The Geometry of the Adjoint Representation of $SE(3)$

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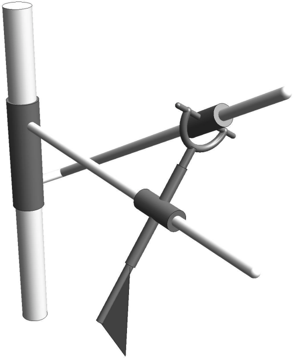
Motivation

Consider this mechanism:



Designed[1] to move a line so it remains in a linear complex of lines. End-effector assumes all displacements that keep line ℓ in a linear line complex.

[1] see Selig & Di Paola, 2024, "Mechanisms generating line trajectories" MMT **191**:105494

Constraint Variety

Set of poses can be thought of as a constraint or displacement variety.

Blaschke showed variety of displacements that keep a line in a linear complex given by the intersection of the Study quadric with another 6-dimensional quadric in $\mathbb{P}^7$.

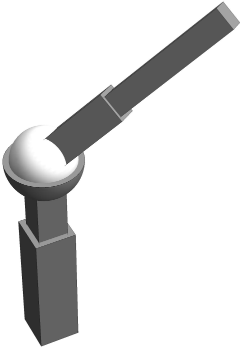
Question

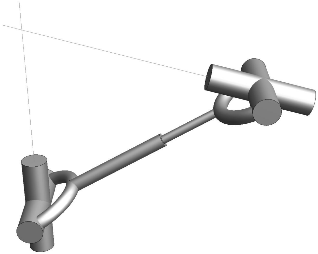
Suppose we have a rigid platform attached to the end-effectors of six such linkages arranged in general position. This will impose six constraints on the platform and hence we can expect a finite number of poses for the platform.

How many?

Other Mechanisms

For special line complexes, set of lines coplanar to a fixed line, other mechanisms possible.





For example the PSP or UPU linkages.

Line Complexes

A linear line complex is the set of lines reciprocal to a fixed screw. Using Plücker coordinates the direction of the line is given by $\boldsymbol{\omega} = (p_{01}, p_{02}, p_{03})^T$ and $\mathbf{v} = (p_{23},\ p_{31},\ p_{12})^T$ is the moment of the line, the equation of a linear line complex is given by a homogeneous linear equation in the Plücker coordinates,

$$\begin{aligned} & u_1 p_{01} + u_2 p_{02} + u_3 p_{03} + \sigma_1 p_{23} + \sigma_2 p_{31} + \sigma_3 p_{12} \\ & \qquad = \mathbf{u} \cdot \boldsymbol{\omega} + \boldsymbol{\sigma} \cdot \mathbf{v} = (\boldsymbol{\sigma}^T\, \mathbf{u}^T) \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{pmatrix} = 0 \end{aligned}$$

where u_i and σ_j are constants.
The set of lines in the complex has cylindrical symmetry about the axis of the screw,

$$\mathbf{s} = \begin{pmatrix} \boldsymbol{\sigma} \\ \mathbf{u} \end{pmatrix}$$

The Study Quadric

The Study quadric is a non-singular, 6-dimensional projective quadric in $\mathbb{P}^7$. If $(a_0 : a_1 : a_2 : a_3 : c_0 : c_1 : c_2 : c_3)$ are homogeneous coordinates, the equation of the Study quadric is

$$a_0 c_0 + a_1 c_1 + a_2 c_2 + a_2 c_3 = 0$$

Points in the Study quadric are in 1-to-1 correspondence with elements of the group of rigid-body displacements $SE(3)$, with the exception of a 3-dimensional projective plane in the quadric, here denoted A_∞.

How not to Solve it

Simple-minded approach: compute the degree of the intersection of six general constraint quadrics with the Study quadric.

Generally, number of solutions to a system of n homogeneous polynomial equations in $\mathbb{P}^n$ is the product of their degrees. Only works when the n varieties meet in finite number of points.

System of equations with this property is called a *complete intersection*. In our case, all the quadrics contain the 3-plane A_∞, which doesn't contain any physical group elements.

Other Approaches

In modern Algebraic Geometry the way out of this difficulty would be to use a technique know as "blowing up".

