Special Semester Presentation

Noncommutative Involutive Bases / Noncommutative Gröbner Walks

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Workshop B2
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Commutative vs. Noncommutative

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- In the commutative case, there is one S-polynomial for every pair of polynomials.
- In the noncommutative case, the number of S-polynomials per pair of polynomials is determined by the overlaps between the lead monomials of the polynomials.
- The commutative algorithm (Buchberger's algorithm) always terminates; the noncommutative algorithm (Mora's algorithm) does not.

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- Popular choices of involutive division include the Thomas, Pommaret and Janet divisions.
- The Involutive Basis algorithm is guaranteed to terminate if the involutive division used satisfies certain properties.

What is an Involutive Basis?

Definition (Thomas)

Let $U = \{u_1, \ldots, u_m\}$ be a set of monomials over a polynomial ring $R[x_1, \ldots, x_n]$, where the monomial $u_j \in U$ (for $1 \leqslant j \leqslant m$) has corresponding multidegree $(e_j^1, e_j^2, \ldots, e_j^n)$.

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The *Thomas* involutive division $\mathcal{T}$ assigns multiplicative variables to elements of U as follows: the variable x_i is multiplicative for monomial u_j (written $x_i \in \mathcal{M}_{\mathcal{T}}(u_j, U)$) if $e_j^i = \max_k e_k^i$ for all $1 \leqslant k \leqslant m$.

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Example

Monomial	x^5y^2z	y^2z	x^2y^2z	xyz^3	xz^3	x^4yz^2	z
Thomas	$\{x, y\}$	$\{y\}$	$\{y\}$	$\{z\}$	$\{z\}$	$\emptyset$	$\emptyset$

What is an Involutive Basis?

Consider the Janet Involutive Basis
$H := \{xy - z,\ yz + 2x + z,\ 2x^2 + xz + z^2,\ 2x^2z + xz^2 + z^3\}$
and the corresponding Gröbner Basis
$G := \{xy - z,\ yz + 2x + z,\ 2x^2 + xz + z^2\}$.

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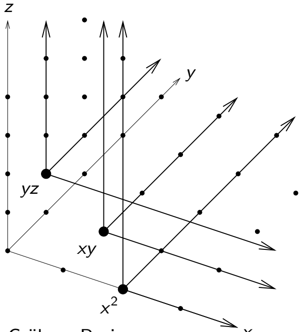
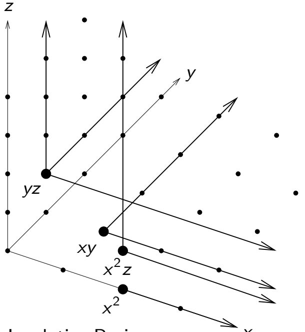


Gröbner Basis



Involutive Basis

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- The Involutive Basis Algorithm can be thought of as an alternative to Buchberger's Algorithm.
- Which is more efficient?
- An Involutive Basis has extra combinatorial properties, e.g. simple deduction of the Hilbert function.
- More information:
  Calmet, Hausdorf and Seiler:
  A Constructive Introduction to Involution;
  Gerdt and Blinkov:
  Involutive Bases of Polynomial Ideals.

Noncommutative Involutive Bases

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  - Thin divisor:

____ ____ ____ ____ ____ ____ ____ ____

✓ ____ ____ ____ ✓

Noncommutative Involutive Bases

- Need left/right multiplicative variables.
- When is a conventional divisor an involutive divisor?
  - Thin divisor:

    ___ ___ ___ ___ ___ ___ ___ ___

    ✓ ___ ___ ___ ✓

  - Thick divisor:

    ___ ___ ___ ___ ___ ___ ___ ___

    ✓ ✓ ___ ___ ___ ✓ ✓ ✓

Noncommutative Involutive Bases

Input: A Basis $F = \{f_1, f_2, \ldots, f_m\}$ for an ideal J over a noncommutative polynomial ring $R\langle x_1, \ldots x_n\rangle$; an admissible monomial ordering O; an involutive division I.

Output: A Locally Involutive Basis $G = \{g_1, g_2, \ldots, g_p\}$ for J (in the case of termination).

