ON A CLASSIFICATION OF Δ-FREE DISTANCE-REGULAR GRAPHS (with a small application of Groebner bases)

Aleksandar Jurišić

Institute of Mathematics, Physics and Mechanics

This is a joint work with J. Koolen and A. Žitnik

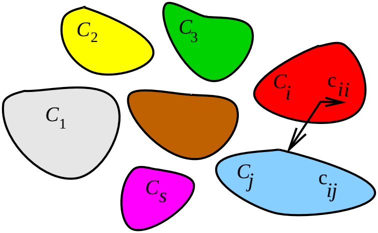
Linz, Austria

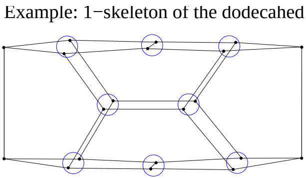
May 6, 2006

An **equitable partition** of a graph Γ is a partition of the vertex set $V(\Gamma)$ into **parts** $C_1, C_2, \ldots, C_s$ s.t.

(a) vertices of each part C_i induce a *regular* graph,

(b) edges between C_i and C_j induce a *half-regular* graph.



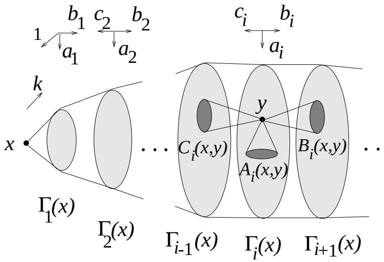


Numbers c_{ij} are the parameters of the partition.

Distance-regularity

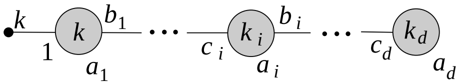
Γ graph, diameter d, $\forall\, x \in V(\Gamma)$

the **distance partition** $\{\Gamma_0(x), \Gamma_1(x), \ldots, \Gamma_d(x)\}$ corresp. to x



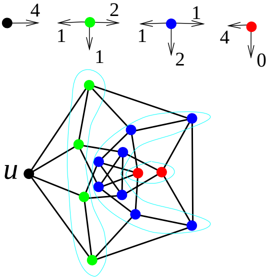
is *equitable* and the **intersection array**

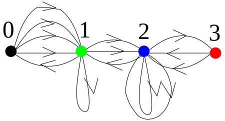
$\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$ is *independent* of x.



A small example of a distance-regular graph

(antipodal, i.e., being at distance diam. is a transitive relation)



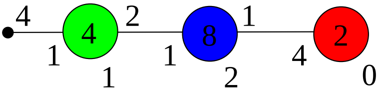


$$\begin{bmatrix} 0 & b_0 & & \\ c_1 & a_1 & b_1 & \\ & c_2 & a_2 & b_2 \\ & & c_3 & a_3 \end{bmatrix} = \begin{bmatrix} 0 & 4 & & \\ 1 & 1 & 2 & \\ & 1 & 2 & 1 \\ & & 4 & 0 \end{bmatrix}$$

The above parameters are the same for each vertex u:

$\{b_0, b_1, b_2; c_1, c_2, , c_3\}$

$= \{4, 2, 1; 1, 1, 4\}$.



Intersection numbers

Set $\boldsymbol{p_{ij}^h} := |\Gamma_i(u) \cap \Gamma_j(v)|$, where $d(u,v)=h$.

Then $a_i = p_{i1}^i$, $b_i = p_{i+1,1}^i$, $c_i = p_{i-1,1}^i$ and $k_i = p_{ii}^0$.

All the intersection numbers p_{ij}^h are determined by the numbers in the intersection array of Γ, so they do not depend on the choice of u and v at distance h.

Proof goes by induction on i, using the following recurrence relation:

$$c_{j+1}p_{i,j+1}^h + a_j p_{ij}^h + b_{j-1}p_{i,j-1}^h = c_{i+1}p_{i+1,j}^h + a_i p_{ij}^h + b_{i-1}p_{i-1,j}^h$$

obtained by a 2-way counting of edges between $\Gamma_i(u)$ and $\Gamma_j(v)$.