An older method: birational transformation. Transform Study quadric to equivalent variety such that constraint quadrics are transformed to hyperplanes. Number of solutions to the original problem just the degree of the variety representing the group $SE(3)$.

Technique used to show number of conformations of a general Gough-Stewart platform is 40. Degree of the variety representing $SE(3)$ found using a computer computation of variety's Hilbert polynomial[2].

[2] Gallet, Nawratil & Schicho, 2015, "Bond theory for pentapods and hexapods", *Journal of Geometry*, **106**:211–228.

Representations of $SE(3)$

An n-dimensional matrix representation of a group is a set of $n \times n$ non-singular matrices, one for each group element, such that multiplication in the group is modelled by multiplication of the corresponding matrices.

If different group elements correspond to different matrices representation is called *faithful*. Different faithful reps. give birational transformations of the group.

Examples

- The homogeneous or standard 4×4 representation embeds $SE(3)$ in a 6-dimensional, degree 8 variety in $\mathbb{P}^{12}$.
- Gallet, Nawratil & Schicho implicitly used 5×5 representation $SE(3)$, sometimes called the conformal representation. Used for pentaspherical coordinates.

Here, want to consider the adjoint representation, uses 6×6 matrices. Embeds the group in $\mathbb{P}^{17}$.

Adjoint Representation

Representation on $SE(3)$ on its Lie algebra—twists. If twist partitioned into,

$$\mathbf{s} = \begin{pmatrix} \boldsymbol{\sigma} \\ \mathbf{u} \end{pmatrix}$$

Then any 6×6 matrix in the adjoint representation can be partitioned as,

$$\mathrm{Ad}(G) = \begin{pmatrix} R & 0 \\ TR & R \end{pmatrix}.$$

where R is the 3×3 rotation matrix of the group element G and T is the 3×3 antisymmetric matrix corresponding to the translational part of G.

Action on the twist is,

$$\begin{pmatrix} \boldsymbol{\sigma} \\ \mathbf{u} \end{pmatrix} \longmapsto \begin{pmatrix} R & 0 \\ TR & R \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma} \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} R\boldsymbol{\sigma} \\ TR\boldsymbol{\sigma} + R\mathbf{u} \end{pmatrix}$$

Constraint Equation

So rigid-body displacements that keep a line,

$$\ell = \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{pmatrix} \quad \text{in a complex determined by} \quad \mathbf{s} = \begin{pmatrix} \boldsymbol{\sigma} \\ \mathbf{u} \end{pmatrix}$$

will satisfy the equation

$$(\boldsymbol{\sigma}^T\, \mathbf{u}^T) \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} R & 0 \\ TR & R \end{pmatrix} \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{pmatrix} = 0$$

Writing the matrix representing the displacements as,

$$\begin{pmatrix} M & 0 \\ N & M \end{pmatrix}$$

and taking the entries m_{ij} and n_{kl} of the 3×3 matrices M and N as homogeneous coordinates in a $\mathbb{P}^{17}$ we see that this gives a linear equation. That is, a hyperplane in $\mathbb{P}^{17}$.

Map from the Study Quadric I

Birational map from the Study quadric, coordinates $a_0, \ldots, c_3$.

$$M = \begin{pmatrix} a_0^2 + a_1^2 - a_2^2 - a_3^2 & 2(a_1a_2 - a_0a_3) & 2(a_1a_3 + a_0a_2) \\ 2(a_1a_2 + a_0a_3) & a_0^2 - a_1^2 + a_2^2 - a_3^2 & 2(a_2a_3 - a_0a_1) \\ 2(a_1a_3 - a_0a_2) & 2(a_2a_3 + a_0a_1) & a_0^2 - a_1^2 - a_2^2 + a_3^2 \end{pmatrix}$$

Multiple of a rotation matrix—determinant is

$$\det(M) = (a_0^2 + a_1^2 + a_2^2 + a_3^2)^3 \quad \text{and}$$
$$M^T M = MM^T = (a_0^2 + a_1^2 + a_2^2 + a_3^2)^2 I.$$

If a_i were homogeneous coordinates for a $\mathbb{P}^3$ this would be a projection of the Veronese 2-fold embedding of $\mathbb{P}^3$ in $\mathbb{P}^9$ to $\mathbb{P}^8$.

