Two Decades (1985-2005) of Gröbner Bases in Multidimensional Systems

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MAY 2006

Special Semester on Gröbner Bases and Related Methods

2006 at Linz and Hagenberg, Austria

Organized by

RICAM (Radon Institute for Computational

and Applied Mathematics)

RISC (Research Insititute for Symbolic Computation)

Directed by

Professor B. Buchberger and Professor H. Engl

~ 1981	B. Buchberger visits me and gives lecture at University of Pittsburgh.
1982	J. P. Guiver receives his Ph.D at University of Pittsburgh
1982	N. K. Bose's text, "Applied Multidimensional Systems Theory," appears.
1985	B. Buchberger writes his famous and well-referenced chapter, "Gröbner Bases: An Algorithm Method in Polynomial Ideal Theory," clarifying and expanding on his doctoral thesis (1965) and his subsequent publications from 1970 (in German)
1988-1989	Ulrich Oberst sends me his voluminous research reports on "Multidimensional Constant Linear Systems", I (June 1988), II (January 1989), III (March 1989)
1990	U. Obersts' seminal work appears in Acta Applicandae Mathematicae.
1998	Gröbner basis talk on its role in multivariate polynomial matrix coprimeness issues and factorization at Padova, Italy, where I again met U. Oberst, before leaving for Poland to participate in the First International Workshop on *n*-D systems. Start collaboration with Zhiping Lin.

2000	Eva Zerz's monograph on "Topics in Multidimensional Linear Systems Theory," appears in the Springer series.
2000	Again in Poland to participate in the Second International Workshop on *n*-D systems. Meet B. Buchberger again as a prelude to visiting Linz in 2003. Met J. F. Pommaret.
2001	J. F. Pommaret's 2 volumes on "Partial Differential Control Theory: Mathematical Tolls (vol. I) and Control Systems (vol. II), appear.
2001	Special Joint Issue, "Applications of Gröbner Bases to Multidimensional Systems and Signal Processing" appears (vol. 12, 3/4, July/October, Mult. Systems and Signal Proc.) Guest edited by Zhiping Lin and Li Xu.
2001	C. Charoenlarpnopparut completes his Ph.D at Penn State University on "Gröbner Bases in Multidimensional Systems and Signal Processing."
2003	Late 2003, the book by N. K. Bose, B. Buchberger and J. P. Guiver entitled "Multidimensional Systems Theory and Applications" appears.
2006	J. of Symbolic Computation, Volume 41, issues 3 – 4, March-April 2006 "Interactions in honor of Bruno Buchbereger."

A basic semialgebraic set is a subset of $\mathbb{R}^n$ defined by a finite number of polynomial equations and inequalities.

Example:

(a) $\left\{(x_1, x_2) \in \mathbb{R}^2 \middle| \frac{x_1^2}{3^2} + \frac{x_2^2}{2^2} \leq 1, x_1^2 - x_2 \leq 0 \right\}$

(b) $f(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]; f(\mathbf{x}) > 0, \forall \mathbf{x}, \mathbf{x} \triangleq (x_1, x_2, \ldots, x_n)$

Approaches for (b) include

- Sum of squares representation, when possible to do (SOSTOOLS, SEDUMI).
- Semidefinite programming to test feasibility of algebraic sets (C.N. Delzell (1980), P. A. Parrilo, B. Sturmfels)
- Elementary decision algebra methods (A. Tarski, Seidenberg, Collins, N. K. Bose, B. D. O. Anderson, E. I. Jury)
- Global lower bound approach (N. Z. Shor)
- Gram matrix method (N. K. Bose, C. C. Li, M. D. Choi, T. Y. Lam, B. Reznik)

Past and Present of Algorithmic Symbolic Computer Algebra in MSSP

Various software used in the past are:

REDUCE (Stanford), SAC (Wisconsin), Scratchpad (IBM), MACSYMA (MIT), Mathematica, Maple (Waterloo), etc.

Theoretical Tools: Multipolynomial Resultant-subresultants (Sylvester, inners, bigradients), Bezoutiants

More recently:

SINGULAR (Kaiserslautern), COCOA (Italy), Macualay 2, Mathematica, Maple

Theoretical Tools: Ideal theory, Gröbner-Buchberger bases (also standard bases of Hironaka etc.)

Algorithmic Algebra pioneered at RISC, Austria by Bruno Buchberger and his group.

QEPCAD (Collins etc.)

Why Gröbner bases?

Examples: Take $K[z_1, z_2]; A(z_1, z_2), B(z_1, z_2) \in K[z_1, z_2]$ are assumed to be relatively prime. Then, the common zeros of $A(z_1, z_2), B(z_1, z_2)$ are always finite in the bivariate case and can be found by resultant theory. But, reduced Gröbner bases by SINGULAR yield the common zeros, usually, with less computational effort.

To wit, let (from a filter bank design problem)

$$A(z_1,z_2) = -0.075(z_1^2 z_2 + z_1 z_2^2 + z_1 + z_2) - 0.0375(z_1^2 + z_2^2 + 1 + z_1^2 z_2^2)$$
$$+ 0.85 z_1 z_2$$
$$B(z_1,z_2) = 0.125(z_1 z_2 + z_1 + z_2 + 1)$$

Resultant based calculations could be messy! The Gröbner basis for $A(z_1,z_2)$ and $B(z_1,z_2)$ is, using SINGULAR (computed with respect to degree reverse lexicographical ordering)

$$G_0(z_1,z_2) = z_2^2 + z_2, G_1(z_1,z_2) = z_1 + z_2 + 1$$

The common zeros of $G_0(z_1,z_2)$ and $G_1(z_1,z_2)$ are easy to find at (-1,0) and (0,-1) and these are also the zeros of $A(z_1,z_2)$ and $B(z_1,z_2)$ (The ideal generated by $G_0(z_1,z_2)$ and $G_1(z_1,z_2)$ is the same as the ideal generated by $A(z_1,z_2)$ and $B(z_1,z_2)$).

Common zeros of $m > 2$ relatively prime multivariate polynomials

Classical approach:	Multipolynomial resultants (Cox, Little, O'Shea, Using Algebraic Geometry)
Gröbner basis approach:	A lex Gröbner basis *G* successively eliminates more and more variables (elimination and extension).