Let $\Gamma = (V, E)$ be a simple graph, diam. d, $V(\Gamma) = \{1, \ldots, n\}$. Then $A = A(\Gamma)$ is $(n \times n)$-dimensional **adjacency matrix** of Γ, when

$$A_{i,j} = \begin{cases} 1, & \text{if } \{i, j\} \in E, \\ 0, & \text{otherwise} \end{cases}$$

For $i \in \{0, \ldots, d\}$ we define the ***i*-th distance graph G_i** by

$$V(G_i) = V(G), \quad x \sim_{G_i} y \iff d_G(x, y) = i.$$

If Γ is distance-regular, then its distance matrices $A_i = A(G_i)$ form an association scheme (symmetric),

i.e., they are symmetric 01-matrices $I = A_0, A_1, \ldots, A_d$ s.t.

$$\sum_{i=0}^{d} A_i = J \quad \text{and} \quad A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h.$$

An association scheme $\mathcal{A}$ is ***P*-polynomial** (called also **metric**) when there there exists a permutation of indices of A_i's, s.t.

$$\exists \text{ polynomials } p_i \text{ of degree } i \text{ s.t. } A_i = p_i(A_1),$$

or equivalently,

the intersection numbers satisfy the **Δ-condition**

(that is, $\forall\, i, j, h \in \{0, \ldots, d\}$

- $p_{ij}^h \neq 0$ implies $h \leq i + j$ and
- $p_{ij}^{i+j} \neq 0$).

Γ distance-regular, diam. d. We say Γ is **primitive**, when all the distance graphs $\Gamma_1, \ldots, \Gamma_d$ are connected (and **imprimitive** otherwise).

Theorem (Smith).

An imprimitive distance-regular graph is either antipodal or bipartite.

The big project of classifying distance-regular graphs:

(a) **find all primitive distance-regular graphs,**

(b) given a distance-regular graph Γ,
find all imprimitive graphs, which give rise to Γ.

Theorem (Van Bon and Brouwer, 1987).

Most classical distance-regular graphs have no antipodal covers.

CONNECTIONS

– **projective and affine planes**,

for $d = 3$, or $d = 4$ and $r = k$ (covers of K_n or $K_{n,n}$),

– **Two graphs** (Q-polynomial), for $d = 3$ and $r = 2$,

– **Moore graphs**, for $d = 3$ and $r = k$,

– **Hadamard matrices**, $d = 4$ and $r = 2$ (covers of $K_{n,n}$),

– **group divisible resolvable designs**, $d = 4$ (covers of $K_{n,n}$),

– finite geometry (e.g. generalized quadrangles),

– coding theory (e.g. perfect codes),

– group theory (classification of finite simple groups),

– orthogonal polynomials.

TOOLS

– graph theory, counting,...

– matrix theory (rank mod p),

– eigenvalue techniques,

– representation theory of graphs,

– geometry (Euclidean, non-Euclidean and finite),

– algebra and association schemes,

– topology (covers and universal objects).

GOALS

– structure of distance-regular graphs (e.g. antipodal covers),

– constructions of new infinite families,

– nonexistence and uniqueness,

– characterizations,

– new techniques, which can be applied to distance-regular graphs (and elsewhere)

Difficult problems:

Find a distance-regular graph $\{15, 12, 1; 1, 2, 15\}$ (7-cover of K_{16}).

Find a distance-regular graph $\{22, 21, 3, 1; 1, 3, 21, 22\}$ (a double-cover of the Higman-Sims graph $\{22, 21; 1, 6\}$).

Bose-Mesner algebra, two bases and duality

Let $E_0, E_1, \ldots, E_d$ be the *primitive idempotents* of the *Bose-Mesner algebra* $\mathcal{M}$ (algebra generated by $A_0, \ldots, A_d$). Then $\exists\, \theta_0, \ldots, \theta_d \in \mathbb{R}$ s.t.

$$A = \sum_{h=0}^{d} \theta_h E_h$$

and $\theta_0, \ldots, \theta_d$ are called the **eigenvalues** of A.

(The matrix of eigenvalues P).

Let $\theta \in \mathrm{ev}(\Gamma)$ and E the associated primitive idempotent. For $\boldsymbol{x} \in E(\mathbb{R}^n) \backslash \{0\}$ we have $A\boldsymbol{x} = \theta \boldsymbol{x}$, i.e., $\forall i$ $\sum_{j \sim i} x_j = \theta x_i$, and

$$E = \frac{m_\theta}{|V\Gamma|} \sum_{h=0}^{d} \omega_h A_h.$$

$\boldsymbol{\omega_0, \ldots, \omega_d}$ is the **cosine sequence** of E (or θ).

