Task

Solution strategy

Division

Leading image

Monomial spaces and operators, and an algorithm

Binomial operators

# Monomial decomposition of linear differential operators

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# A glimpse of the next 18 minutes.

1 Task

2 Solution strategy

3 Division

4 Leading image

5 Monomial spaces and operators, and an algorithm

6 Binomial operators

# What do we want to do?

Given $T \in \mathbb{K}[x, \partial]$, $h \in \mathbb{K}[x]$ decide whether there exists $g \in \mathbb{K}[x]$, s.t.

$$Tg = h.$$

In the case of existence construct some $g$ (this is not very precise (but can be fixed)).

**Remark:** analogue constructions for

$$T \in \mathbb{K}[[x]][\partial], h \in \mathbb{K}[[x]] \text{ and } g \in \mathbb{K}[[x]].$$

Task
Solution strategy
Division
Leading image
Monomial spaces and operators, and an algorithm
Binomial operators

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Task

Solution strategy

Division

Leading image

Monomial spaces and operators, and an algorithm

Binomial operators

# Example

Consider $T = x\partial_y + \partial_x \in \mathbb{K}[x, y, \partial_x, \partial_y]$. For each

$$h \in \{x^{31} + x^{15}, y^{26} - y^{19}\}$$

there exists a unique solution $g$ of the particular equation,

$$Tg = h.$$

In one case we have $|\text{supp}(g)| = 2$, in the other case $|\text{supp}(g)| = 47$.

There exist infinitely many linear independent solutions $g$ of the homogenous equation $Tg = 0$.

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Task

Solution strategy

Division

Leading image

Monomial spaces and operators, and an algorithm

Binomial operators

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# Strategy

We interpret $T = \sum c_{\alpha\beta} x^{\alpha} \partial^{\beta} \in \mathbb{K}[x, \partial]$ as the linear operator

$$T: \quad \mathbb{K}[x] \longrightarrow \mathbb{K}[x]$$

$$g = \sum c_{\gamma} x^{\gamma} \longmapsto Tg = \sum c_{\alpha\beta} c_{\gamma} \gamma^{\underline{\beta}} x^{\gamma+\alpha-\beta}$$

# Strategy

A polynomial $g$ s.t. $Tg = h$ exists iff $h \in \mathrm{Im}(T)$.

- We will specify a $\mathbb{K}$-vectorspace $J$ generated by monomials, s.t.

$$\mathbb{K}[x] = \mathrm{Im}(T) \oplus J.$$

- for $h$ we compute a unique representation

$$h = q + r$$

where

$$q \in \mathrm{Im}(T), r \in J.$$

- If $r \neq 0$ there exists no $g$ s.t. $Tg = h$.

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Task

Solution strategy

Division

Leading image

Monomial spaces and operators, and an algorithm

Binomial operators

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Task

Solution strategy

Division

Leading image

Monomial spaces and operators, and an algorithm

Binomial operators

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Task

Solution strategy

Division

Leading image

Monomial spaces and operators, and an algorithm

Binomial operators

# Division

We divide $h \in \mathbb{K}[x]$ by $T \in \mathbb{K}[x, \partial]$, to obtain a representation

$$h = Tg + r$$

with unique polynomials $Tg \in \mathrm{Im}(T)$, $r \in J$.

If $r = 0$ the "quotient" $g$ fulfills

$$Tg = h.$$

Task

Solution strategy

Division

Leading image

Monomial spaces and operators, and an algorithm

Binomial operators

# Ordering

We fix a total ordering $<_\lambda$ on $\mathbb{Z}^n$ induced by an injective linear form $\lambda : \mathbb{Z}^n \longrightarrow \mathbb{K}$ with positive, $\mathbb{Q}$-linear independent coefficients. We set

$$\nu <_\lambda \mu \Leftrightarrow \lambda\mu < \lambda\nu.$$

with associated monomial ordering

$$x^\nu <_\lambda x^\mu \Leftrightarrow \nu <_\lambda \mu.$$

Such $<_\lambda$ is artinian on $\mathbb{N}^n$.

Task

Solution strategy

Division

Leading image

Monomial spaces and operators, and an algorithm

Binomial operators

# Division algorithm

Divide $h \in \mathbb{K}[x]$ with $x^\delta = \mathrm{lt}(h)$ by $T \in \mathbb{K}[x, \partial]$.