Variant of Hilbert's Nullstellensatz in n-D FIR perfeect reconstruction filter bank design problem

For *N*-channel analysis $\left(\{H_i(\mathbf{z})\}_{i=1}^{N}\right)$ and synthesis $\left(\{F_i(\mathbf{z})\}_{i=1}^{N}\right)$ filters in a PR filter bank,

$$\sum_{i=1}^{N} H_i(\mathbf{z})F_i(\mathbf{z}) = \mathbf{z}^{\mathbf{m}}, \mathbf{m} \in Z_{+}^{\mathbf{m}}$$

$\mathbf{z} = [z_1, z_2, \ldots, z_n]^T$ and $\mathbf{z}^{\mathbf{m}}$ is the notation for a *n*-variate delay $z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n}$. Note that $\mathbf{m}$ is not known a priori and given $\{H_i(\mathbf{z})\}$ the set $\{F_i(\mathbf{z})\}$ plus $\mathbf{m}$ has to be found satisfying the PR constraint.

WLOG, consider the *N*=2, *n*=2 (2-channel bivariate case). In this case, $H_1(z_1, z_2)$ and $H_2(z_1, z_2)$ could have common zeros at (0,0) $(0, \beta_i), \beta_i \neq 0$ or $(\alpha_i, 0), \alpha_i \neq 0$. The "Rabinowitsch trick" could be used as done to solve the more general problem (L. Xu, O. Saito, and K. Abe, MSSP, 1, January 1994, pp. 41 – 60):

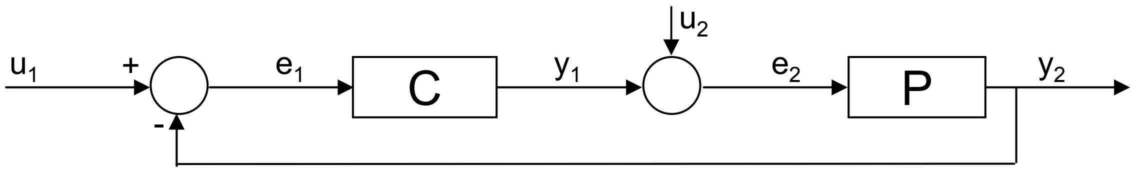
$$H_1(z_1, z_2)F_1(z_1, z_2) + H_2(z_1, z_2)F_2(z_1, z_2) = S^r(z_1, z_2)$$

No previous knowledge of the positive integer *r* is needed. Let z_3 be a new indeterminate. Then $(1 - z_3 S(z_1, z_2))$ and $H_i(z_1, z_2), i = 1,2$ are zero coprime. According to Hilberts' Nullstellensatz, the ideal generated by these polynomials must be the unit ideal, i.e there exist polynomials $\hat{G}_i(z_1, z_2, z_3), i = 1,2,3$ such that

$$\sum_{i=1}^{2} H_i(z_1, z_2)\hat{G}_i(z_1, z_2, z_3) + (1 - z_3 S(z_1, z_2))\hat{G}_3(z_1, z_2, z_3) = 1$$

Then, substitute $S(z_1, z_2)$ for z_3 and clear out denominator to obtain an equation of the form desired.

Stabilization of Scalar (and Matrix) Feedback Systems (2-D) (J. P. Guiver and N. K. Bose, 1985)



Fact 1: Let n and d be relatively prime polynomials with no common zeros in bidisc $\overline{U}^2$. Let there exist $x, y \in \mathbb{R}[z_1, z_2]$ such that $yd + xn \in \mathbb{R}_S[z_1, z_2]$ the ring of bivariate polynomials which have no zero in $\overline{U}^2$. Then, the stabilizing compensators are characterized by $C = -\dfrac{s_1 d + s_2 x}{s_1 n + s_2 y}$, where $s_1 \in \mathbb{R}[z_1, z_2], s_1 \in \mathbb{R}_S[z_1, z_2]$ are arbitrary.

Example 1: Let $n(z_1, z_2) = z_1 z_2 - z_1 - 2z_2 \overset{\Delta}{=} G_1(z_1, z_2)$

$$d(z_1, z_2) = z_1^2 z_2^2 - 2z_1 z_2^2 - 2z_1 z_2 + 4z_1 + 4 \overset{\Delta}{=} G_2(z_1, z_2)$$

$$S_{pol}(G_1, G_2) = G_2 - z_1 z_2 G_1 = z_1^2 z_2 - 2z_1 z_2 + 4z_1 + 4$$

$$\xrightarrow{-z_1 G_1} z_1^2 + 4z_1 + 4 \overset{\Delta}{=} G_3(z_1, z_2)$$

$G_3(z_1, z_2)$ is in normal form, i.e. no term of G_3 is a multiple of the head terms of $G_i, i < 3$. Backtracking

$$d(z_1, z_2) - z_1(z_2 + 1)n(z_1, z_2) = z_1^2 + 4z_1 + 4 \in \mathbb{R}_S(z_1, z_2)$$

Example 2: $n(z_1, z_2) = z_1 + z_2, d(z_1, z_2) = \dfrac{z_1 + z_2}{1 - z_1 + z_2}$

Common zeros at $\left(\tfrac{1}{2}, -\tfrac{1}{2}\right) \in \overline{U}^2$

$\Rightarrow$ Not stabilizable by causal compensators (but stabilizable by weakly causal systems (1985)). Also in N. K. Bose, B. Buchberger, and J. P. Guiver (BBG), "Multidimensional Systems Theory And Application," Kluwer 2003 (now Springer, Dordrecht, Netherland).

Fact 2: In the MIMO case of $D_L^{-1}N_L$ is a minor coprime left MFD (maximal order minors of $[D_L N_L]$ have no nontrivial common factor) of an unstable plant P, and $X_R Y_R^{-1}$ is a minor comprime right MFD (maximal order minors of $[X_R^t Y_R^t]^t$ have no common factor) of a compensator C , then C stabilizes P if and only if det($D_L Y_R + N_L X_R$) is zero free in $\overline{U}^2$.

For stabilization of MIMO Weakly Causal systems, and updates on other results, including computational methods for determining strong stabilizability of *n*-D systems, see [BBG] and work of Ying (refs. [135], [136] in [BBG]). Research counterparts in the *n*-D case (*n* > 2) for constructing a stabilizing compensator for a MIMO plant using Gröbner bases (following work of Xu, Saito and Abe in 1994 (ref [133] in [BBG]) is worth pursuing.

***n*-D System Stabilizability:** (J. Q. Ying et al, J. Symb. Computation, 27, 1999, 479 – 499) In the generic feedback system configuration, plant $p(\mathbf{z}) = f(\mathbf{z})/g(\mathbf{z})$, compensator $c(\mathbf{z}) = h(\mathbf{z})/k(\mathbf{z})$

Q. Do there exist polynomials $h(\mathbf{z})$ and $k(\mathbf{z})$ such that (strong stabilizability), $k(\mathbf{z}) \neq 0$ and $f(\mathbf{z})h(\mathbf{z}) + g(\mathbf{z})k(\mathbf{z}) \neq 0, \forall \mathbf{z} \in \overline{U}^n$?

Procedure based on cylindrical algebraic decomposition of semialgebraic sets.

