What can algebraic invariants tell us about robot kinematics?

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Kinematic Aspects of Robotics

Outline

Joint work over 35+ years with Chris Gibson, Jon Selig, Deborah Crook, Mohammed Daher, Petros Hadjicostas...

Invariant theory history

- Boole, Cayley 1840s: find quantities determined by coefficients of homogeneous forms that remain constant under arbitrary linear change of variables.
- **Example**: Discriminant $\Delta = b^2 - ac$ of a homogeneous quadratic $p(x, y) = ax^2 + 2bxy + cy^2$.

Proof of invariance. Write:

$$p(x,y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{x}^T P \mathbf{x} \qquad (1)$$

So $\Delta = b^2 - ac = -\det P$.
If $Q \in SL(2)$ (2×2 matrix with $\det Q = 1$) then

$$\det(Q^T P Q) = \det P \Rightarrow \Delta(p(Q\mathbf{x})) = \Delta(p(\mathbf{x})).$$

- The group $SL(2)$ acts on vector space of symmetric matrices by *congruence*, and on space of quadratics by *change of variable*.

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- Gordan (1868) proved invariants of binary ($n = 2$) quantics are finitely generated...
- Klein (1880): invariants of finite groups acting on $\mathbb{C}^2$ and began Erlangen programme
- Hilbert (1890,1893) proved a more general finite generation theorem
- Noether (1916), Weyl (1939): invariants of the *classical groups*
- Nagata (1960): counterexample to Hilbert's 14th problem
- Mumford (1960s): *geometric invariant theory*
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Classical group example

The general linear group $GL(n)$—non-singular $n \times n$ matrices = change of coordinates on $\mathbb{R}^n$ (or $\mathbb{C}^n$).

$GL(n)$ acts on $M(n)$, $n \times n$ matrices or linear transformations $\mathbb{R}^n \to \mathbb{R}^n$ by similarity/conjugacy: if $A \in M(n)$, $P \in GL(n)$ then

$$A \mapsto PAP^{-1}$$

The characteristic polynomial is invariant.

For all P:

$$\det(\lambda I - PAP^{-1}) = \det P(\lambda I - A)P^{-1} = \det(\lambda I - A) \qquad (2)$$

so all the coefficients (eg trace, determinant) of the characteristic polynomial—polynomials in the entries A_{ij} of A—are invariant.

Invariant polynomials in general

Given a group G, a group representation is a realisation of G as linear transformations of a vector space V (over a field k, usually $\mathbb{R}$ or $\mathbb{C}$), ie a group homomorphism $G \to GL(V)$ (group of automorphisms of V).

With respect to a basis for V, $g \in G$ can be written as a $n \times n$ matrix ($n = \dim V$).

The polynomial functions (with respect to a choice of basis for V having coordinate functions $x_1, \ldots, x_n$) form a ring (and k-algebra) $k[x_1, \ldots, x_n]$.

G acts on $k[\mathbf{x}] = k[x_1, \ldots, x_n]$ by:

$$(g \cdot f)(\mathbf{x}) = f(g^{-1}.\mathbf{x}) \tag{3}$$

An invariant (polynomial) of the representation is $f \in k[\mathbf{x}]$ such that for all $g \in G$, $g \cdot f = f$ (as polynomial functions).

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Fundamental theorems of invariant theory

Denote the set of invariants by $k[\mathbf{x}]^G$.

It is a **subalgebra** but not, in general, an **ideal** in the polynomial ring $k[\mathbf{x}]$.

First Fundamental Theorem (1FT).
Find a (minimal) set of polynomials $f_1, \ldots, f_r \in k[\mathbf{x}]^G$, (finite set of generators) so that for any $f \in k[\mathbf{x}]^G$, there is an r-variable polynomial g such that

$$f = g(f_1, \ldots, f_r).$$

Generators may not be *algebraically independent*: a polynomial g such that $g(f_1, \ldots, f_r) = 0$ is called a *syzygy*.

