x_2 + 1)$$
$$+m5 * (x_2^2 + x_2)$$
$$+m4 * (x_2 + 1)$$
$$+m3 * (x_2^4 x_1)$$
$$+m2 * (x_2^2 x_1)$$
$$+m1 * (x_2^5 x_1 + x_2^4 x_1^2 + x_2^4 x_1 + x_2^3 x_1^2);$$

$m10 * m7$
$$+m7 * (x_2 x_1 + x_1)$$
$$+m3 * (x_2^2 x_1)$$
$$+m2 * (x_2^3 x_1 + x_2^2 x_1^2 + x_2^2 x_1)$$
$$+m1 * (x_2^4 x_1^2 + x_2^3 x_1^3);$$

$m10 * m8$
$$+m8 * (x_2 x_1 + x_1)$$
$$+m7 * x_2^2$$
$$+m5 * x_2^2$$
$$+m4 * x_2^2;$$

$m10 * m9$
$$+m9 * (x_2 x_1 + x_1)$$
$$+m7 * x_2^2$$
$$+m4 * (x_2^2 x_1 + x_2^2);$$

$m10 * m10$
$$+m10 * (x_2 x_1 + x_1)$$
$$+m4 * (x_2^2 x_1);$$

$m6 * x_1 + m5 * x_2$
$$+m7 + m4;$$