Example: Let

$$V(f)\cap V(g)=\left\{\left(-\frac{1+\sqrt{2}}{2},\frac{1-\sqrt{2}}{2}\right),\left(\frac{-1+\sqrt{2}}{2},\frac{1+\sqrt{2}}{2}\right)\right\}$$

$\Rightarrow f$ and g do not have common zeros in $\overline{U}^2$. Let, as in Guiver-Bose

$$s(z_1,z_2)=\left(z_1+\frac{1+\sqrt{2}}{2}\right)\left(z_2-\frac{1+\sqrt{2}}{2}\right)^{\dagger}$$

Then, by Rabinowitsch's trick, since $f(z_1,z_2), g(z_1,z_2)$, and $1-z_3 s(z_1,z_2)$ do not have any common zeros in $R[z_1,z_2,z_3]$,using Gröbner bases (by Hilbert's Nullstellensatz)

$$-\frac{1}{4}z_3 f(z_1,z_2)-\frac{1+\sqrt{2}}{2}z_3 g(z_1,z_2)+(1-z_3 s(z_1,z_2))=1$$

Setting $z_3=\dfrac{1}{s(z_1,z_2)}$

$$f+2(1+\sqrt{2})g=-4s$$

As $V(s)\cap\overline{U}^2=\Phi$, therefore

$$h=\frac{1}{2(1+\sqrt{2})}$$

is a stable stabilizer.

†Does not work in 3-D

SOS Decomposition, SDP etc.

Gram Matrix Method [Bose-Li (1968), Choi-Lam-Reznick (1995), Parrilo (2000)]

Fact: A multivariate real coefficient polynomial $p(\mathbf{x})$ in *n* real variables $\mathbf{x} \triangleq (x_1, x_2, \ldots, x_n)$ and of total degree 2*d* is a SOS if and only if it is representable as $p(\mathbf{x}) = \mathbf{v}^T Q \mathbf{v}$, where the $\binom{n+d}{d}$ - vector of monomials,

$$\mathbf{v}^T = \begin{pmatrix} 1 & z_1 & z_1 & \cdots & z_n & z_1 z_2 & \cdots & \cdots & z_n^d \end{pmatrix}$$

and Q is a symmetric PSD matrix.

Comment:

(a) Q can be found by SDP; it is *not* unique.

(b) $\binom{n+d}{d}$ can be very large; but $p(\mathbf{x})$ usually has some structure e.g. lacunary.

Positivstellensatz [Stengle (1974)]:

Given polynomials $\{f_1, \ldots, f_r\}, \{g_1, \ldots, g_k\}$, and $\{h_1, \ldots, h_l\}$ in $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, the following are equivalent

1. $\left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \begin{array}{l} f_i(\mathbf{x}) \geq 0, i = 1,2,\ldots,r \\ g_i(\mathbf{x}) \neq 0, i = 1,2,\ldots,k \\ h_i(\mathbf{x}) = 0, i = 1,2,\ldots,l \end{array} \right\}$ is the empty set

2. There exist polynomials $f \in$ (cone generated by $\{f_1, \ldots, f_r\}$),
   $g \in$ (cone generated by $\{g_1, \ldots, g_k\}$), and
   $h \in$ (cone generated by $\{h_1, \ldots, h_l\}$) such that $f + g^2 + h = 0$.

Comments:

a) The multiplicative monoid M generated by $\{g_i\}_{i=1}^k$ is the set of all finite products of g_i's including 1.

e.g. $M(g_1, g_2) = \left\{ g_1^{k_1}, g_2^{k_2} \middle| k_1, k_2 \in \mathbb{Z}_+ \cup \{0\} \right\}$

b) The cone P generated by $\{f_i\}_{i=1}^r$ is

$$P(f_1, \ldots, f_r) = \left\{ s_0 + \sum_{i=1}^{l} s_i b_i \middle| l \in \mathbb{Z}_+, s_i \in \Sigma_n, b_i \in M(f_1, \ldots, f_r) \right\}$$

where Σ_n denotes the set of SOS polynomials in *n*-variables. Note that $f_i^2 s_i \in \Sigma_n$ as well.

c) Positivstellensatz gives a characterization of the *infeasibility* of polynomial equations and inequalities over the reals.

Nonnegativity of a polynomial $f(\mathbf{x})$ on an algebraic variety $h_i(\mathbf{x}) = 0$

$f(\mathbf{x}) + \sum_i \lambda_i(\mathbf{x}) h_i(\mathbf{x})$ is a SOS in *n*-variate polynomial ring $\mathbb{R}[\mathbf{x}]$

$\longleftrightarrow f(\mathbf{x})$ is a SOS in quotient ring $\mathbb{R}[\mathbf{x}]/\mathbf{I}$ where polynomial ideal $\mathbf{I} = \langle h_1(\mathbf{x}), \ldots, h_l(\mathbf{x}) \rangle$

Example (Parrilo) Is $f(\mathbf{x}) = 10 - x_1^2 - x_2$ nonnegative on $x_1^2 + x_2^2 - 1 = 0$?

$\mathbf{I} = \langle x_1^2 + x_2^2 - 1 \rangle$ in this case of one constraint equation, $h(\mathbf{x})$ is the Gröbner basis of the corresponding ideal

$$10 - x_1^2 - x_2 = \begin{pmatrix} 1 & x_1 & x_2 \end{pmatrix} \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix}$$

$$= q_{11} + q_{22}x_1^2 + q_{33}x_2^2 + 2q_{12}x_1 + 2q_{13}x_2 + 2q_{23}x_1x_2$$

$$\equiv (q_{11} + q_{33}) + (q_{22} - q_{33})x_1^2 + 2q_{12}x_1 + 2q_{13}x_2 + 2q_{23}x_1x_2 \text{(mod } \mathbf{I})$$

$$Q = \begin{pmatrix} q & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1 \end{pmatrix} = L^T L, L = \frac{1}{\sqrt{2}} \begin{pmatrix} 3 & 0 & -1/6 \\ 0 & 0 & \sqrt{35}/6 \end{pmatrix}$$

$$\Rightarrow 10 - x_1^2 - x_2 \equiv \left(3 - \frac{x_2}{6}\right)^2 + \frac{35}{36}x_2^2 \text{(mod } \mathbf{I})$$

$\Rightarrow f(x_1, x_2)$ is a SOS on $\mathbb{R}[x_1, x_2]/\mathbf{I}$

Therefore, SOS on quotient ring $\mathbb{R}[\mathbf{x}]/\mathbf{I}$ is needed, where $\mathbf{I} = \langle h_i(\mathbf{x}) \rangle_{i=1}^{l}$ is the ideal generated by equality constraints. The computations can be effectively done in $\mathbb{R}[\mathbf{x}]/\mathbf{I}$ after computing the Gröbner basis for $\mathbf{I}$ (details in Parrilo, Positive Polynomials in Control, eds. D. Henrion and A. Garulli, Springer, 2005, pp. 181 – 194.)