$G = \emptyset$;
$F = \text{Autoreduce}(F)$;
while $(G == \emptyset)$ **do**
  $S = \{x_i f \mid f \in F,\ x_i \notin \mathcal{M}_I^L(f, F)\} \cup \{f x_i \mid f \in F,\ x_i \notin \mathcal{M}_I^R(f, F)\}$;
  $s' = 0$;
  while $(S \neq \emptyset)$ **and** $(s' == 0)$ **do**
    Let s be a polynomial in S whose lead monomial is minimal with respect to O;
    $S = S \setminus \{s\}$;
    $s' = \text{Rem}_I(s, F)$;
  end while
  if $(s' \neq 0)$ **then**
    $F = \text{Autoreduce}(F \cup \{s'\})$;
  else
    $G = F$;
  end if
end while
return G;

Noncommutative Involutive Bases

Example

Let $F := \{f_1, f_2\} = \{x^2y^2 - 2xy^2 + x^2,\ x^2y - 2xy\}$ be a basis for an ideal J over the polynomial ring $\mathbb{Q}\langle x, y\rangle$, and let the monomial ordering be DegLex.

Noncommutative Involutive Bases

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Assume multiplicative variables for F as follows.

Polynomial	$\mathcal{M}_I^L(f_i, F)$	$\mathcal{M}_I^R(f_i, F)$
$f_1 = x^2y^2 - 2xy^2 + x^2$	$\{x, y\}$	$\{y\}$
$f_2 = x^2y - 2xy$	$\{x, y\}$	$\{x\}$

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Autoreduction does not alter the set, so we construct the set of prolongations
$S = \{f_1x, f_2y\} = \{x^2y^2x - 2xy^2x + x^3,\ x^2y^2 - 2xy^2\}$.

Noncommutative Involutive Bases

As $x^2y^2 < x^2y^2x$ in the DegLex monomial ordering, we involutively reduce the element $f_2y \in S$ first.

$$\begin{aligned} f_2y = x^2y^2 - 2xy^2 & \xrightarrow[I]{}_{f_1} & x^2y^2 - 2xy^2 - (x^2y^2 - 2xy^2 + x^2) \\ & = & -x^2. \end{aligned}$$

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As the prolongation did not involutively reduce to zero, we now exit from the second while loop of the algorithm and proceed by autoreducing the set
$F \cup \{f_3 := -x^2\} = \{x^2y^2 - 2xy^2 + x^2,\ x^2y - 2xy,\ -x^2\}$.

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$F \cup \{f_3 := -x^2\} = \{x^2y^2 - 2xy^2 + x^2,\ x^2y - 2xy,\ -x^2\}$.
(This of course requires a new assignment of multiplicative variables; the algorithm eventually terminates with the set
$G = \{-x^2,\ -2xy,\ -2xy^2,\ -2xyx,\ -2xy^2x\}$ as output.)

Noncommutative Involutive Bases

Input set F

Noncommutative Involutive Bases

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$\downarrow$ apply algorithm

Locally Involutive Basis

Noncommutative Involutive Bases

$$\begin{array}{c} \text{Input set } F \\ \Big\downarrow \text{apply algorithm} \\ \text{Locally Involutive Basis} \\ \Big\downarrow \text{continuity} \\ \text{Involutive Basis} \end{array}$$

Noncommutative Involutive Bases

Input set F

↓ apply algorithm

Locally Involutive Basis

↓ continuity

Involutive Basis

↓ strong

Gröbner Basis

Noncommutative Involutive Bases

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Gröbner ↓ ↓ strong

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Noncommutative Involutive Bases

Definition (The Left Division)

Given any monomial u, the left division $\triangleleft$ assigns no left nonmultiplicative variables to u, and assigns no right multiplicative variables to u (in other words, all variables are left multiplicative and right nonmultiplicative for u).

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Given any monomial u, the left division $\lhd$ assigns no left nonmultiplicative variables to u, and assigns no right multiplicative variables to u (in other words, all variables are left multiplicative and right nonmultiplicative for u).

Remark

The Left Division is strong and continuous.