Is there some $g_1 \in \mathbb{K}[x]$ s.t. $x^\delta = \mathrm{lt}(Tg_1)$ ?

yes, then

$$h = Tg_1 - (Tg_1 - h) + 0$$

no, then

$$h = (h - x^\delta) + x^\delta$$

After this division step:

$$\deg(Tg_1 - h), \deg(h - x^\delta) <_\lambda \delta$$

Task

Solution strategy

Division

Leading image

Monomial spaces and operators, and an algorithm

Binomial operators

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Task
Solution strategy
Division
Leading image
Monomial spaces and operators, and an algorithm
Binomial operators

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Task

Solution strategy

Division

Leading image

Monomial spaces and operators, and an algorithm

Binomial operators

# Division algorithm

After finitely many division steps we obtain

$$h = Tg + r.$$

No monomial of the “remainder” $r$ occurs as leading monomial of a polynomial in the image of $T$.

Task

Solution strategy

Division

Leading image

Monomial spaces and operators, and an algorithm

Binomial operators

# Leading image

To perform the division process it is necessary to know the leading monomials of polynomials in the image of

$$\begin{array}{rccc} T: & \mathbb{K}[x] & \longrightarrow & \mathbb{K}[x] \\ & g & \longmapsto & Tg. \end{array}$$

We define

$$\operatorname{lm}(\operatorname{Im}(T)) =_{\mathbb{K}} \langle\, x^{\delta}; \exists g \in \mathbb{K}[x] \text{ mit } x^{\delta} = \operatorname{lm}(Tg)\rangle$$

as the *leading image* of $T$.

Task
Solution strategy
Division
Leading image
Monomial spaces and operators, and an algorithm
Binomial operators

# Leading image

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Task

Solution strategy

Division

Leading image

Monomial spaces and operators, and an algorithm

Binomial operators

# Leading image

We define a $\mathbb{K}$-vectorspace $J$, generated by monomials, by

$$\mathbb{K}[x] = \mathrm{lm}(\mathrm{Im}(T)) \oplus J.$$

The sketched division process yields the direct sum decomposition

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Task

Solution strategy

Division

Leading image

Monomial spaces and operators, and an algorithm

Binomial operators

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Task

Solution strategy

Division

Leading image

Monomial spaces and operators, and an algorithm

Binomial operators

# Constructing the leading image is the crucial point know

- solutions of differential equations
- division algorithm
- decide if $x^\delta \in \mathrm{lm}(\mathrm{Im}(T))$ for given $x^\delta$
- we want to know the leading image of $T$
- construction of leading image?

Task

Solution strategy

Division

Leading image

Monomial spaces and operators, and an algorithm

Binomial operators

# Construction of leading image

To describe the leading image, we define

- *algebraic monomial* spaces
- *monomial decomposition* of differential operators

and give an algorithm for the

- approximation of the leading image by algebraic monomial spaces.

Task
Solution strategy
Division
Leading image
Monomial spaces and operators, and an algorithm
Binomial operators

# Algebraic monomial spaces

A subspace $M$ of $\mathbb{K}[x]$ is called *algebraic monomial*, if there exists $\Sigma \subset \mathbb{N}^n$ given by finitely many algebraic equations and inequations, such that $M$ consists of all polynomials with support in $\Sigma$.

$$M =_{\mathbb{K}} \langle\, x^a;\ a^2 - 10a + 21 = 0,\ a^2 + 1 \neq 50 \rangle \subseteq \mathbb{K}[x]$$

Task
Solution strategy
Division
Leading image
Monomial spaces and operators, and an algorithm
Binomial operators

# Monomial differential operators

A *monomial differential operator* is of the form

$$T = \sum_{\alpha - \beta = \tau} c_{\alpha\beta} x^{\alpha} \partial^{\beta}$$

for a $\tau \in \mathbb{Z}^n$. Application of $T$ on $x^\gamma$ yields

$$T(x^\gamma) = c(\gamma) \cdot x^{\gamma + \tau},$$

where

$$c(\gamma) = \sum_{\alpha - \beta = \tau} c_{\alpha\beta} \gamma^{\underline{\beta}} .$$

$c$ is the (polynomial) *coefficient function* of $T$,
$\tau$ is the *shift* of $T$.

