The LEX Game

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Standard monomials

K field, $K[x_1, \ldots, x_n] = K[\mathbf{x}]$ polynomial ring in n variables
$\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{N}^n$, a monomial $\mathbf{x}^{\mathbf{w}} = x_1^{w_1} \ldots x_n^{w_n}$

$\prec$ lexicographic ordering

Let $I \trianglelefteq K[\mathbf{x}]$ be an ideal.
$\mathsf{Lm}(I) = \langle \ell\mathsf{m}(f) \ : \ f \in I \rangle$;
standard monomials of I: the other monomials:

$$\mathsf{Sm}(I) = \mathsf{Mon}_n \setminus \mathsf{Lm}(I)$$

Theorem $\mathsf{Sm}(I)$ *is a linear basis of the vector space* $K[\mathbf{x}]/I$.

Primary decomposition of zero dimensional ideals

Let $I \trianglelefteq K[\mathbf{x}]$ be zero dimensional.

Theorem *There are uniquely determined ideals Q_j such that*

$$I = \prod_{j=1}^{k} Q_j$$

and $M_j := \sqrt{Q_j}$ are pairwise different maximal ideals.

I is a splitting ideal if all $M = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$
for some $\mathbf{a} \in K^n$.

From now on, I is a zero dimensional splitting ideal, thus

$$I = \prod_{\mathbf{a} \in V} Q_{\mathbf{a}}$$

for a finite $V \subseteq K^n$.

Vanishing ideal of a finite set of points

Let $V \subseteq K^n$ be finite.

$I(V) := \{f(\mathbf{x}) \in K[\mathbf{x}] \ : \ f$ vanishes on $V\}$

Then

$$I = \prod_{\mathbf{a} \in V} M_{\mathbf{a}}$$

$M_{\mathbf{a}} = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$.

The LEX Game

$$I = \prod_{\mathbf{a} \in V} Q_{\mathbf{a}}$$

Lex$(I; \mathbf{w})$:

1 Lea chooses w_n (not necessarily different) elements of K.

Stan picks an $a_n \in K$.

They set r_n to be the multiplicity of a_n among Lea's guesses.

2 Lea chooses w_{n-1} elements of K.

Stan picks an $a_{n-1} \in K$.

They set the result r_{n-1}.

$\ldots$ (The game goes on in the same fashion.)

n Lea chooses w_1 elements of K.

Stan picks an $a_1 \in K$.

They set the result r_1.

Put $\mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{N}^n$.
Stan wins iff $\mathbf{a} \in V$ and $\mathbf{x}^{\mathbf{r}} \in \mathrm{Sm}\,(Q_{\mathbf{a}})$.

Example game

If $I = I(V)$ then all $\mathsf{Sm}\,(Q_{\mathbf{a}}) = \mathsf{Sm}\,(M_{\mathbf{a}}) = \{1\}$ so Stan wins iff $\mathbf{r} = \mathbf{0}$.

Let $n = 5$,
$V \subseteq \{\alpha, \beta\}^5$ s.t. in $\mathbf{a} \in V$ there is exactly 1, 2 or 3 α.
$I := I(V)$
With $\mathbf{w} = (11100)$ Lea has winning strategy.
With $\mathbf{w} = (01110)$ Stan has winning strategy.

The theorem

$\mathsf{Stan}\,(I) := \{\mathbf{x}^{\mathbf{w}} \;:\; \text{Stan wins } \mathsf{Lex}(I;\mathbf{w})\}$ the Stan monomials.

Theorem $\mathsf{Stan}\,(I(V)) = \mathsf{Sm}\,(I(V))$.

Example

With $\mathbf{w} = (11100)$ Lea wins, thus $x_1x_2x_3 \in \mathsf{Lm}\,(I(V))$;
but with $\mathbf{w} = (01110)$ Stan can win, so $x_2x_3x_4 \in \mathsf{Sm}\,(I(V))$.

A part of the proof

$\mathsf{Stan}\,(I(V)) \supseteq \mathsf{Sm}\,(I(V))$, ie

Lea wins with $\mathbf{w} \Rightarrow \mathbf{x}^{\mathbf{w}} \in \mathsf{Lm}\,(I(V))$

Proof:

$$f(\mathbf{x}) = \prod_{j=1}^{n} \left(\prod_{i=1}^{w_j} \left(x_j - f_{j,i}(x_{j+1}, \ldots, x_n) \right) \right)$$

where $f_{j,1}, f_{j,2} \ldots f_{j,w_j}$ are Lea's guesses for a_j.

$f_{j,i}$ can be considered as a polynomial	$\Rightarrow$	$f(\mathbf{x}) \in K[\mathbf{x}]$
$f(\mathbf{x})$ vanishes on V	$\Rightarrow$	$f(\mathbf{x}) \in I(V)$
$\ell\mathsf{m}\,(f(\mathbf{x})) = \mathbf{x}^{\mathbf{w}}$	$\Rightarrow$	$\mathbf{x}^{\mathbf{w}} \in \mathsf{Lm}\,(I(V))$

Standard monomials of permutations

Let $n = 4$ and $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in K$ distinct elements.

