Fundamental Theorem of Symmetric Polynomials:

Fix \( d \). For the polynomial equations

\[
    x^d + c_1 x^{d-1} + \cdots + c_{d-1}x + c_d = 0
\]

the symmetric polynomials of the roots (in \( \mathbb{C} \)) are exactly the polynomials in the coefficients \( c_1, \ldots, c_{d-1}, c_d \).

What about systems of polynomial equations in several variables with finitely many solutions?
The **diagonally symmetric polynomials** are the polynomials in the entries of the matrix

\[
\begin{bmatrix}
  x_1(a_1) & x_1(a_2) & \cdots & x_1(a_n) \\
x_2(a_1) & x_2(a_2) & \cdots & x_2(a_n) \\
\vdots & \vdots & \ddots & \vdots \\
x_r(a_1) & x_r(a_2) & \cdots & x_r(a_n)
\end{bmatrix}
\]

that are invariant under all permutations of the columns.

They are the polynomial functions of multisets of \( n \) points, \( a_1, a_2, \ldots, a_n \), of the affine \( r \)-dimensional space!

**Ex:** the **diagonally symmetric power sums**

\[
p_\alpha = \sum_{i=1}^{n} x_1^{\alpha_1}(a_i) = \sum_{i=1}^{n} (x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_r^{\alpha_r})(a_i).
\]
These objects appear naturally when studying systems of equations with finitely many solutions, e.g. as coefficients of the Chow form, as traces of monomials, or multidimensional residues . . .

They are not new !!! They were first introduced by Cayley, MacMahon, . . . in the XIX\textsuperscript{th} century.
The case of Gröbner bases

Let \( x = (x_1, \ldots, x_r) \). Choose some monomial order, and monomials \( x^{\alpha_1}, x^{\alpha_2}, \ldots, x^{\alpha_r} \), with among them a pure power of each variable.

Consider all systems

\[
\begin{align*}
F_1 &= x^{\alpha_1} + \text{smaller terms} \\
F_2 &= x^{\alpha_2} + \text{smaller terms} \\
&\vdots \\
F_k &= x^{\alpha_k} + \text{smaller terms}
\end{align*}
\]

that are a Gröbner basis.

**THEOREM** (E.B.): the diagonally symmetric polynomials of the roots of these systems are polynomial functions of the coefficients.
An example:
Lexicographic degree order,
\( x_1 > x_2 \).
leading monomials:
\( x^{2,0}, x^{1,1}, x^{0,2} \).

\[
\begin{align*}
F_1 &= x_2^2 + bx_1x_2 + cx_2^2 + dx_1 + ex_2 + f, \\
F_2 &= x_1x_2 + gx_2^2 + hx_1 + jx_2 + k, \\
F_3 &= x_2^2 + qx_1 + sx_2 + t.
\end{align*}
\]

is a Gröbner basis iff
\[
\begin{align*}
f &= -cs + es + dj + \text{other terms \ldots}, \\
k &= -gh^2 + jh - g^3q^2 + \text{other terms \ldots}, \\
t &= -h^2 + qd + sh + \text{other terms \ldots}.
\end{align*}
\]

Then the system has \( n = 3 \) zeros \( a_1, a_2, a_3 \) in \( \mathbb{C}^2 \). One has, for instance, that the diagonally symmetric power sum
\[
p_{11} = x_1^{1,1}(a_1) + x_1^{1,1}(a_2) + x_1^{1,1}(a_3)
\]

admits as an expression
\[
-bh^2 + dh - jh - gs^2 - cqh \\
+3eq + gjq - 3bjq - 3cqs - g^2qs - dgq \\
+cq^2g - bg^2q^2 + 2bgqh + 3bgqs + ghns + js.
\]
If the zeros are all simple, one can find back the coefficients of the reduced Gröbner basis

\[
\begin{cases}
F_1 = x_1^2 + d x_1 + e x_2 + f, \\
F_2 = x_1 x_2 + h x_1 + j x_2 + k, \\
F_3 = x_2^2 + q x_1 + s x_2 + t.
\end{cases}
\]

by means of Lagrange interpolation–like formulas, e.g.:

\[
d = \frac{1 \ x_2(a_1) \ x_2^2(a_1)}{1 \ x_2(a_2) \ x_2^2(a_2)} \frac{1 \ x_2(a_3) \ x_2^2(a_3)}{1 \ x_1(a_1) \ x_2(a_1)} \frac{1 \ x_1(a_2) \ x_2(a_2)}{1 \ x_1(a_3) \ x_2(a_3)}
\]

that can be also written as quotients of diagonally symmetric polynomials:

\[
d = \frac{3 \ p_{01} \ p_{20}}{p_{10} \ p_{11} \ p_{30}} \frac{p_{01} \ p_{02} \ p_{21}}{3 \ p_{10} \ p_{01}} \frac{p_{10} \ p_{20} \ p_{11}}{p_{01} \ p_{11} \ p_{02}}.
\]
Gröbner bases with as many equations as unknowns

Choose a monomial order. Consider all systems

\[
\begin{align*}
F_1 &= x_1^{d_1} + \text{smaller terms} \\
F_2 &= x_2^{d_2} + \text{smaller terms} \\
&\vdots \\
F_r &= x_r^{d_r} + \text{smaller terms}
\end{align*}
\]

A formula from the Theory of multidimensional residues, due to Aizenberg and Kytmanov, provides the power sums as the coefficients in some series expansion:

\[
\left| \frac{x_1 \cdot x_2 \cdots x_r}{F_1 \cdot F_2 \cdots F_r} \left( \frac{\partial F_i}{dx_j} \right) \right| = \sum_{\alpha \in \mathbb{N}^r} \frac{p_\alpha}{x^\alpha} + \cdots
\]

that corresponds to diagonally symmetric analogs of Newton's recursion formulas between coefficients and power sums.
References

**Ludwig Schläfli.** Über die Resultante eines systems mehrerer algebraischen Gleichungen. *Vienna Academy Denkschriften*, 4, 1852.


**Emmanuel Briand.** *Polynômes multisymétriques*. Thèse de doctorat, Université de Rennes 1, 2002.

See also [http://emmanuel.jean.briand.free.fr/publications/polms/](http://emmanuel.jean.briand.free.fr/publications/polms/)