Task

Solution strategy

Division

Leading image

Monomial spaces and operators, and an algorithm

Binomial operators

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$c$ is the (polynomial) *coefficient function* of $T$,

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# Examples

$T = x\partial_y$:

$$T(x^a y^b) = b \cdot x^{a+1} y^{b-1},\ \tau = (1, -1),\ c(a,b) = b.$$

$T = \partial_x$:

$$T(x^a y^b) = a \cdot x^{a-1} y^b,\ \tau = (-1, 0), c(a,b) = a.$$

$T = x^4 y^2 \partial_x^3 \partial_y + x^2 y^3 \partial_x \partial_y^2 + x$:

$$T(x^a y^b) = \big(a(a-1)(a-2)b + ab(b-1) + 1\big) \cdot x^{a+1} y^{b+1}$$
$$\tau = (1, 1),\ c(a,b) = a(a-1)(a-2)b + ab(b-1) + 1.$$

# Monomial differential operators and algebraic monomial spaces

- images and kernels, and
- finite intersections and sums of algebraic monomial spaces

are algebraic monomial spaces.

Task
Solution strategy
Division
Leading image
Monomial spaces and operators, and an algorithm
Binomial operators

# Monomial decomposition of differential operators

We write $T \in \mathbb{K}[x, \partial]$ as a sum of monomial differential operators,

$$T = \sum_{\tau \in S} \sum_{\alpha - \beta = \tau} c_{\alpha\beta} x^{\alpha} \partial^{\beta},$$

where the finite set $S \subseteq \mathbb{Z}^n$ consists of the *shifts* of $T$.

# Approximation of $\mathrm{lm}(\mathrm{Im}(T))$ by algebraic monomial spaces

In the following we describe an algorithm to approximate $\mathrm{lm}(\mathrm{Im}(T))$ by algebraic monomial spaces.

Basic idea: for $g_1, g_2 \in \mathbb{K}[x]$ with

$$x^\delta = \mathrm{lt}(Tg_1) = \mathrm{lt}(Tg_2).$$

the application of $T$ to $g_1 - g_2$ yields

$$x^\mu = \mathrm{lm}(T(g_1 - g_2)) = \mathrm{lm}(Tg_1 - Tg_2) \neq x^\delta.$$

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# Approximation of $\mathrm{lm}(\mathrm{Im}(T))$ by algebraic monomial spaces

Here is a sketch of the algorithm:

Initialization:

- $U^1 = \{x^\gamma; \gamma \in \mathbb{N}^n\}$
- $I^1 = \mathrm{lm}(T(U^1))$

$I^1$ is an algebraic monomial space.

Iteration:

given algebraic monomial spaces $I^j$ und sets $U^j$, $j = 1, \ldots, k-1$, compute

- $U^k = \{g_1 - g_2; g_1 \in U^{k-1}, g_2 \in U^j, j \leq k-1, \mathrm{lt}(Tg_1) = \mathrm{lt}(Tg_2)\}$
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$I^k$ is an algebraic monomial space.

# Approximation of $\mathrm{lm}(\mathrm{Im}(T))$ by algebraic monomial spaces (the woes)

Clearly, we have

$$I^1 \subseteq I^1 + I^2 \subseteq \cdots \subseteq I^1 + \cdots + I^k \subseteq \mathrm{lm}(\mathrm{Im}(T)).$$

Termination:

- there exists a termination criterion for the given algorithm, but hard to verify in practice
- in general the algorithm is not finite ( i.e. there is no $k_0 \in \mathbb{N}$ s.t. $I^1 + \cdots + I^{k_0} = \mathrm{lm}(\mathrm{Im}(T))$)

Task

Solution strategy

Division

Leading image

Monomial spaces and operators, and an algorithm

Binomial operators

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# Approximation of $\mathrm{lm}(\mathrm{Im}(T))$ by algebraic monomial spaces the profit

Completeness:

- if $x^\delta \in \mathrm{lm}(\mathrm{Im}(T))$ then there exists $k \in \mathbb{N}$ s.t. $x^\delta \in I^k$.
- if $g \in \mathrm{Ker}(T)$ then there exists $k \in \mathbb{N}$ s.t. $g \in \cup_{j=1}^{k} U^j$.