Noncommutative Involutive Bases

To illustrate the difference between the overlapping cones of a noncommutative Gröbner Basis and the disjoint cones of a noncommutative Involutive Basis with respect to the left division, consider the following example.

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Example

Let $F := \{2xy + y^2 + 5, x^2 + y^2 + 8\}$ be a basis over the polynomial ring $\mathbb{Q}\langle x, y\rangle$, and let the monomial ordering be DegLex.

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Let $F := \{2xy + y^2 + 5,\ x^2 + y^2 + 8\}$ be a basis over the polynomial ring $\mathbb{Q}\langle x, y\rangle$, and let the monomial ordering be DegLex.
Applying Mora's algorithm to F, we obtain the Gröbner Basis $G := \{2xy+y^2+5,\ x^2+y^2+8,\ 5y^3-10x+37y,\ 2yx+y^2+5\}$.

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Applying the noncommutative Involutive Basis algorithm to F (with respect to the left involutive division), we obtain the Involutive Basis $H := \{2xy + y^2 + 5,\ x^2 + y^2 + 8,\ 5y^3 - 10x + 37y,\ 5xy^2 + 5x - 6y,\ 2yx + y^2 + 5\}$.

Noncommutative Involutive Bases

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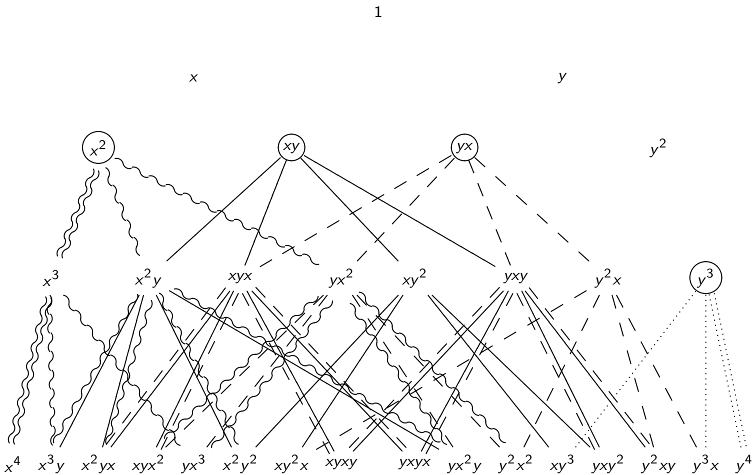

Noncommutative Involutive Bases

Involutive Basis $H = \{2xy + y^2 + 5,\ x^2 + y^2 + 8,$
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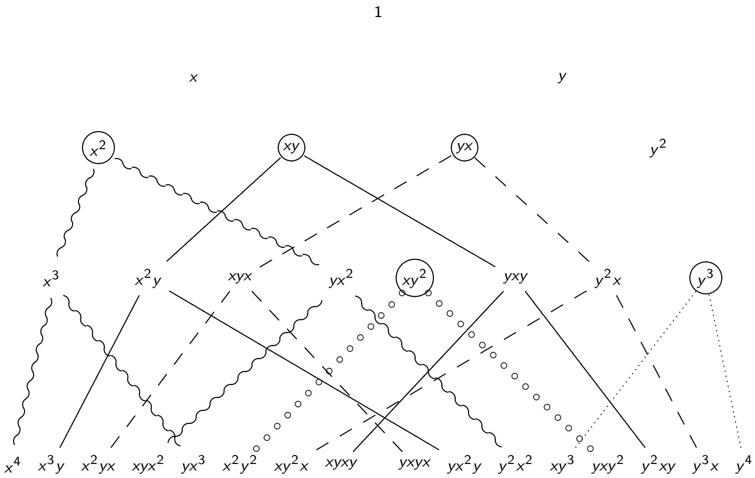

Noncommutative Involutive Bases

Application: Complete Rewrite Systems for Groups.

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Example

Let $C := \langle Y, X, y, x \mid x^3 \to \varepsilon,\ y^2 \to \varepsilon,\ (xy)^2 \to \varepsilon,\ Xx \to \varepsilon,\ xX \to \varepsilon,\ Yy \to \varepsilon,\ yY \to \varepsilon \rangle$ be a monoid rws for S_3.