Second Fundamental Theorem (2FT). Find a finite set of syzygies for an invariant generating set.

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The Euclidean group

We are interested in *isometries* of a Euclidean space E over an inner product space V with $\dim V = n$, say.

Theorem 1

Every isometry $f : E \to E$ can be written in the form

$$f(a) = \tau_{\mathbf{t}} \circ \rho_R(a), \tag{4}$$

where $o \in E$ is an arbitrary choice of origin, $t \in V$, $R \in O(V)$ and $\tau_{\mathbf{t}}(a) = a + t$, $\rho_R(a) = o + R(a - o)$.

Choose orthonormal coordinates for V, so that $V \cong \mathbb{R}^n$, then R can be written as a matrix satisfying $R^{\mathrm{T}} R = I_n$, hence $\det R = \pm 1$.
To preserve orientation, fix $\det R = +1$ so $R \in SO(n)$.

Then the **Euclidean group** of orientation-preserving isometries is isomorphic to the *semi-direct product* $SE(n) = SO(n) \ltimes \mathbb{R}^n$.
We have for $\mu_i = (R_i, \mathbf{t}_i) \in SE(n)$, $i = 1, 2$:

$$\mu_2 \cdot \mu_1 = (R_2, \mathbf{t}_2) \cdot (R_1, \mathbf{t}_1) = (R_2 R_1, R_2 \mathbf{t}_1 + \mathbf{t}_2). \tag{5}$$

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Change of coordinates

An isometry $\mu \in SE(n)$ transforms under changes of origin and orthonormal coordinates, $\rho \in SE(n)$, by conjugacy:

$$\mu \mapsto \rho\mu\rho^{-1}. \tag{6}$$

Relevant physical properties of μ must be invariant under conjugacy.

Example. If $\mu = (I_n, \mathbf{t}) \in SE(n)$ is a pure a translation then it remains so under change of coordinates.

$SE(n)$ as a Lie group

The Euclidean group $SE(n)$ is a *Lie group* of dimension $\frac{1}{2}n(n+1)$.

The derivatives along paths through the identity $e = (I, \mathbf{0}) \in SE(n)$ comprise the *tangent space* at e, denoted $\mathfrak{se}(n)$. This is its Lie algebra.

Differentiating conjugacy $\mu \mapsto \rho\mu\rho^{-1}$ with respect to μ gives the adjoint action of $SE(n)$ on $\mathfrak{se}(n)$:

$$\mathrm{Ad}(\rho).x = \rho x \rho^{-1}$$

Differentiating again, w.r.t. ρ, gives the Lie bracket:

$$\mathrm{ad}(y).x = [y, x] = yx - xy$$

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Twist space

Fix $n = 3$. We call $\mathfrak{se}(3)$ the twist space and elements are twists.
It decomposes as

$$\mathfrak{se}(3) = \mathfrak{so}(3) \oplus \mathbb{R}^3$$

where elements of $\mathfrak{so}(3)$ are skew-symmetric matrices W or, equivalently, the associated 3-vectors $\boldsymbol{\omega}$.
Write a twist as $\mathbf{s} = (\boldsymbol{\omega}, \mathbf{v})$—a pair of 3-vectors or a single 6-vector.

The adjoint representation is:

$$\mathrm{Ad}(R, \mathbf{t}).(\boldsymbol{\omega}, \mathbf{v}) = \begin{pmatrix} R & O \\ TR & R \end{pmatrix} \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{pmatrix} \tag{7}$$

where T is the skew-symmetric representation of $\mathbf{t}$.

The Lie bracket is

$$[(\boldsymbol{\omega}_1, \mathbf{v}_1), (\boldsymbol{\omega}_2, \mathbf{v}_2)] = (\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2, \boldsymbol{\omega}_1 \times \mathbf{v}_2 + \mathbf{v}_1 \times \boldsymbol{\omega}_2). \tag{8}$$

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Twist invariants

Relevant physical properties of twists must be invariant under the adjoint representation.