Structure: for $g$ fulfilling $Tg = h$

- the support of $g$ is contained in the grid in $\mathbb{Z}^n$ generated by the differences $\tau_i - \tau_j$ of shifts of $T$ (roughly speaking)
- the coefficients of $g$ are rational expressions in the coefficient functions of the monomial operators of $T$

Task
Solution strategy
Division
Leading image
Monomial spaces and operators, and an algorithm
Binomial operators

# Example 1/2

For $f_1, \dots, f_d \in \mathbb{K}[x]$ and new variables $t_1, \dots, t_d$ we consider

$$T = f_1 \cdot \partial_{t_1} + \cdots + f_d \cdot \partial_{t_d} \in \mathbb{K}[x, \partial_t] \subseteq \mathbb{K}[t, x, \partial_t, \partial_x].$$

The leading image of the $\mathbb{K}[x]$-linear mapping

$$\begin{aligned} T: \quad t_1 \cdot \mathbb{K}[x] \oplus \cdots \oplus t_d \cdot \mathbb{K}[x] &\longrightarrow \mathbb{K}[x] \\ t_1 \cdot a_1 + \cdots + t_d \cdot a_d &\longmapsto f_1 \cdot a_1 + \cdots + f_d \cdot a_d. \end{aligned}$$

can be computed in finitely many steps of the sketched algorithm.

Note, that $\mathrm{lm}(\mathrm{Im}(T))$ coincides with $\mathrm{lm}(f_1, \dots, f_d)$, the ideal of leading monomials of the ideal $(f_1, \dots, f_d)$.

# Example 2/2

For $T = x\partial_y + \partial_x$ we obtain

$$I^1 = x \cdot \mathbb{K}[x, y] \oplus \mathbb{K},$$
$$I^k = \mathbb{K} \cdot y^{k-1}, \qquad k \geq 2.$$

Consequently, we have $\operatorname{lm}(\operatorname{Im}(T)) = \mathbb{K}[x, y]$ and

$$I^1 + \ldots + I^k \subsetneq \operatorname{lm}(\operatorname{Im}(T))$$

for any $k \in \mathbb{N}$.

Task

Solution strategy

Division

Leading image

Monomial spaces and operators, and an algorithm

Binomial operators

# Binomial operators

Frustration: even for the simple example $T = x\partial_y + \partial_x$ the algorithm does not terminate.

Consolation: for *binomial operators* (as in this last example) the monomial stuff yields enough information for deciding the existence and explicitly construct solutions $g$ of $Tg = h$ in finite time.

Task
Solution strategy
Division
Leading image
Monomial spaces and operators, and an algorithm
Binomial operators

# Binomial Operators

We consider

$$T = T_1 + T_2$$

with monomial operators $T_1, T_2$ with shifts $\tau_1 >_\lambda \tau_2$ and coefficient functions $c_1, c_2$.

Task (still): for a given monomial $x^\delta$ decide, if

$$x^\delta \in \mathrm{lm}(\mathrm{Im}(T)).$$

# Binomial Operators

For binomial operators there exist two prototypes of polynomials $g$ mit $x^\delta = \operatorname{lm}(Tg)$.

- $g = x^{\delta - \tau_1}$ where $c_1(\delta - \tau_1) \neq 0$.
- $g = \sum g_\gamma x^\gamma$ where
  - initial monomial $x^{\delta - \tau_2}$
  - $\operatorname{supp}(g)$ contained in straight line $\mathcal{L} = \{\delta - \tau_2 + l \cdot (\tau_1 - \tau_2), l \in \mathbb{Z}\} \subseteq \mathbb{Z}^n$
  - for $\gamma \in \operatorname{supp}(g)$ the following conditions hold:
    - $c_1(\gamma) = 0, c_2(\gamma) \neq 0$ if $\gamma = \deg(g)$
    - $c_1(\gamma) \neq 0, c_2(\gamma) \neq 0$ otherwise

Finite criteria for $x^\delta \in \operatorname{lm}(\operatorname{Im}(T))$: Check if one of the two prototypes exists.

Task

Solution strategy

Division

Leading image

Monomial spaces and operators, and an algorithm

Binomial operators

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Task

Solution strategy

Division

Leading image

Monomial spaces and operators, and an algorithm

Binomial operators

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Finite criteria for $x^{\delta} \in \mathrm{lm}(\mathrm{Im}(T))$: Check if one of the two prototypes exists.