PROBLEM:

Let ring $R = K[z_1, z_2, z_3]$. Let $M \subset R^3$ be the *R*-module $\langle f_1, f_2, f_3 \rangle$ where the columns f_i are the Koszul relations.

$$\mathbf{f}_1 = \begin{pmatrix} z_2 \\ -z_1 \\ 0 \end{pmatrix}, \mathbf{f}_2 = \begin{pmatrix} z_3 \\ 0 \\ -z_1 \end{pmatrix}, \mathbf{f}_3 = \begin{pmatrix} 0 \\ z_3 \\ -z_2 \end{pmatrix}$$

Consider the (1 x 3) matrix $A = \begin{pmatrix} z_1 & z_2 & z_3 \end{pmatrix}$. Clearly

$$M = \textbf{ker } A = \{\mathbf{f} \in R^3, A\mathbf{f} = 0\}$$

Consider $\alpha_i \in R$; then

$$\alpha_1 \mathbf{f}_1 + \alpha_2 \mathbf{f}_2 + \alpha_3 \mathbf{f}_3 = 0 \text{ for } \alpha_1 = z_3, \alpha_2 = -z_2, \alpha_3 = z_1$$

$\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ are *R*-linearly dependent but any $\{\mathbf{f}_i, \mathbf{f}_j\}, 1 \le i < j \le 3$ is R-linearly independent.

But $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ are minimal in the sense that $M \ne \langle \mathbf{f}_i, \mathbf{f}_j \rangle, 1 \le i < j \le 3$

Fact (well-known): The maximal set of linearly independent elements of the reduced Gröbner basis for module M may only generate a proper submodule of M.

$\Rightarrow$ Problem in GCLF (GCRF) extraction, even if such factor exists from a *n*-variate polynomial matrix, *n*>2.

Note: From linear algebra, every vector space over a field has a basis; not every module has a basis e.g. *R* – module generated by x_1^2 and x_1^3, which are not *R* – linear independent.

Example (Chalie, 2000): It is known that

$$C = \left(\begin{array}{c:cc} z_1 z_2^2 z_3 & 0 & -z_1^2 z_2^2 - 1 \\ z_1^2 z_3^2 + z_3 & -z_3 & z_1^3 z_3 - z_1 \end{array}\right) \triangleq \hat{G}(A_1 \,\vdots\, B_1)$$

(note that the columns of $\hat{G}$ generate the same module as the columns of C) has zero coprime reduced minors and has a GCLF of the submatrices A_1, B_1. The 2 x 2 minors are:

$$m_1 = -z_1 z_2^2 z_3^2 \quad m_2 = -z_1^2 z_2^2 z_3 - z_3 \quad m_3 = z_1^2 z_3^2 + z_3$$

So, the reduced minors, after extracting the gcd $d = z_3$, have no common zeros. To wit, a factorization is,

$$G = \begin{pmatrix} -z_1^2 z_2^2 - 1 & z_1 z_2^2 z_3 \\ -z_1 & z_3 \end{pmatrix} A_1 = \begin{pmatrix} z_1^3 z_2^2 z_3^2 \\ z_1^4 z_2^2 z_3 + z_1^2 z_3 + 1 \end{pmatrix}$$

$$B = \begin{pmatrix} -z_1 z_2^2 z_3 & -z_1^4 z_2^2 z_3 + 1 \\ -z_1^2 z_2^2 - 1 & -z_1^3 (z_1^2 z_2^2 + 1) \end{pmatrix}$$

Using degree reverse lexicographical ordering with $z_1 \succ z_2 \succ z_3$, the matrix G whose columns are the reduced Gröbner basis vectors of the module generated by the columns of C are (using SINGULAR)

$$G = \begin{pmatrix} 0 & z_3 & z_1^2 z_2^2 + 1 \\ z_3 & 0 & z_1 \end{pmatrix}$$

However, $\hat{G}$ cannot be computed from G by applying the algorithmic theory of Gröbner bases because it can be shown that no proper linearly independent subset of the columns of G can generate the column space of C.

Q. But, can a factorization be found by another algorithm?

A. Yes; M. Wang and C. P. Kwong, "On multivariate polynomial matrix factorization problem," Math. Control Signals Systems (2005), 17, 297 – 311. Let s_i be the i$^{\text{th}}$ row of $\begin{pmatrix} C \\ -z_3 I_3 \end{pmatrix}$, since $d = z_3$. They use CoCoA, under the default module term ordering (Deg Revlex and ToPos) to get three generators of a syzygy module of s_i, i = 1, … , 5.

$$f_1 = \begin{pmatrix} 0 & z_3 & z_1^2 z_3^2 + z_3 & -z_3 & -z_1^3 z_3 - z_3 \end{pmatrix}$$
$$f_2 = \begin{pmatrix} z_3 & -z_1 z_2^2 z_3 & -z_1^3 z_2^2 z_3^2 & z_1 z_2^2 z_3 & z_1^4 z_2^2 z_3 - 1 \end{pmatrix}$$
$$f_3 = \begin{pmatrix} z_1 & -z_1^2 z_2^2 - 1 & -z_1^4 z_2^2 z_3 - z_1^2 z_3 - 1 & z_1^2 z_2^2 + 1 & z_1^5 z_2^2 + z_1^3 \end{pmatrix}$$

Let M be the module generated by f_1, f_2, f_3. Using a CoCoA command "Minimalized (M)," they find that M has the following *two* generators $(C \in \mathbb{R}^{l \times m}[z_1, z_2, z_3], l < m)$

$$f_2 = \begin{pmatrix} z_3 & -z_1 z_2^2 z_3 & -z_1^3 z_2^2 z_3^2 & z_1 z_2^2 z_3 & z_1^4 z_2^2 z_3 - 1 \end{pmatrix}$$
$$f_3 = \begin{pmatrix} z_1 & -z_1^2 z_2^2 - 1 & -z_1^4 z_2^2 z_3 - z_1^2 z_3 - 1 & z_1^2 z_2^2 + 1 & z_1^5 z_2^2 + z_1^3 \end{pmatrix}$$

Factorization:

$$C = \begin{pmatrix} z_1^2 z_2^2 + 1 & z_1 z_2^2 z_3 \\ z_1 & z_3 \end{pmatrix} \cdot \begin{pmatrix} -z_1^3 z_2^2 z_3^2 & z_1 z_2^2 z_3 & z_1^4 z_2^2 z_3 - 1 \\ z_1^4 z_2^2 z_3 + z_1^2 z_3 + 1 & -z_1^2 z_2^2 - 1 & -z_1^5 z_2^2 - z_1^3 \end{pmatrix}$$

Limitation: Does not work if the system of generators of the syzygy module does not have l (2 in this case) elements.