(The matrix of dual eigenvalues Q).

Lemma 1. Γ *distance-regular, diam.* $d \geq 2$,

E *is a primitive idempotent of* Γ *corresponding to* θ,
$\omega_0, \ldots, \omega_d$ *is the cosine sequence of* θ.

For $x, y \in V\Gamma$, $i = \partial(x, y)$ *we have*

(i) $\langle Ex, Ey \rangle = xy$*-entry of* $E = \omega_i \dfrac{m_\theta}{|V\Gamma|}$.

(ii) ω_i *is the cosine of the angle between the vectors* Ex *and* Ey.

(ii) $\omega_0 = 1$ *and* $c_i\omega_{i-1} + a_i\omega_i + b_i\omega_{i+1} = \theta\omega_i$ *for* $0 \leq i \leq d$.

$$\omega_1 = \frac{\theta}{k}, \qquad \omega_2 = \frac{\theta^2 - a_1\theta - k}{kb_1}, \qquad 1 - \omega_2 = \frac{(k-\theta)(\theta + k - a_1)}{kb_1}$$

Let I be the identity matrix and J the all ones matrix of dimension $k \times k$. It is easy to see that for any scalars a and b

$$\mathbf{det}(aI + bJ) = a^{k-1}(a + kb).$$

Let Γ be a **Δ-free** distance-regular graph, $E = E_\theta$ and $\theta \neq \pm k$.

For $\Gamma(x) = \{x_1, \ldots, x_k\}$ and $G_{ij} = \langle Ex_i, Ex_j \rangle$ we have

$$G = \frac{m_\theta}{|V\Gamma|}\left(I_k + \omega_2(J_k - I_k)\right) = \frac{m_\theta}{|V\Gamma|}\left((1-\omega_2)I_k + \omega_2 J_k\right) \text{ and}$$

$$(|V\Gamma|/m_\theta)^k \ \mathbf{det}(G) = (1-\omega_2)^{k-1}(1 + (k-1)\omega_2).$$

By $a_1 = 0$ and $\theta \neq \pm k$ we have $1 = \omega_0 \neq \omega_2$, so $\det(G) \neq 0$ if and only if $\omega_2 \neq -1/(k-1)$, i.e., $\theta \neq 0$. ... Hence,

$$m_\theta \geq k$$

(this inequality follows also from Terwilliger tree-bound).

Let us assume the equality holds, i.e., $\boldsymbol{m_\theta = k}$.

Distance-regular graphs coming from a **formally self-dual** association schemes satisfy this property (used in knot theory).

Characterization [JKM'2004]

In a **triangle- and pentagon-free** *(i.e., $a_1 = a_2 = 0$) distance-regular graph Γ with diam. $d \geq 2$ and valency $k \geq 3$*

$$\boldsymbol{m_\theta = k} \iff \Gamma \text{ is one of the following}$$

(i) $K_{k+1,k+1} \backslash$*a matching,* $\mathbf{Q_d}$, *folded* $\mathbf{Q_d}$ *($7 \leq d$ odd),*

(ii) a Hadamard graph, i.e.,

$$\{4g, 4g-1, 2g, 1; 1, 2g, 4g-1, 4g\}, \quad g \in \mathbb{N},$$

(iii) a distance-regular graph with intersection array

$$\{k, k-1, k-c, c, 1; 1, c, k-c, k-1, k\},$$

$k = g(g^2+3g+1)$, $c = g(g+1)$ *and* $g \in \mathbb{N}$.

[JKM'2004] $\theta \neq 0$ and the neighbourhood $\Gamma(x)$, $\forall x \in V\Gamma$, projected to the eigenspace of θ, forms a basis of this space.

Lemma 2. Γ **Δ-free** *nonbip. DR, diam.* $d \geq 2$, *valency* $k \geq 3$.

E *a primitive idempotent of* Γ *s.t.* $\boldsymbol{m_E = k}$.

$\omega_0, \ldots, \omega_d$ *is the cosine sequence of* E.