$V = \{(\pi(\alpha_1), \pi(\alpha_2), \pi(\alpha_3), \pi(\alpha_4))$:

. π is a permutation of $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ }.

(Thus $4! = |V|$.)

Who wins the game $\mathrm{Lex}(I(V); \mathbf{w} = (0, 1, 2, 3))$?

And the games

$\mathrm{Lex}(I(V); (0, 0, 0, 4))$,

$\mathrm{Lex}(I(V); (0, 0, 3, 0))$,

$\mathrm{Lex}(I(V); (0, 2, 0, 0))$,

$\mathrm{Lex}(I(V); (1, 0, 0, 0))$?

Answering these, we have determined the set $\mathrm{Sm}\,(I(V))$.

Field independence of the Gröbner basis

Corollary

Let V be such that $V \subseteq \{0,1\}^n$. Then for all fields K
(1) the set $\mathrm{Sm}\,(I_K(V))$ is the same,
(2) the reduced Gröbner basis G of $I_K(V)$ is the same.

Proof: (2)

Sufficient: the reduced Gb G of $I_{\mathbb{Q}}(V)$ has integer coefficients.
Suppose that $g \in G$ and $z \in \mathbb{Z}$ minimal such that $z \cdot g(\mathbf{x}) \in \mathbb{Z}[\mathbf{x}]$.
If $p \mid z$ a prime then consider $zg(x)$ in $\mathbb{F}_p$:

$$\ell\mathrm{m}_{\mathbb{F}_p}(zg(x)) \prec \ell\mathrm{m}_{\mathbb{Q}}(zg(x))$$

Thus $\ell\mathrm{m}_{\mathbb{F}_p}(zg(x)) \in \mathrm{Sm}\,(I(V))$ a contradiction.

Recursive structure of $\mathrm{Sm}\,(I(V))$

$V_\alpha = \{(a_1, \ldots, a_{n-1}) \;:\; (a_1, \ldots, a_{n-1}, \alpha) \in V\}$ (prefixes of V)

If $r_n = 0$ in Lex$(I(V); \mathbf{w}) \Rightarrow$ Lex$(I(V_\alpha); (w_1, \ldots, w_{n-1}))$.

Corollary

Let $n > 1$. Then $\mathbf{x}^{\mathbf{w}} \in \mathrm{Sm}\,(I(V)) \iff$ there exist at least $w_n + 1$ such α that $x_1^{w_1} \ldots x_{n-1}^{w_{n-1}} \in \mathrm{Sm}\,(I(V_\alpha))$.

Proof:

Stan wins Lex$(I(V); (w_1, \ldots, w_n)) \iff$
there exist at least $w_n + 1$ such α that
Stan wins Lex$(I(V_\alpha); (w_1, \ldots, w_{n-1}))$. $\square$

$\Rightarrow$ fast algrorithm for computing $\mathrm{Sm}\,(I(V))$.

Other splitting ideals

$$I = \prod_{\mathbf{a} \in V} Q_{\mathbf{a}}$$

Theorem *Suppose that all $Q_{\mathbf{a}}$ is generated by polinomials of the form $(\mathbf{x}-\mathbf{a})^{\mathbf{w}}$ (with $\mathbf{w} \in \mathbb{N}^n$) ($I$ is locally monomial). Then* $\mathrm{Stan}\,(I) = \mathrm{Sm}\,(I)$.

Theorem *Suppose that V is such that $\mathbf{a}, \mathbf{b} \in V$ and $a_{n-1} = b_{n-1}$, $a_n = b_n$ implies $\mathbf{a} = \mathbf{b}$. Then* $\mathrm{Stan}\,(I) = \mathrm{Sm}\,(I)$.

Example

Let $Q_{(0,0,0)} = \left\langle x^3, y^2, z^3, xy + z^2 \right\rangle$ and
$Q_{(1,0,0)} = \left\langle (x-1)^3, y^2, z^3, (x-1)y + z^2 \right\rangle$,
$I = Q_{(0,0,0)} \cdot Q_{(1,0,0)}$.
Then $x^2y \in \mathrm{Sm}\,(I) \setminus \mathrm{Stan}\,(I)$ and $x^4z^2 \in \mathrm{Stan}\,(I) \setminus \mathrm{Sm}\,(I)$.

Properties of $\mathsf{Stan}\,(I)$

$$I = \prod_{\mathbf{a}\in V} Q_{\mathbf{a}}$$

Let

$$Q'_{\mathbf{a}} := \langle (\mathbf{x}-\mathbf{a})^{\mathbf{w}} \ : \ \mathbf{x}^{\mathbf{w}} \in \mathsf{Lm}\,(Q_{\mathbf{a}}) \rangle$$

and

$$I' := \prod_{\mathbf{a}\in V} Q'_{\mathbf{a}}$$

Then I' is locally monomial, so $\mathsf{Sm}\,(I') = \mathsf{Stan}\,(I') = \mathsf{Stan}\,(I)$.

Conjecture

$\mathsf{Stan}\,(I)$ *is a monomial linear basis of* $K\,[\mathbf{x}]\,/I$.