Theorem 2 (Gibson, D 1991)

$\mathbb{R}[\boldsymbol{\omega}, \mathbf{v}]^{SE(3)}$ *is the polynomial algebra* $\mathbb{R}[\boldsymbol{\omega}.\boldsymbol{\omega}, \boldsymbol{\omega}.\mathbf{v}]$.

This establishes 1FT for the action. There are no syzygies, hence 2FT holds trivially.

Note: $\omega.\omega$ is the Killing form is a standard invariant for both $SO(3)$ and $SE(3)$.

The pitch of a non-zero twist $h = \begin{cases} \dfrac{\boldsymbol{\omega}.\mathbf{v}}{\boldsymbol{\omega}.\boldsymbol{\omega}} & \boldsymbol{\omega} \neq \mathbf{0} \\ \infty & \boldsymbol{\omega} = \mathbf{0} \end{cases}$ is an invariant of twists <u>and</u> screws—elements of the projective space $P(\mathfrak{se}(3))$.

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Orbits and normal forms

Away from the nullcone, $\boldsymbol{\omega} = \mathbf{0}$, the invariants $\boldsymbol{\omega}.\boldsymbol{\omega}, \boldsymbol{\omega}.\mathbf{v}$ separate orbits (equivalence classes) of twists. Also h separates orbits of screws.

A twist $(\boldsymbol{\omega}, \mathbf{v})$ with invariants $\alpha = \boldsymbol{\omega}.\boldsymbol{\omega} > 0$ and $\beta = \boldsymbol{\omega}.\mathbf{v}$ is equivalent to the normal form

$$(\sqrt{\alpha}, 0, 0, \beta/\sqrt{\alpha}, 0, 0)^{\mathrm{T}}$$

If $\alpha = 0$, hence $\beta = 0$, then

$$(0, 0, 0, \sqrt{\gamma}, 0, 0)^{\mathrm{T}}, \gamma = \mathbf{v}.\mathbf{v} \geq 0$$

suffices.

Note. A polynomial f is an invariant of the adjoint action (Ad) of a connected Lie group if and only if it is invariant under the adjoint action (ad) of the Lie algebra. This makes computation simpler.

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Dual numbers and $SE(3)$

The *dual numbers* $\mathbb{D}$ are the 2-dimensional real algebra of quantities $a + \varepsilon b$, $a, b \in \mathbb{R}$, where $\varepsilon^2 = 0$.

Theorem 3

The Euclidean group $SE(3)$ is isomorphic to the Lie group $SO(3, \mathbb{D})$.

There is a double cover of $SO(3, \mathbb{D})$ by the unit dual quaternions

$$\mathbb{D}S^3 = \{\check{q} = q + \varepsilon r : q, r \in \mathbb{H}, \check{q}\check{q}^* = 1\}$$

The Lie algebras of $SO(3, \mathbb{D})$ and $\mathbb{D}S^3$ are $\mathbb{D}^3$—pure dual quaternions—with Lie bracket the dual vector product.
Identify $(\boldsymbol{\omega}, \mathbf{v}) \in \mathfrak{se}(3)$ with pure dual quaternion $\check{\omega} = \boldsymbol{\omega} + \varepsilon \mathbf{v}$.
Then

$$\check{\omega}^2 = \boldsymbol{\omega}.\boldsymbol{\omega} + 2\,\varepsilon\,\boldsymbol{\omega}.\mathbf{v} \tag{9}$$

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Multi-twists and invariants

A multi-twist (or k-twist) is a k-tuple $(\mathbf{s}_1, \ldots, \mathbf{s}_k) \in \mathfrak{se}(3)^k$.

The adjoint action of $SE(3)$ induces an action on k-twists component-wise.