Otherwise, more research needed in constructive (algorithmic algebra) *n*-variate factorization problems, *n* > 2.

OPEN PROBLEM IN GENERAL

Minimax Controller Design Using Rate Feedback

Gröbner bases were used to solve special cases of a general minimax control problem using optimal rate feedback.

Mathematical Problem: Given plant consisting of a fixed set of coupled oscillators:

$$m(s) = \prod_{i=1}^{n}\left(s^2 + \beta_i\right), 0 < \beta_i < \beta_{i+1}$$

The problem is to find from an uncountably infinite set of odd degree polynomials $n_i(s)$, whose generic element has the form

$$n_i(s) = k_i s \prod_{l=1}^{n-1}\left(s^2 + \gamma_{l(i)}\right), k_i > 0, \beta_l < \gamma_{l(i)} < \beta_{l+1}$$

the one denoted for brevity by

$$n(s) = ks \prod_{l=1}^{n-1}\left(s^2 + \gamma_l\right), k > 0, \beta_l < \gamma_l < \gamma_{l+1}$$

so that the characteristic polynomial, $n(s) + m(s)$ of the resulting optimal rate feedback system has the fastest slowest mode among the set of strict Hurwitz polynomials, $\{m(s) + n_i(s)\}$. In other words, the rightmost roots (root) of $n(s) + m(s)$ are (is) required to the farthest to the left of the imaginary axis in comparison to similar roots for any polynomial in the set $\{m(s) + n_i(s)\}$.

<u>Solutions available:</u>

n=3 case	N. K. Bose and C. Charoenlarpnopparut, “Minimax controller design using rate feedback,” Circuits, Systems and Signal Processing, 18, 1, 1999, pp. 17-25
n=4 case	Presented at Hagenberg Castle, RISC, Austria, August 18, 2003.
Tool used	Gröbner bases
n>4 case	Solution still unknown.

Main problem: Size of polynomial in Gröbner basis very large and “intractable?”

Bihermitian Forms and Associated Linear Maps

$$S \in \mathbb{C}^{m\times m} \xrightarrow{L} L(S) \in \mathbb{C}^{n\times n}$$

is n.n.d. Hermitian (nndH)	is n.n.d. Hermitian (nndH)

For such a linear operator $L : S$ nndH $\to L(S)$ nndH when do there exist matrices $V_k \in \mathbb{C}^{n\times m}$ so that $\forall S$ nndH there exists a finite sum of congruences representation for $L(S)$

$$L(S) = \sum_{k=1}^{p} V_k S V_k^* \qquad (1)$$

(J. de Pillis, Pac. J. Maths., 23, 1967, 129-137).

That (1) does not hold over the field of complex numbers follows from the counterexample.

Counterexample: Consider $L(S) = (\mathrm{tr}\, S)I - S$. Then $L: S$ nndH $\rightarrow L(S)$ is nndH. Hypothesize, $L(S) = \sum_{k=1}^{p} V_k S V_k^*$. Choose $S = \mathbf{x}\mathbf{x}^*$, where $\mathbf{x}$ is a column vector representation of a complex *m*-tuple such that $\mathbf{x}^*\mathbf{x} = 1$, but could be arbitrary otherwise.

$L(\mathbf{x}\mathbf{x}^*) = I - \mathbf{x}\mathbf{x}^*$ and $\mathbf{x}\mathbf{x}^* L(\mathbf{x}\mathbf{x}^*) = \sum_k \mathbf{x}\mathbf{x}^* V_k \mathbf{x}\mathbf{x}^* V_k = 0$.

$$\mathrm{tr} \sum_k \mathbf{x}\mathbf{x}^* V_k \mathbf{x}\mathbf{x}^* V_k = \sum_k \mathbf{x}^* V_k \mathbf{x}\, \overline{\mathbf{x}^* V_k \mathbf{x}} = 0$$

$$\Rightarrow \qquad \mathbf{x}^* V_k \mathbf{x} = 0, \forall k \quad \textbf{CONTRADICTION}$$

Definition 1: A scalar valued function $H(\mathbf{x}, \mathbf{y}) \in \mathbb{C}^n \times \mathbb{C}^m$ is called a *bihermitian form* if $H(\mathbf{x}, \mathbf{y})$ is a hermitian form in $\mathbf{x}$ (i.e expressible as $\mathbf{x}^* B\mathbf{x}, B = B^*$), for each $\mathbf{y} \in \mathbb{C}^m$ and a hermitian form in $\mathbf{y}$ for each $\mathbf{x} \in \mathbb{C}^n$.

A bihermitian form clearly assumes only real values. A bihermitian form $H(\mathbf{x}, \mathbf{y})$ is nnd, provided $H(\mathbf{x}, \mathbf{y}) \geq 0$ for each $\mathbf{x} \in \mathbb{C}^n$ and each $\mathbf{y} \in \mathbb{C}^m$.

Definition 2: A scalar valued function $H(\mathbf{x},\mathbf{y}) \in \mathbb{C}^n \times \mathbb{C}^m$ is a *hermitian bilinear form* provided $H(\mathbf{x},\mathbf{y})$ is linear in $\mathbf{x}$ for each $\mathbf{y} \in \mathbb{C}^m$ and $\overline{H(\mathbf{x},\mathbf{y})}$ is linear in $\mathbf{y}$ for each $\mathbf{x} \in \mathbb{C}^n$.

Question: Given a nnd bihermitian form $H(\mathbf{x},\mathbf{y})$ can it be expressed as

$$H(\mathbf{x},\mathbf{y}) = \sum_{j=1}^{k} \phi_j(\mathbf{x},\mathbf{y})\overline{\phi_j(\mathbf{x},\mathbf{y})} \qquad (2)$$

where, $\phi_j(\mathbf{x},\mathbf{y})$ is a hermitian bilinear form in $\mathbf{x}$ and in $\mathbf{y}$ for each $j \in \{1,2,\ldots,k\}$.

When $\mathbb{C}^m$ is replaced by $\mathbb{R}^m$ and $\mathbb{C}^n$ by $\mathbb{R}^n$, Eq. (2) becomes the *sum of squares* (SOS) *representation*

$$H(\mathbf{x},\mathbf{y}) = \sum_{j=1}^{k} \phi_j^2(\mathbf{x},\mathbf{y})$$

Lemma [Koga(1968), Calderon(1973), LAA, 7, 1973, 175-173]: Let f be a polynomial with real coefficients in p real variables $\mu_1, \mu_2, \ldots, \mu_p$. If given values $\mu_j (j \neq i)$, f is quadratic in μ_i for each $i = 1, 2, \ldots, p$ and if f assumes only nonnegative values, then it can be decomposed into the SOS, $f = \sum_u L_u^2$, where L_u is linear in each μ_i separately.