Then for $x, y \in V\Gamma$, $\partial(x,y) = i$, $2 \leq i \leq d$ *we have*

$$Ey = C_i \sum_{z \in \Gamma(x) \cap \Gamma_{i-1}(y)} Ez + A_i \sum_{z \in \Gamma(x) \cap \Gamma_i(y)} Ez + B_i \sum_{z \in \Gamma(x) \cap \Gamma_{i+1}(y)} Ez,$$

where $C_i = \dfrac{\omega_1\omega_{i-1} - \omega_2\omega_i}{\omega_1(1-\omega_2)}$, $A_i = \dfrac{(\omega_1 - \omega_2)\omega_i}{\omega_1(1-\omega_2)}$, $B_i = \dfrac{\omega_1\omega_{i+1} - \omega_2\omega_i}{\omega_1(1-\omega_2)}$.

Proof.

$\exists\{\alpha_z \in \mathbb{R}, z \in \Gamma(x)\}$, s.t.

$$Ey = \sum_{z\in\Gamma(x)} \alpha_z Ez.$$

Taking the scalar product with $Ex,\ Ez,\ z \in \Gamma(x)$ resp., we get

$$\begin{aligned}
\omega_1 \sum_{z\in\Gamma(x)} \alpha_z &= \omega_i, \\
\alpha_z(\omega_0 - \omega_2) + \omega_i\omega_2/\omega_1 &= \begin{cases} \omega_{i-1} & ; z \in \Gamma(x) \cap \Gamma_{i-1}(y), \\ \omega_i & ; z \in \Gamma(x) \cap \Gamma_i(y), \\ \omega_{i+1} & ; z \in \Gamma(x) \cap \Gamma_{i+1}(y). \end{cases}
\end{aligned}$$

QED

Many examples

- the folded $(2m+1)$-cube, for an integer $m \geq 2$, whose least eigenvalue has multiplicity equal to its valency,
- $\{21, 20, 16; 1, 2, 12\}$ is the coset graph of doubly truncated binary Golay code, $v = 512$, $\mathrm{ev}(\Gamma) = 21^1, 5^{210}, (-3)^{280}, (-11)^{21}$,
- $\{21, 20, 16, 6, 2, 1; \dots\}$ is the coset graph of once shortened and once truncated binary Golay code, $v = 1024$, eigenvalues $21^1, 9^{56}, 5^{210}, 1^{336}, (-3)^{280}, (-7)^{120}, (-11)^{21}$,
- $\{21, 20, 16, 9, 2, 1; 1, 2, 3, 16, 20, 21\}$ is the coset graph of a subcode of the doubly truncated binary Golay code, $v = 2048$, eigenvalues $21^1, 9^{168}, 5^{210}, 1^{1008}, (-3)^{280}, (-7)^{360}, (-11)^{21}$,
- an infi nite (feasible) family of 2-covers ($d = 5$):

$$\{2\mu^2 + \mu, 2\mu^2 + \mu - 1, \mu^2, \mu, 1; \dots\}.$$

Known examples with $m_\theta = k$, $a_1 = 0$, $d = 3$

- $K_{k+1,k+1}$ with a perfect matching deleted.
- distance-regular r-covers of complete graphs K_n:
  with parameters $(n, r, c_2) = (2^{2s}, 2^{2s-1}, 2)$, where $s \in \mathbb{N}$,
  constructed by de Caen, Mathon and Moorhouse.
- the coset graph of the doubly truncated binary Golay code
  with intersection array $\{21, 20, 16; 1, 2, 12\}$ and spectrum

  $$21^1, \ \ (-11)^{21}, \ \ 5^{210}, \ \ (-3)^{280}.$$

  Uniqueness shown by Ivanov and Shpectorov,
  as a special case of the infinite family of
  the Hermitean forms graphs over GF(2^2).
- the folded 7-cube
  (a special case of the folded $(2m+1)$-cube, $m \geq 2$).

The case $m_\theta = k$, $a_1 = 0$, $d = 3$

We start with 4 parameters of intersection array

$$\{k, k-1, b_2; 1, c_2, c_3\}.$$

Let $k = \theta_0 > \theta_1 > \theta_2 > \theta_3$ be the eigenvalues of Γ. We introduce labels u, v and t such that $\{u, v, t\} = \{\theta_1, \theta_2, \theta_3\}$.