For $\mu \in SE(3)$, $\mu \cdot (\mathbf{s}_1, \ldots, \mathbf{s}_k) = (\mu \cdot \mathbf{s}_1, \ldots, \mu \cdot \mathbf{s}_k)$

Theorem 4 (Crook 2009)

The invariant subalgebra for 2-twists $(\mathbf{s}_1, \mathbf{s}_2)$, where $\mathbf{s}_i = (\boldsymbol{\omega}_i, \mathbf{v}_i)$, $i = 1, 2$, under the adjoint action is the polynomial algebra generated by:

$$\boldsymbol{\omega}_i \cdot \boldsymbol{\omega}_j, \quad \boldsymbol{\omega}_i \cdot \mathbf{v}_i, \quad \boldsymbol{\omega}_1 \cdot \mathbf{v}_2 + \boldsymbol{\omega}_2 \cdot \mathbf{v}_1 \;\; (1 \leq i, j \leq 2). \tag{10}$$

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Multi-twist and invariants

Given a real polynomial f, its dualisation $\check{f}$ is the same polynomial but with real variables replaced by dual variables. We have:

$$\check{f}(x_1+\varepsilon y_1, \ldots, x_k+\varepsilon y_k) = f(x_1, \ldots, x_n)+\varepsilon \nabla f(x_1, \ldots, x_k).(y_1, \ldots, y_k)^{\mathrm{T}}.$$

Theorem 5 (Study 1903, Daher 2013)

If f is k-vector invariant of the adjoint action of $SO(3)$, then the primal and dual parts of $\check{f}$ are (real) k-twist invariants of the induced action of $SE(3)$.

Furthermore, the primal and dual parts of the dualisation of any syzygy among $SO(3)$ k-vector invariants are syzygies for $SE(3)$ invariants.

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3-twist invariants and syzygies

Following Weyl, for 3-twists, 14 invariants come from the primal and dual parts of the dualisations of 7 generating 3-vector invariants for $SO(3)$:

$$I_{ij} = \boldsymbol{\omega}_i \cdot \boldsymbol{\omega}_j, \qquad \tilde{I}_{ij} = \boldsymbol{\omega}_i \cdot \mathbf{v}_j + \mathbf{v}_i \cdot \boldsymbol{\omega}_j, \quad 1 \le i \le j \le 3$$

$$I_{123} = [\boldsymbol{\omega}_1\ \boldsymbol{\omega}_2\ \boldsymbol{\omega}_3], \quad \tilde{I}_{123} = \sum_{\sigma \in C_3} [\boldsymbol{\omega}_{\sigma(1)}\ \boldsymbol{\omega}_{\sigma(2)}\ \mathbf{v}_{\sigma(3)}]. \tag{11}$$

where C_3 denotes the cyclic group of order 3 and $[\cdots]$ denotes determinant.

A single 3-vector $SO(3)$-invariant syzygy gives two syzygies for these 14 invariants:

$$I_{123}^2 = I_{11}I_{22}I_{33} - I_{11}I_{23}^2 + 2I_{12}I_{13}I_{23} - I_{12}^2 I_{33} - I_{13}^2 I_{22}$$

$$\begin{aligned} 2I_{123}\tilde{I}_{123} = {} & \tilde{I}_{11}I_{22}I_{33} + I_{11}\tilde{I}_{22}I_{33} + I_{11}I_{22}\tilde{I}_{33} \\ & - \tilde{I}_{11}I_{23}^2 - 2I_{11}I_{23}\tilde{I}_{23} + 2\tilde{I}_{12}I_{13}I_{23} + 2I_{12}\tilde{I}_{13}I_{23} + 2I_{12}I_{13}\tilde{I}_{23} \\ & - 2I_{12}\tilde{I}_{12}I_{33} - I_{12}^2\tilde{I}_{33} - 2I_{13}\tilde{I}_{13}I_{22} - I_{13}^2\tilde{I}_{22}. \end{aligned} \tag{12}$$