<u>**Representation of bihermitian forms:**</u>

Theorem 1: A bihermitian form $H(\mathbf{x}, \mathbf{y})$ is representable as

$$H(\mathbf{x}, \mathbf{y}) = \sum_{u=1}^{m} \sum_{v=1}^{m} (\mathbf{x}^* A^{uv} \mathbf{x}) \overline{y_u} y_v \qquad (3)$$

where $\mathbf{y}^* = (\overline{y_1}, \overline{y_2}, \ldots, \overline{y_m}), \mathbf{x}^* = (\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n})$ and $A^{uv} = ((a_{ij}^{uv})) \in \mathbb{C}^{n \times n}$ are complex matrices such that $(A^{uv})^* = A^{vu}$.

Furthermore, if A_{ij} represents a matrix of order $m \times m$ with a_{ij}^{uv} as its element in the $(u,v)^{\text{th}}$ position, $H(\mathbf{x},\mathbf{y})$ can be equivalently expressed as

$$H(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n}\sum_{j=1}^{n}(\mathbf{y}^{*}A_{ij}\mathbf{y})\overline{x_i}x_j \qquad (4)$$

Proof: We note from the definition of a bihermitian form

$$H(\mathbf{x},\mathbf{y}) = \mathbf{y}^{*}A(\mathbf{x})\mathbf{y} = \sum_{u=1}^{m}\sum_{v=1}^{m}a^{uv}(\mathbf{x})\overline{y_u}y_v \qquad (5)$$

where, for each $\mathbf{x}, A(\mathbf{x}) = (a^{uv}(\mathbf{x}))$ is a hermitian matrix in $\mathbb{C}^{m\times m}$. Let $\mathbf{e}_{uv}$ be column vectors defined for *m*-tuples instead of *n*-tuples. For $u \le v$, $\mathbf{e}_{uv}$ has the u^{th} and v^{th} coordinates are equal to 1, other coordinates are equal to zero.

For $u > v$, the v^{th} coordinate of $\mathbf{e}_{uv}$ is 1, the u^{th} coordinate is the imaginary number $i = \sqrt{-1}$ and the rest are zero.

Put $\mathbf{y} = \mathbf{e}_{uu}$ on both sides of (5) to get

$$a^{uu}(\mathbf{x}) = H(\mathbf{x}, \mathbf{e}_{uu}) = \mathbf{x}^* A^{uu} \mathbf{x} \qquad (6)$$

since $H(\mathbf{x}, \mathbf{e}_{uu})$, an hermitian form in $\mathbf{x}$ can always be so expressed.

Also, for $u < v$

$$H(\mathbf{x}, \mathbf{e}_{vu}) = a^{uu}(\mathbf{x}) + a^{vv}(\mathbf{x}) + i\left(a^{uv}(\mathbf{x}) - a^{vu}(\mathbf{x})\right) \qquad (7)$$

$$H(\mathbf{x}, \mathbf{e}_{uv}) = a^{uu}(\mathbf{x}) + a^{vv}(\mathbf{x}) + a^{uv}(\mathbf{x}) + a^{vu}(\mathbf{x}) \qquad (8)$$

Hence

$$\begin{aligned} a^{uv}(\mathbf{x}) = .&\frac{1}{2}\left(H(\mathbf{x}, \mathbf{e}_{uv}) - H(\mathbf{x}, \mathbf{e}_{uu}) - H(\mathbf{x}, \mathbf{e}_{vv})\right) \\ &- \frac{\sqrt{-1}}{2}\left(H(\mathbf{x}, \mathbf{e}_{vu}) - H(\mathbf{x}, \mathbf{e}_{uu}) - H(\mathbf{x}, \mathbf{e}_{vv})\right) \end{aligned} \qquad (9)$$

$$\begin{aligned} a^{vu}(\mathbf{x}) = .&\frac{1}{2}\left(H(\mathbf{x}, \mathbf{e}_{uv}) - H(\mathbf{x}, \mathbf{e}_{uu}) - H(\mathbf{x}, \mathbf{e}_{vv})\right) \\ &+ \frac{\sqrt{-1}}{2}\left(H(\mathbf{x}, \mathbf{e}_{vu}) - H(\mathbf{x}, \mathbf{e}_{uu}) - H(\mathbf{x}, \mathbf{e}_{vv})\right) \end{aligned} \qquad (10)$$

Since $\frac{1}{2}\left(H(\mathbf{x},\mathbf{e}_{uv})-H(\mathbf{x},\mathbf{e}_{uu})-H(\mathbf{x},\mathbf{e}_{vv})\right)$ and $\frac{1}{2}\left(H(\mathbf{x},\mathbf{e}_{vu})-H(\mathbf{x},\mathbf{e}_{uu})-H(\mathbf{x},\mathbf{e}_{vv})\right)$ are hermitian forms in $\mathbf{x}$, these can be expressed as $(\mathbf{x}^* B^{uv}\mathbf{x})$ and $(\mathbf{x}^* B^{vu}\mathbf{x})$ respectively, where B^{uv} and B^{vu} are hermitian matrices.

We have, therefore, $a^{uv}(\mathbf{x}) = \mathbf{x}^* A^{uv}\mathbf{x}, a^{vu}(\mathbf{x}) = \mathbf{x}^* A^{vu}\mathbf{x},$ where $A^{uv} = B^{uv} - \sqrt{-1}B^{vu}$ and $A^{vu} = B^{uv} + \sqrt{-1}B^{vu}$. Hence $(A^{uv})^* = A^{vu}$. It follows in an analogous manner that $H(\mathbf{x},\mathbf{y})$ can be equivalently represented as

$$H(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n}\sum_{j=1}^{n}(\mathbf{y}^* A_{ij}\mathbf{y})\overline{x_i}x_j$$

where, $A_{ij} \in \mathbb{C}^{m\times m}$ with $(A_{ij})^* = A_{ji}$. The proof is now complete.

Dual maps underlying a bihermitian form

Next, consider the matrices A_{ij} and A^{uv} associated with a bihermitian form *H*. For $B = ((b_{ij})) \in \mathbb{C}^{n\times n}$

$$L_{nm}(B) = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} b_{ij} \qquad (11)$$

defines a linear mapping from $\mathbb{C}^{n\times n}$ to $\mathbb{C}^{m\times m}$.