$$c_2 = -\frac{k(1+u+v+t)+uv+vt+tu+uvt}{k-1},$$

$$c_3 = -\frac{k(u+v+t)+uvt}{c_2}, \quad a_2 = u+v+t+c_3.$$

and $$m_t = \frac{k(k-u)(k-v)(k+vu+v+u)}{(t-v)(t-u)c_2c_3}.$$

Lemma 3. Γ *nonbipartite distance-regular,* $d = 3$*,*

$ev(\Gamma) = \{k, u, v, t\}$ *and* $m_t = k$*. Then* $t = -1$ *if and only if*

(i) Γ *is antipodal,*

(ii) its eigenvalues are integral,

(iii) $c_2 = -(u+v)$, $k = -uv$, $r = -\dfrac{(u-1)(v-1)}{u+v}$ *and*

$$m_u = \frac{(k+1)(r-1)v}{v-u}, \qquad m_v = \frac{(k+1)(r-1)u}{u-v}.$$

If Γ is bipartite and antipodal, then
$\Gamma = K_{k+1,k+1}$ with a perfect matching deleted.

(The case of Γ being bipartite and not antipodal is impossible.)

Theorem. Γ Δ-*free nonbipartite DR, diam.* $d \geq 2$ *and* $k \geq 3$.
Let $\theta \in ev(\Gamma)$ *s.t.* $m_\theta = k$.
Let $\omega_0, \ldots, \omega_d$ *be the cosine sequence of* θ. *Then*

$$1 = (U_i + V_i)\omega_2 + V_i(1 - \omega_2)$$

for $V_i = c_i C_i^2 + a_i A_i^2 + b_i B_i^2, \quad U_i + V_i = (c_i C_i + a_i A_i + b_i B_i)^2.$

Set $\quad f(k,t,x) := k^2 + kt - kx + t^2x - x^2 - tx^2$.

Lemma 4. Γ *primitive DR,* $\quad d=3$, $ev(\Gamma)=\{k,u,v,t\}$, $m_t=k$.
Then $\quad$ *(i)* $k\omega_2 \in \{u,v\}$, $\quad$ *(ii)* $f(k,t,v)=0$ $\;$ *if* $v \neq k\omega_2$.

From now on we assume WLOG $\quad u = k\omega_2 = \dfrac{t^2 - k}{k-1}$,
$f(k,t,v) = k^2 - v^2 - (k+vt)(v-t) = 0$.

Theorem. *Γ primitive Δ-free DR, $d=3$, $ev(\Gamma)=\{k,u,v,t\}$, $m_t=k$. Let $\omega_0,\dots,\omega_d$ be the cosine sequence of θ. Then*

(i) all the eigenvalues are integral,

(ii) $k>t>u>0>v$ or $k>u>0>v>t$, where $u=k\omega_2$,

(iii) Γ is formally self-dual (in particular also Q-polynomial, $ev(\Gamma)=\{k\omega_0,k\omega_1,k\omega_2,k\omega_3\}$).

We can now use the cosines ω_1 in ω_3 to express k and ω_2:

$$k=-\frac{1+\omega_1-\omega_3-\omega_3^2}{\omega_1\omega_3(\omega_1-\omega_3)} \quad \text{and} \quad \omega_2=\frac{\omega_1(\omega_1-\omega_3^2)}{1+(\omega_1-1)\omega_3-\omega_3^2},$$

where we obtained k from $f(k,t,v)=0$ and ω_2 directly from the recursion relation on the cosines sequence and $a_1=0$.

Proof.

(a) $\boldsymbol{m_u = k}$**.** By Lemma 4, we have $(u^2 - k)/(k-1) \in \{t, v\}$.

Suppose first $t = (u^2 - k)/(k-1)$.

Since $u = (t^2 - k)/(k-1)$, this equation transforms (after multiplying by $(k-1)^3/((t+1)(k-t))$) to

$$t^2 + kt - t + k^2 - 3k + 1 = 0, \text{ i.e., } t = \frac{1}{2}\left(1 - k \pm \sqrt{-3 + 10k - 3k^2}\right).$$

The expression for t is a complex number for $k > 3$ and it is equal to -1 for $k = 3$. A contradiction!

Therefore, $v = (u^2 - k)/(k-1)$ and, by Lemma 4(ii), also $f(k,u,t) = 0$, i.e., $k^2 + ku - kt + u^2t - t^2 - ut^2 = 0.$

Since $u = (t^2 - k)/(k-1)$, this equation transforms (after multiplying by $(k-1)^2/(k-t)$) to

$$(p_2(k,t) :=)\ \ t^4 + t^3 - kt^2 - t - k^3 + 3k^2 - 2k = 0$$

and $f(k,t,v) = 0$ (after multiplying by $(k-1)^6/(k-t)$) to a polynomial $p_1(k,t) = 0$ of 7th degree in k and 8th degree in t.