# Binomial Operators

For binomial operators there exist two prototypes of polynomials $g$ mit $x^\delta = \mathrm{lm}(Tg)$.

- $g = x^{\delta-\tau_1}$ where $c_1(\delta - \tau_1) \neq 0$.
- $g = \sum g_\gamma x^\gamma$ where
  - initial monomial $x^{\delta-\tau_2}$
  - $\mathrm{supp}(g)$ contained in straight line $\mathcal{L} = \{\delta - \tau_2 + l \cdot (\tau_1 - \tau_2), l \in \mathbb{Z}\} \subseteq \mathbb{Z}^n$
  - for $\gamma \in \mathrm{supp}(g)$ the following conditions hold:
    - $c_1(\gamma) = 0, c_2(\gamma) \neq 0$ if $\gamma = \deg(g)$
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Finite criteria for $x^\delta \in \mathrm{lm}(\mathrm{Im}(T))$: Check if one of the two prototypes exists.

Task

Solution strategy

Division

Leading image

Monomial spaces and operators, and an algorithm

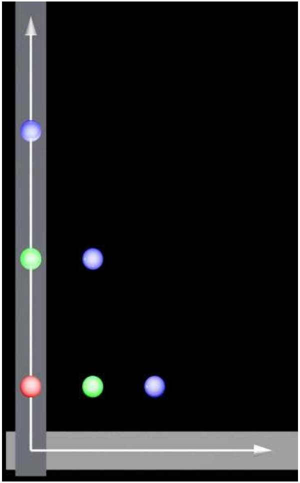
Binomial operators

# Example

For $T = x\partial_y + \partial_x$ check if $x^{a+1}y^b$, $1$, $y$, $y^2 \in \mathrm{lm}(\mathrm{Im}(T))$. We find

- $x^{a+1}y^b = \mathrm{lm}(T(x^a y^{b+1}))$
- $1 = \mathrm{lm}(T(x))$
- $y = \mathrm{lm}(T(x^3 - 3xy))$
- $y^2 = \mathrm{lm}(T(x^5 - 5x^3y + \frac{15}{2}xy^2))$

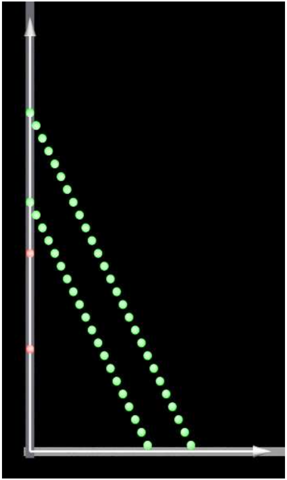
# Example, illustrated



$\mathbb{N}^2$ with powers of $x$ up, powers of $y$ left, kernel of $c_1(a,b) = b$ dark, kernel of $c_2(a,b) = a$ light gray

Task
Solution strategy
Division
Leading image
Monomial spaces and operators, and an algorithm
Binomial operators

# Example, reminder



Example from the introduction: $T = x\partial_y + \partial_x$,
$Tg = x^{31} + x^{15}$ and $Tg = y^{26} - y^{19}$

Task

Solution strategy

Division

Leading image

Monomial spaces and operators, and an algorithm

Binomial operators

# Binomial operators

What if we do not longer want to solve particular but homogenous equations?

Prototype of polynomial $g$ in the kernel of $T = T_1 + T_2$ is $g = \sum g_\gamma x^\gamma$, where

- the support of $g$ is contained in the straight line

$$\mathcal{L} = \{\mathrm{ord}(g) + l \cdot (\tau_1 - \tau_2), l \in \mathbb{Z}\}$$
$$(= \{\deg(g) + l \cdot (\tau_1 - \tau_2), l \in \mathbb{Z}\})$$

- for $\gamma \in \mathrm{supp}(g)$ the following conditions hold:
  - $c_1(\mathrm{ord}(g)) \neq 0, c_2(\mathrm{ord}(g)) = 0$
  - $c_1(\deg(g)) = 0, c_2(\deg(g)) \neq 0$
  - $c_1(\gamma) \neq 0, c_2(\gamma) \neq 0$ otherwise