When *B* is nonnegative definite of rank 1, say $B = \mathbf{x}\mathbf{x}^*$

$$L_{nm}(B) = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} \overline{x_i} x_j$$

is hermitian. Since a herimitian matrix is a linear combination of non-negative definite matrices of rank 1 each with coefficients ±1, it is seen that whenever B is hermitian so is $L_{nm}(B)$. For $C = ((c_{uv})) \in \mathbb{C}^{m\times m}$

$$L^{mn}(C) = \sum_{u=1}^{m}\sum_{v=1}^{m} A^{uv} c_{uv} \qquad (12)$$

likewise defines a linear map from $\mathbb{C}^{m\times m}$ to $\mathbb{C}^{n\times n}$ such that hermitian matrices in $\mathbb{C}^{m\times m}$ are so mapped into hermitian matrices in $\mathbb{C}^{n\times n}$.

The linear maps L^{mn} and L_{nm} introduced in (11) and (12) are said to be the *dual of one another, the duality being interpreted in terms of the common bihermitian form to which both correspond*. Indeed, every linear map from $\mathbb{C}^{m\times m}$ to $\mathbb{C}^{n\times n}$ that preserves hermitian symmetry corresponds uniquely to a bihermitian form $H(\mathbf{x},\mathbf{y})$ and to a dual linear map from $\mathbb{C}^{n\times n}$ into $\mathbb{C}^{m\times m}$ with a like property. In fact

$$H(\mathbf{x},\mathbf{y}) = \mathbf{y}^* L_{nm}(\mathbf{x}\mathbf{x}^*)\mathbf{y} = \mathbf{x}^* L^{mn}(\mathbf{y}\mathbf{y}^*)\mathbf{x}$$

for $B = \mathbf{x}\mathbf{x}^*, C = \mathbf{y}\mathbf{y}^*$. $L^{mn}(C)$ is hermitian because $C = \mathbf{y}\mathbf{y}^*$ is hermitian. The same argument applies to $L_{nm}(B)$.

Complete Positivity (Stinespring)

Digression and Background

For compactness of notation, we denote $\mathbb{C}^{n\times n}$ by M_n, the algebra of $n\times n$ complex matrices. $E_{jk} \in M_{n,}$ is the $n\times n$ matrix with 1 at j,k component and zeros elsewhere. $M_n(M_m) = M_m \otimes M_n$ is the collection of all $n\times n$ block matrices with $m\times m$ matrices as entries. Clearly, $M_n(M_m)$ is $\mathbb{C}^{nm\times nm}$

Definition:

$\phi : M_n \to M_n$ is positive if ϕ maps the set of positive definite matrices into itself (similarly $\phi : M_n \to M_p$)

ϕ is *p*-positive definite if $\phi \otimes I_p$ is positive on $M_m \otimes M_p$

($\phi \otimes I_p : M_p(M_n) \to M_p(M_m)$ is given by

$$\phi \otimes I_p((A_{jk}))_{1\le j,k\le p} = \phi((A_{jk}))_{1\le j,k\le p}$$

ϕ is completely positive if ϕ is *p*-postive for every p i.e.

$\phi \otimes I_p$ is positive for all integers *p*.

Example: For each $n \times m$ matrix V the map $\phi : M_n \to M_n$ with $A \in M_n \to V^* AV \in M_m$ is completely positive.

Fact: Let $\phi : M_n \to M_m$. Then ϕ is completely positive iff ϕ is of the form $\sum_i V_i^* AV_i$ for all $A \in M_n$ where V_i are $n \times m$ matrices.

An equivalent result is:

Let $\phi : M_n \to M_m$. Then ϕ is completely positive if $(\phi(E_{jk}))_{1 \le j,k \le n}$ is positive.

Comment: Each linear map $\phi : M_n \to M_m$ is determined by its values on $E_{jk} \in M_n, 1 \le j, k \le n$. Hence ϕ is completely determined by the single element $(\phi(E_{jk}))_{1 \le j,k \le n} \in M_n(M_m)$.

Comment: A completely positive map is a positive map which keeps is positivity when the system it acts on is embedded as a subsystem in an arbitrary larger system.

Comment: In the real case $\phi : M_n(\mathbb{R}) \rightarrow M_n(\mathbb{R})$ iff the block matrix $(\phi(E_{ij}))_{1 \leq i, j \leq n}$ is positive definite. However, the results on positive maps are slightly different because a positive map imposes no condition on the space of skew-symmetric matrices.

When *H* is nonnegative definite, the associated linear maps L^{mn} and L_{nm} also preserve positivity in the sense that whenever $C \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{n\times n}$ are nonnegative definite, so are $L^{mn}(C)$ and $L_{nm}(B)$. Such maps are called positive. L_{nm} is called completely positive if for every integer p and the partitioned form $((B_{\lambda\mu}))$ of a nonnegative definite matrix $B \in \mathbb{C}^{np\times np}$ with $B_{\lambda\mu} \in \mathbb{C}^{n\times n}$ the partitioned matrix $((L_{nm}(B_{\lambda\mu})))$ is also nonnegative definite. The bihermitian form associated with the map $L(S) = (\mathrm{tr}\, S)I - S$ in the counterexample (consider the *m*=*n* case) is

$$H(\mathbf{x},\mathbf{y}) = \mathbf{x}^*\mathbf{y}\mathbf{y}^*\mathbf{x} - \mathbf{y}^*\mathbf{x}\mathbf{x}^*\mathbf{y}$$
$$H(\mathbf{x},\mathbf{y}) = 0$$

The associated linear maps are identical in the *m*=*n* case.

If (1) holds, let us write $\phi_r(\mathbf{x},\mathbf{y}) = \mathbf{y}^* Z_r^* \mathbf{x}$ for some $Z_r \in \mathbb{C}^{n\times m}$. Then, using $B = \mathbf{x}\mathbf{x}^*, C = \mathbf{y}\mathbf{y}^*$

$$L^{mn}(C) = \sum_{r=1}^{k} Z_r C Z_r^* \qquad (13)$$

$$L_{nm}(B) = \sum_{r=1}^{k} Z_r^* B Z_r \qquad (14)$$

both of which are seen to be completely positive linear maps. Note that though $H(\mathbf{x},\mathbf{y})$ cannot be expressed as in (1) in the complex case, the condition can always be met in the real case by requiring that Z_r be skew symmetric and indeed when $\mathbf{x}$ and $\mathbf{y}$ are the real *n*-tuples, the corresponding biquadratic form $\mathbf{x}^t\mathbf{x}\mathbf{y}^t\mathbf{y} - \mathbf{x}^t\mathbf{y}\mathbf{y}^t\mathbf{x}$ can always be expressed as $\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i y_j - x_j y_i)^2.$

Linear Maps Preserving Hermitian Symmetry

For a nonnegative definite bihermitian form H, the most that be claimed in general is that H has a representation:

$$H(\mathbf{x},\mathbf{y}) = \sum_{r=1}^{k} \pm \phi_r(\mathbf{x},\mathbf{y})\overline{\phi_r(\mathbf{x},\mathbf{y})} \tag{15}$$

where $\phi_1, \phi_{2,\ldots,}\phi_k$ are hermitian bilinear forms in $(\mathbf{x},\mathbf{y})$.