3-twist invariants and syzygies

Following Weyl, for 3-twists, 14 invariants come from the primal and dual parts of the dualisations of 7 generating 3-vector invariants for $SO(3)$:

$$I_{ij} = \boldsymbol{\omega}_i \cdot \boldsymbol{\omega}_j, \qquad \tilde{I}_{ij} = \boldsymbol{\omega}_i \cdot \mathbf{v}_j + \mathbf{v}_i \cdot \boldsymbol{\omega}_j, \quad 1 \le i \le j \le 3$$

$$I_{123} = [\boldsymbol{\omega}_1 \ \boldsymbol{\omega}_2 \ \boldsymbol{\omega}_3], \quad \tilde{I}_{123} = \sum_{\sigma \in C_3} [\boldsymbol{\omega}_{\sigma(1)} \ \boldsymbol{\omega}_{\sigma(2)} \ \mathbf{v}_{\sigma(3)}]. \tag{11}$$

where C_3 denotes the cyclic group of order 3 and $[\cdots]$ denotes determinant.

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6-twist invariants and syzygies

Q. Do 1FT and 2FT hold for 3-twist invariants?

Theorem 6 (Selig, D 2008)

Given 6 twists where $\mathbf{s}_i = (\boldsymbol{\omega}_i, \mathbf{v}_i)$, $i = 1, \ldots, 6$, *the Jacobian:*

$$J = [\mathbf{s}_1 \quad \mathbf{s}_2 \quad \mathbf{s}_3 \quad \mathbf{s}_4 \quad \mathbf{s}_5 \quad \mathbf{s}_6]. \tag{13}$$

is a 6-twist invariant of the induced adjoint action.

J *satisfies the degree* 12 *syzygy:*

$$J^2 = \det \begin{pmatrix} \tilde{I}_{11} & \cdots & \tilde{I}_{16} \\ \vdots & \ddots & \vdots \\ \tilde{I}_{16} & \cdots & \tilde{I}_{66} \end{pmatrix} \tag{14}$$

Q. Are there any more invariants for multi-twists?

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Product of exponentials

Let m twists, $\mathbf{s}_1, \dots, \mathbf{s}_m$, represent joints of a serial manipulator in its home configuration. The **forward kinematics** for the end-effector is

$$f : \mathbb{R}^m \to SE(3),\ f(u_1, \dots, u_m) = \exp(u_1\mathbf{s}_1)\exp(u_2\mathbf{s}_2)\cdots\exp(u_m\mathbf{s}_m)$$

Here $u_1, \dots, u_m$ are the joint variables.

Updating the twists at a configuration $\mathbf{u} = (u_1, \dots, u_m)$:

$$\mathbf{s}_2(\mathbf{u}) = \mathrm{Ad}(\exp(u_1\mathbf{s}_1))\mathbf{s}_2, \dots,$$
$$\mathbf{s}_m(\mathbf{u}) = \mathrm{Ad}(\exp(u_1\mathbf{s}_1)\cdots\exp(u_{m-1}\mathbf{s}_{m-1}))\mathbf{s}_m.$$

In addition to the multi-twist action induced by the adjoint, there is an additional action of prior joints on later joints.

Q. Which multi-twist (static) invariants remain (kinematically) invariant under this additional action?

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$$\mathbf{s}_2(\mathbf{u}) = \mathrm{Ad}(\exp(u_1 \mathbf{s}_1))\mathbf{s}_2, \ldots, \\ \mathbf{s}_m(\mathbf{u}) = \mathrm{Ad}(\exp(u_1 \mathbf{s}_1) \cdots \exp(u_{m-1} \mathbf{s}_{m-1}))\mathbf{s}_m.$$

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Denavit–Hartenberg parameters

DH parameters:

- Pitch h_i of joint i (usually $h = 0, \infty$)
- Link length $l_{i,i+1}$ and link angle $\theta_{i,i+1}$ between joints i, $i+1$
- Offset along joint i, $d_{i-1,i,i+1}$, for successive links.