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If we apply the Knuth-Bendix algorithm to C with respect to the DegLex (word) ordering, we obtain the complete rewrite system

$$C' := \langle Y, X, y, x \mid xyx \to y,\ yxy \to X,\ x^2 \to X,\ Xx \to \varepsilon,\ y^2 \to \varepsilon,\ Xy \to yx,\ xX \to \varepsilon,\ yX \to xy,\ X^2 \to x,\ Y \to y \rangle.$$

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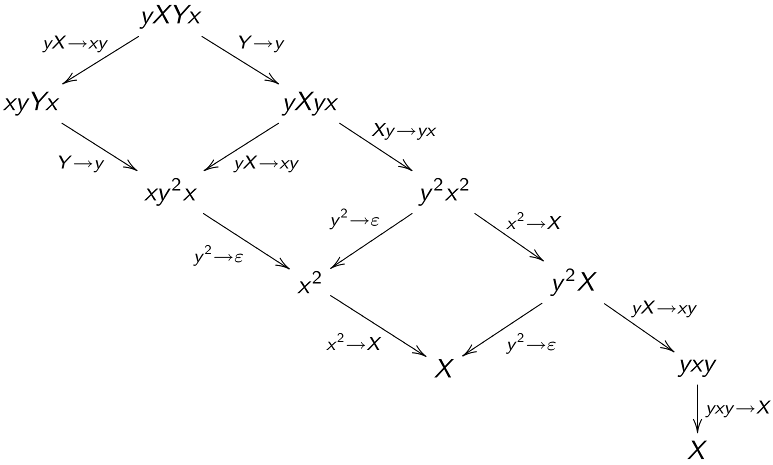
The corresponding involutive complete rewrite system is

$$C'' := \langle Y, X, y, x \mid y^2 \rightarrow \varepsilon,\ Xx \rightarrow \varepsilon,\ xX \rightarrow \varepsilon,\ Yy \rightarrow \varepsilon,\ y^2x \rightarrow x,\ Y \rightarrow y,\ Yx \rightarrow yx,\ Xxy \rightarrow y,\ Yyx \rightarrow x,\ x^2 \rightarrow X,\ X^2 \rightarrow x,\ xyx \rightarrow y,\ Xy \rightarrow yx,\ Xyx \rightarrow xy,\ x^2y \rightarrow yx,\ yX \rightarrow xy,\ yxy \rightarrow X,\ Yxy \rightarrow X,\ YX \rightarrow xy\rangle.$$

Noncommutative Involutive Bases

Consider the word $yXYx$. Using the 10 element complete rewrite system C', there are several reduction paths for this word, as illustrated by the following diagram.



Noncommutative Involutive Bases

However, by involutively reducing the word $yXYx$ with respect to the 19 element involutive complete rewrite system C'', there is only one reduction path, namely

$$yXYx \xrightarrow{Yx \to yx} yXyx \xrightarrow{Xyx \to xy} yxy \xrightarrow{yxy \to X} X$$

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With respect to the left division, the noncommutative Involutive Basis algorithm does not always terminate, given the existence of a noncommutative Gröbner Basis.

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Solution:

Try defining a different division!

Noncommutative Involutive Bases

Definition (The Left Overlap Division $\mathcal{O}$)

Let $U = \{u_1, \ldots, u_m\}$ be a set of monomials, and assume that all variables are left and right multiplicative for all elements of U to begin with.

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(a) For all possible ways that a monomial $u_j \in U$ is a subword of a (different) monomial $u_i \in U$, so that

$$\text{Subword}(u_i, k, k + \deg(u_j) - 1) = u_j$$

for some integer k, if u_j is not a suffix of u_i, assign the variable $\text{Subword}(u_i, k + \deg(u_j), k + \deg(u_j))$ to be right nonmultiplicative for u_j.