Using Groebner bases, we find polynomials $q_1(k,t)$ and $q_2(k,t)$ s.t.

$$q_1(k,t) \cdot p_1(k,t) + q_2(k,t) \cdot p_2(k,t) = (k-2)(k-1)^{10}k.$$

In the above equality the LHS is zero, while the RHS is non-zero. A contradiction!

(b) $m_v = k$.

We derive a contradiction similarly as in the case (a).

(c) $m_u = m_v$. In this case t is integral and hence

$u = (t^2 - k)/(k - 1)$ is rational. It follows that also v is rational.

But since they are zeros of a monic polynomial with integer coefficients, they must be integral.

Therefore, all the eigenvalues are integral.

Now we directly verify the remaining part of the statement.

QED

List of feasible parameters

Based on the above result we can list small feasible parameters of primitive distance-regular graphs with $d = 3$, $a_1 = 0$, $a_2 \neq 0$ and $m_t = k$.

In the case $k > t > u > v$ we have

$$k \in \{70, 105, 161, 611, 1425, 2668, \dots\},$$

while in the case $k > u > v > t$ we obtain

$$k \in \{21, 175, 276, 345, 793, 2541, \dots\}.$$

Homogeneous property in the sense of Nomura

Γ graph, diam. d, $x, y \in V(\Gamma)$, s.t. $\partial(x,y) = h$, $i, j \in \{0, \ldots, d\}$.

Set $\boldsymbol{D_i^j = D_i^j(x,y)} := \Gamma_i(x) \cap \Gamma_j(y)$ and note $|D_i^j| = p_{ij}^h$.

The graph Γ is ***h*-homogeneous** when the partition

$$\{ D_i^j \mid 0 \le i, j \le d,\ D_i^j \neq \emptyset \}$$

is *equitable* for every $x, y \in V(\Gamma)$, $\partial(x,y) = h$, and
the parameters corresponding to equitable partitions are
independent of x and y.

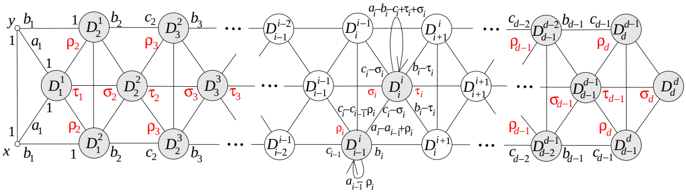
(0-homogeneous $\Longleftrightarrow$ distance-regular)

By calculating various scalar products, it was shown that Γ is 1-homogeneous for $d = 3$ or $d \geq 4$ and $a_4 = 0$.

Let $x, y \in V\Gamma$ be adjacent and set for $0 \leq i, j \leq d$

$$\mathbf{D_i^j = D_i^j(x,y)} := \{z \in V\Gamma \mid \partial(z,x) = i,\ \partial(z,y) = j\}$$

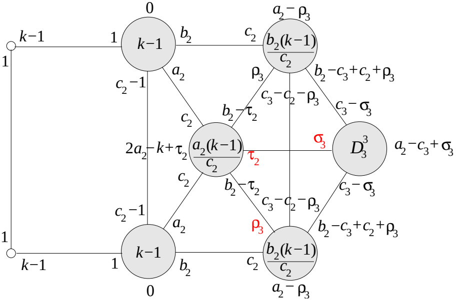
Note $|D_i^j| = p_{ij}^1$. Thus $D_i^j = \emptyset$ when $|i - j| > 1$ by Δ-inequality and $D_i^{i-1} \neq \emptyset$ and $D_{i-1}^i \neq \emptyset$ when $i \in \{1, \ldots, d\}$.