In terms of adjoint invariants (Crook, Daher, D 2012, 2015):

$$h_i = \tilde{I}_{ii}/I_{ii}$$

$$l_{ij} = \left(I_{ij}(I_{ii}\tilde{I}_{jj} + \tilde{I}_{ii}I_{jj}) - I_{ii}I_{jj}\tilde{I}_{ij}\right)/I_{ii}I_{jj}\sqrt{I_{ii}I_{jj} - I_{ij}^2}$$

$$\cos\theta_{ij} = I_{ij}/\sqrt{I_{ii}I_{jj}}$$

$$l_{ij}\sin\theta_{ij} = \tilde{I}_{ij}/\sqrt{I_{ii}I_{jj}}$$

$$d_{ijk} = \frac{\tilde{I}_{ijk}I_{jj}(I_{ij}I_{jk} - I_{ik}I_{jj}) + I_{ijk}\left(\frac{1}{2}\tilde{I}_{jj}(I_{ij}I_{jk} + I_{ik}I_{jj})I_{jj}(\tilde{I}_{ik}I_{jj} - I_{ij}\tilde{I}_{jk} - \tilde{I}_{ij}I_{jk})\right)}{\sqrt{I_{jj}}(I_{ii}I_{jj} - I_{ij}^2)(I_{jj}I_{kk} - I_{jk}^2)}$$

Note: $I_{ii}I_{jj} - I_{ij}^2 = \|\boldsymbol{\omega}_i \times \boldsymbol{\omega}_j\|^2$

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In terms of adjoint invariants (Crook, Daher, D 2012, 2015):

$$\begin{aligned}
h_i &= \tilde{I}_{ii}/I_{ii} \\
l_{ij} &= \left(I_{ij}(I_{ii}\tilde{I}_{jj} + \tilde{I}_{ii}I_{jj}) - I_{ii}I_{jj}\tilde{I}_{ij} \right) / I_{ii}I_{jj}\sqrt{I_{ii}I_{jj} - I_{ij}^2} \\
\cos\theta_{ij} &= I_{ij}/\sqrt{I_{ii}I_{jj}} \\
l_{ij}\sin\theta_{ij} &= \tilde{I}_{ij}/\sqrt{I_{ii}I_{jj}} \\
d_{ijk} &= \frac{\begin{array}{c}\tilde{I}_{ijk}I_{jj}(I_{ij}I_{jk} - I_{ik}I_{jj}) + \\ I_{ijk}\left(\frac{1}{2}\tilde{I}_{jj}(I_{ij}I_{jk} + I_{ik}I_{jj})I_{jj}(\tilde{I}_{ik}I_{jj} - I_{ij}\tilde{I}_{jk} - \tilde{I}_{ij}I_{jk})\right)\end{array}}{\sqrt{I_{jj}}(I_{ii}I_{jj} - I_{ij}^2)(I_{jj}I_{kk} - I_{jk}^2)}
\end{aligned}$$

Note: $I_{ii}I_{jj} - I_{ij}^2 = \|\boldsymbol{\omega}_i \times \boldsymbol{\omega}_j\|^2$

Kinematic invariants

Theorem 7 (D 2021)

For a serial manipulator with (home) joint twists $\mathbf{s}_1, \ldots, \mathbf{s}_m$*,*

1. *the only known static invariants that are kinematically invariant are* $I_{ii}, \tilde{I}_{ii}$*,* $1 \leq i \leq m$ *and* $I_{i,i+1}, \tilde{I}_{i,i+1}$*,* $1 \leq i \leq m-1$*.*
2. *if* $\tilde{I}_i = 0$ *then* $I_{i-1,i,i+1}, \tilde{I}_{i-1,i,i+1}$ *for* $2 \leq i \leq m-1$ *are also kinematically invariant.*

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