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(b) For all possible ways that a proper prefix of a monomial $u_i \in U$ is equal to a proper suffix of a (not necessarily different) monomial $u_j \in U$, so that

$$\text{Prefix}(u_i, k) = \text{Suffix}(u_j, k)$$

for some integer k and u_i is not a subword of u_j or vice-versa, assign the variable $\text{Subword}(u_i, k + 1, k + 1)$ to be right nonmultiplicative for u_j.

Noncommutative Involutive Bases

Example

Consider the set of polynomials $F := \{xy - z,\ x + z,\ yz - z,\ xz,\ zy + z,\ z^2\}$. Here are the left and right multiplicative variables for $\mathrm{LM}(F)$ with respect to the left overlap division $\mathcal{O}$.

u	$\mathcal{M}^L_{\mathcal{O}}(u, \mathrm{LM}(F))$	$\mathcal{M}^R_{\mathcal{O}}(u, \mathrm{LM}(F))$
xy	$\{x, y, z\}$	$\{x, y\}$
x	$\{x, y, z\}$	$\{x\}$
yz	$\{x, y, z\}$	$\{x\}$
xz	$\{x, y, z\}$	$\{x\}$
zy	$\{x, y, z\}$	$\{x, y\}$
z^2	$\{x, y, z\}$	$\{x\}$

Noncommutative Involutive Bases

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More information:
Evans: Noncommutative Involutive Bases (PhD Thesis, University of Wales, Bangor, 2005). Available on the arXiv.

Commutative Gröbner and Involutive Walks

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- It works with the matrices associated to the source and target monomial orderings. Example:

$$\text{DegLex} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ \text{Lex} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

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- The walk takes place on the line segment between the first two rows of the source and target matrices.

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- The 'walk' converts a Gröbner or Involutive Basis with respect to one monomial ordering to a Gröbner or Involutive Basis with respect to another monomial ordering.
- It works with the matrices associated to the source and target monomial orderings. Example:

$$\text{DegLex} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \text{Lex} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- The walk takes place on the line segment between the first two rows of the source and target matrices.
- Each step of the walk computes a Gröbner or Involutive Basis for a set of 'initials', determined by the first row of the current matrix.

Example

Input: $\{xy - z,\ yz + 2x + z,\ 2x^2 + xz + z^2\}$, DegLex.

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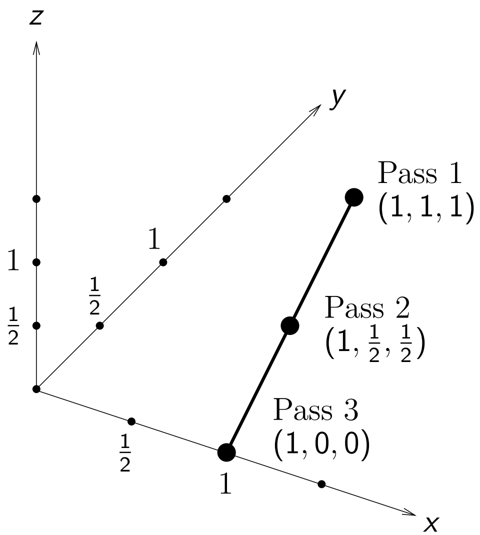

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Commutative	Noncommutative
Matrices	Functional Decompositions
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Definition

The functional decomposition $\Theta = \{\theta_1, \theta_2, \ldots\}$ corresponding to the DegLex monomial ordering is defined (for an arbitrary monomial m) as follows.

$$\theta_i(m) = \begin{cases} \deg(m) & \text{if } i = 1. \\ n + 1 - \mathrm{val}_{i-1}(m) & \text{if } i > 1. \end{cases}$$

Noncommutative Walks

Results

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Problems

- How to find the next step on the walk?
- In particular, how to define an intermediate monomial ordering that is admissible.

Noncommutative Walks

More information (Commutative):

Amrhein, Gloor and Küchlin:
On the Walk;
Collart, Kalkbrener and Mall:
Converting Bases with the Gröbner Walk;
Golubitsky:
Involutive Gröbner Walk.

More information (Noncommutative):

Evans:
Noncommutative Involutive Bases (PhD Thesis, University of Wales, Bangor, 2005).
Available on the arXiv.