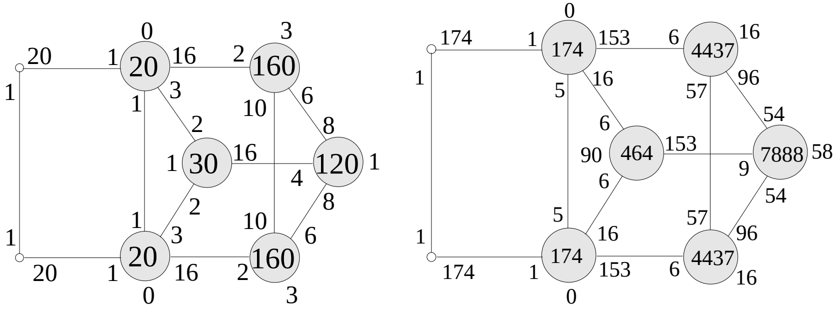


Lemma 5 [JKM'2004]. *Γ primitive Δ-free DR, $d = 3$, $a_2 \neq 0$, $m_t = k$, eigenvalues k, $u > 0$, $v < 0$, t.*
Let x and y be adjacent vertices of Γ.
Then we have $\forall z \in D_2^2(x,y)$ and $g(k,t) := k^2 - 2k + t^3 + t^2 - t$ the following formulae for $b_2 - \tau_2 = |\Gamma(z) \cap D_3^2(x,y)|$

$$\frac{g(k,t)\,(k^2 - 2k + vt^2 + t^2 - v)^2(k^2 - k + tk + t + 1 + vt^2 + v + 2vt + t^2)}{\big((v+t)g(k,t) + (t^2-k)(1-t^2)\big)^2(k-1)^2},$$

where the above denominator is nonzero.





(a) (b)

The distance partition corresponding to an edge of

(a) the coset graph of the doubly truncated binary Golay code

(the second subconstituent of the second graph consists of 21 disjoint Petersen graphs),

(b) the case $k = 175$, $n = 17{,}576$ is open.

The 1-parameter family

Lemma. Γ *primitive* Δ*-free DR,* $d=3$, $a_2 \neq 0$, $m_t=k$, $ev(\Gamma)=\{k,u,v,t\}$. *Let* x *and* y *be adjacent vertices of* Γ. *Then* $\mathbf{b_2 = \tau_2}$ *if and only if*

$$b_2 = q^2(2-q-q^2+q^3), \quad a_2 = (q-1)^2(q+1),$$

$$c_2 = q(q-1), \quad c_3 = q^2(q^2-q+1).$$

for some integer $q \geq 2$.

In the case $q=2$ we have the coset graph of the doubly truncated binary Golay code with intersection array $\{21, 20, 16; 1, 2, 12\}$.

$$x = \frac{b_2(a_2 - \rho_3) + a_2(b_2 - \tau_2)}{c_2} = \frac{a_2(2b_2 - \tau_2) - b_2\rho_3}{c_2}.$$

$\Gamma_2(y)$

(y, z_1) \ (y, z_2)	D_1^2	D_2^2	D_3^2	Σ
D_1^2	$c_2 - 1$	a_2	b_2	$k-1$
D_2^2	a_2	$\frac{a_2(a_2-1)}{c_2}$	$\frac{a_2 b_2}{c_2}$	$\frac{(k-1)a_2}{c_2}$
D_3^2	b_2	$\frac{a_2 b_2}{c_2}$	$\frac{b_2(b_2-1)}{c_2}$	$\frac{(k-1)b_2}{c_2}$
Σ	$k-1$	$\frac{(k-1)a_2}{c_2}$	$\frac{(k-1)b_2}{c_2}$	$\frac{(k-1)k}{c_2}$

Theorem. Γ *primitive* Δ*-free DR,* $d \geq 2$, $a_2 \neq 0$, $m_t = k$.
Let Γ *be* 1*-homogeneous. Then*

(i) $c_2 \mid \gcd\big(\; a_2(2b_2 - \tau_2) - b_2\rho_3,$

$$a_2(a_2 - 1 - b_2 + \tau_2) + b_2\rho_3,$$

$$b_2(b_2 - 1) - a_2(b_2 - \tau_2) + b_2\rho_3 \; \big),$$

(ii) If $\tau_2 = b_2$*, then the size of a connected component of a second subconstituent of* Γ *is* $c_2 - 1 + 2a_2 + a_2(a_2 - 1)/c_2$*, and divides* $k - 1 + a_2(k - 1)/c_2$.

Corollary. Γ *primitive* Δ*-free DR,* $d = 3$, $a_2 \neq 0$, $m_t = k$.
Then $b_2 = \tau_2$ *if and only if* Γ *is*
the coset graph of the doubly truncated binary Golay code.

In particular, there are no examples of graphs with parameters $k \in \{175, 793, 2541\}$.

Open cases: $k \in \{276, 345, \ldots\}$.