This follows from the following result due to de Pillis.

Fact 1 (de Pillis): Let L_{nm} be a linear map from $\mathbb{C}^{n\times n}$ into $\mathbb{C}^{m\times m}$ such that if B is hermitian, so is $L_{nm}(B)$. Then there exist complex matrices $Z_r \in \mathbb{C}^{n\times m}, r = 1,2,\ldots,k$ such that

$$L_{nm}(B) = \sum_{r=1}^{k} \pm Z_r^* B Z_r \tag{16}$$

The representation given by de Pillis involves B^t in place of B in (16).

Both forms of representation are, however, clearly equivalent since if L_{nm} preserves hermitian symmetry, so does the linear map defined by the correspondence $B \to L_{nm}(B^t)$. An elementary proof of this fact can be constructed but is omitted here.

Linear Maps Preserving Positivity – Real Symmetric Case

Theorem 2: Let L_{nm} be a linear map from $\mathbb{R}^{n\times n}$ into $\mathbb{R}^{m\times m}$ such that if B is nonnegative definite symmetric, so is $L_{nm}(B)$, then there exist $Z_r \in \mathbb{R}^{n\times m}$ such that

$$L_{nm}(B) = \sum_r Z_r^t B Z_r \tag{17}$$

if and only if the corresponding nonnegative definite biquadratic form $Q(\mathbf{x},\mathbf{y})$ in *n* by *m* real variables can be expressed as a finite sum of squares of bilinear forms

$$Q(\mathbf{x},\mathbf{y}) = \sum_j \phi_j^2(\mathbf{x},\mathbf{y}) \tag{18}$$

Proof: (*Only if part*) As B is nonnegative definite, $B = DD^t$ where $D \in \mathbb{R}^{n \times k}$. Let $D = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k]$, where $\mathbf{x}_i \in \mathbb{R}^n, i = 1,2,\ldots,k.$

$$\begin{aligned}
\mathbf{y}^t L_{nm}(B)\mathbf{y} &= \mathbf{y}^t \left(\sum_r Z_r^t B Z_r \right) \mathbf{y} \\
&= \mathbf{y}^t \left(\sum_r \sum_i Z_r^t \mathbf{x}_i \mathbf{x}_i^t Z_r \right) \mathbf{y} \\
&= \sum_j \phi_j^2(\mathbf{x}, \mathbf{y})
\end{aligned}$$

(*If part*): As $\phi_j(\mathbf{x}, \mathbf{y})$ in (18) is a bilinear form, $\phi_j(\mathbf{x}, \mathbf{y}) = \mathbf{x}^t Z_j \mathbf{y}$. So $\sum_j \phi_j^2(\mathbf{x}, \mathbf{y}) = \mathbf{y}^t \left(\sum_j Z_j^t \mathbf{x}\mathbf{x}^t Z_j \right) \mathbf{y}$.

The rest follows.

An Example

$$\pi(x_1, x_2) = \begin{pmatrix} x_1^2 + x_1x_2 + x_2^2 & x_2^2 \\ x_2^2 & x_1^2 - x_1x_2 + x_2^2(1+\varepsilon) \end{pmatrix}$$

where, $\varepsilon > 0$.

It is stated that $\pi(x_1, x_2) \geq 0, \forall$ real x_1, x_2. $\pi(x_1, x_2)$ can be viewed as a linear mapping of the class of real symmetric matrices

$$A = \begin{pmatrix} x_1^2 & x_1x_2 \\ x_1x_2 & x_2^2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \end{pmatrix}$$

A sum of congruences representation for $\pi(x_1, x_2)$ is expected in this case, because of the Koga-Calderon results. Note that $m = n = 2$, $p = mn = 4$. For the example considered here

$$L(A) = \pi(x_1, x_2) = \sum_{i=1}^{4} V_i A V_i^t$$

where,

$$V_1 = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} \qquad V_2 = \begin{pmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{pmatrix}$$

$$V_3 = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \dfrac{1-\sqrt{2\varepsilon}}{2} \end{pmatrix} \qquad V_4 = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \dfrac{1+\sqrt{2\varepsilon}}{2} \end{pmatrix}$$

Also, $P(x_1) = \pi(x_1, x_2)\big|_{x_2=1} = MM^t$

where

$$M = \begin{pmatrix} x_1 + \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -x_1 + \frac{1}{2} & \dfrac{1-\sqrt{2\varepsilon}}{2} & \dfrac{1+\sqrt{2\varepsilon}}{2} \end{pmatrix}$$

Algorithm for construction of the representation as a sum of congruences remains to be developed.

Well-Known Specialization:

Fact: There exists a positive linear map such that

$$\phi : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$$

does not admit the representation of $\phi(A), A \in \mathbb{R}^{3\times 3}$

as a finite sum of congruences.

$$\phi(A) = \sum_k V_k^t A V_k, V_k \in \mathbb{R}^{3\times 3}, \forall k$$

Fact: The positive definite biquadratic form

$$Q(\mathbf{x},\mathbf{y}) = \sum_j \sum_k \sum_r \sum_l q_{jkrl} x_j x_k y_r y_l, (j \le k, r \le l)$$

$$\mathbf{x} = \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix}, \mathbf{y} = \begin{pmatrix} y_1 & y_2 & \dots & y_m \end{pmatrix}$$

may not be representable as a sum of squares of bilinear forms for all combinations of *n* and *m*.

Comment: Such an infeasibility in the *m* = *n* = 3 case is a stumbling block to passive synthesis of a multiport multivariate positive real matrix.

Completely positive maps provide the answer to SOS representation

Fact: Let $\phi : \mathbb{C}^{n\times n} \to \mathbb{C}^{m\times m}$. Then ϕ is positive IFF there exist completely positive linear maps ϕ_1 and ϕ_2 such that for all $A \in \mathbb{C}^{n\times n}, \phi(A) = \phi_1(A) + \phi_2(A^t)$ i.e. $\phi(A)$ is of the form

$$\phi(A) = \sum_i V_i^* A V_i + \sum_j W_j^* A^t W_j$$

where V_i and W_j are *n* x *m* matrices.

Problem: Characterization of positive maps which are not completely positive.

Answer: Difference of 2 completely positive maps?

(Sixia Yu, The Institute of Theoretical Physics, Academica Sinica, Beijing, June 5, 2005)