Task

Solution strategy

Division

Leading image

Monomial spaces and operators, and an algorithm

Binomial operators

# Binomial operators

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Task

Solution strategy

Division

Leading image

Monomial spaces and operators, and an algorithm

Binomial operators

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Task

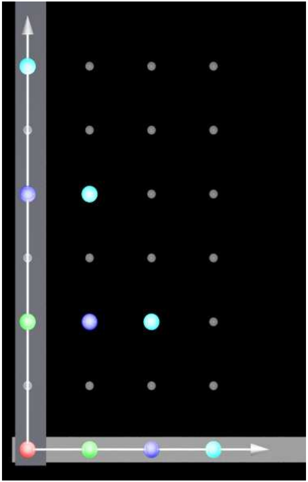
Solution strategy

Division

Leading image

Monomial spaces and operators, and an algorithm

Binomial operators



For $T = x\partial_y + \partial_x$ and $m \in \mathbb{N}$ there exists a polynomial in the kernel of $T$ with leading monomial $x^{2m}$, but no such polynomial with leading monomial $x^{2m+1}$.

Task
Solution strategy
Division
Leading image
Monomial spaces and operators, and an algorithm
Binomial operators

# Binomial systems: Appell $F_1(a, b, b', c)$

For complex parameters $a, b, b', c$ and (binomial operators)

$$R = -abx - bxy\partial_y - (a+b+1)x^2\partial_x - x^2y\partial_x\partial_y - x^3\partial_x^2 + cx\partial_x + xy\partial_x\partial_y + x^2\partial_x^2,$$
$$S = -ab'y - b'xy\partial_x - (a+b+1)y^2\partial_y - xy^2\partial_x\partial_y - y^3\partial_y^2 + cy\partial_y + xy\partial_x\partial_y + y^2\partial_y^2,$$
$$T = b\partial_y + x\partial_x\partial_y - b'\partial_x - y\partial_x\partial_y.$$

the system

$$Rg = Sg = Tg = 0$$

is called *Appell differential equation $F_1(a, b, b', c)$*.

# Binomial systems: Appell $F_1(a, b, b', c)$

For $(a, b, b', c) = (2, -3, -2, 5)$

$$g = x^3y^2 - 3x^3y - \frac{9}{2}x^2y^2 + \frac{12}{5}x^3 + \frac{72}{5}x^2y + \frac{36}{5}xy^2 - \frac{63}{5}x^2 - \frac{126}{5}xy - \frac{21}{5}y^2 + \frac{126}{5}x + \frac{84}{5}y - 21$$

is the only polynomial solution. It can easily be computed applying the techniques already used for binomial operators.

# Extended binomial systems: GKZ-systems

For $A$ a matrix of dimension $d \times n$ with integer entries $a_{ij}$ of rank $d$ and a vector $b \in \mathbb{K}^d$ we define

$$S_{\mu,\nu} = \partial^\mu - \partial^\nu \qquad \text{für alle } \mu, \nu \in \mathbb{N}^n \text{ mit } A\mu = A\nu,$$

$$T_i = \sum_{j=1}^n a_{ij} x_j \partial_j - b_j \qquad \text{für } i = 1, \ldots, d.$$

The associated *GKZ-System* $H_A(b)$ is given by

$$S_{\mu,\nu} g = 0 \qquad \text{for all } \mu, \nu \in \mathbb{N}^n \text{ s.t. } A\mu = A\nu,$$

$$T_i g = 0 \qquad \text{für } i = 1, \ldots, d,$$

for $g$.

Task

Solution strategy

Division

Leading image

Monomial spaces and operators, and an algorithm

Binomial operators

# Extended binomial systems: GKZ-systems

For $n = 2,\ d = 1,\ A = (1 \quad a)$ with $a \in \mathbb{N}$ and $b \in \mathbb{K}$ we find

$$\begin{aligned} S &= \partial_y - \partial_x^a, && \text{binomial operator} \\ T &= x\partial_x + ay\partial_y - b, && \text{monomial operator.} \end{aligned}$$

$Sg = Tg = 0$ has a unique polynomial solution (up to multiplication with constants) given by

$$g = \sum_{i=0}^{r} e_i \cdot x^{b-ia} y^i,$$

with coefficients

$$e_i = \begin{cases} 1 & \text{für } i = 0 \\ -e_{i-1} \frac{c_{S_1}((b-(i-1)a), i-1)}{c_{S_2}(b-ia, i)} & \text{für } i > 0, \end{cases}$$

where $r = \lfloor \frac{b}{a} \rfloor$.