Map from the Study Quadric II

For the bottom left of the adjoint matrix the columns of N are,

$$\begin{pmatrix} n_{11} \\ n_{21} \\ n_{31} \end{pmatrix} = 2 \begin{pmatrix} -2(a_2c_2 + a_3c_3) \\ a_0c_3 + a_1c_2 + a_2c_1 + a_3c_0 \\ -a_0c_2 + a_1c_3 - a_2c_0 + a_3c_1 \end{pmatrix}$$

$$\begin{pmatrix} n_{12} \\ n_{22} \\ n_{32} \end{pmatrix} = 2 \begin{pmatrix} -a_0c_3 + a_1c_2 + a_2c_1 - a_3c_0 \\ -2(a_1c_1 + a_3c_3) \\ a_0c_1 + a_1c_0 + a_2c_3 + a_3c_2 \end{pmatrix}$$

$$\begin{pmatrix} n_{13} \\ n_{23} \\ n_{33} \end{pmatrix} = 2 \begin{pmatrix} a_0c_2 + a_1c_3 + a_2c_0 + a_3c_1 \\ -a_0c_1 - a_1c_0 + a_2c_3 + a_3c_2 \\ -2(a_1c_1 + a_2c_2) \end{pmatrix}$$

Notice that, map is undefined on A_∞ where all a_is vanish. This is only place where the map is undefined.

The Variety in $\mathbb{P}^{17}$ I

The group $SE(3)$ is six dimensional, so not all of $\mathbb{P}^{17}$. Call the image of the map V_6. Defined by an ideal of polynomials. Many possible bases for the ideal.
From the fact that both M^TM and MM^T are diagonal matrices we get six quadratic equations,

$$0 = m_{11}m_{12} + m_{21}m_{22} + m_{31}m_{32},$$
$$0 = m_{11}m_{13} + m_{21}m_{23} + m_{31}m_{33},$$
$$0 = m_{12}m_{13} + m_{22}m_{23} + m_{32}m_{33},$$

$$0 = m_{11}m_{21} + m_{12}m_{22} + m_{13}m_{32},$$
$$0 = m_{11}m_{31} + m_{12}m_{32} + m_{13}m_{33},$$
$$0 = m_{21}m_{31} + m_{22}m_{32} + m_{23}m_{33}.$$

The Variety in $\mathbb{P}^{17}$ II

Expressing the fact that the diagonal elements are all equal gives 15 quadratic equations, but only 5 are linearly independent, for example we could choose,

$$
\begin{aligned}
m_{11}^2 + m_{21}^2 + m_{31}^2 &= m_{12}^2 + m_{22}^2 + m_{32}^2, \\
m_{11}^2 + m_{21}^2 + m_{31}^2 &= m_{13}^2 + m_{23}^2 + m_{33}^2, \\
m_{11}^2 + m_{21}^2 + m_{31}^2 &= m_{11}^2 + m_{21}^2 + m_{13}^2, \\
m_{11}^2 + m_{21}^2 + m_{31}^2 &= m_{21}^2 + m_{22}^2 + m_{23}^2, \\
m_{11}^2 + m_{21}^2 + m_{31}^2 &= m_{31}^2 + m_{32}^2 + m_{33}^2.
\end{aligned}
$$

Relation between the determinant and trace of M gives a degree 6 equation,

$$\mathrm{T}r(M^T M)^3 = 27 \det(M)^2$$

The Variety in $\mathbb{P}^{17}$ III

We also need equations for the components of N. Expect NM^T to be anti-symmetric, since $N = TR$ and T anti-symmetric. The fact that the diagonal element of NM^T vanish gives three equations,

$$
\begin{aligned}
0 &= n_{11}m_{11} + n_{12}m_{12} + n_{13}m_{13},\\
0 &= n_{21}m_{21} + n_{22}m_{22} + n_{23}m_{23},\\
0 &= n_{31}m_{31} + n_{32}m_{32} + n_{33}m_{33}.
\end{aligned}
$$

Another three equations express the fact that pairs of elements, symmetric about the diagonal have opposite signs,

$$
\begin{aligned}
0 &= n_{11}m_{21} + n_{12}m_{22} + n_{13}m_{23} + n_{21}m_{11} + n_{22}m_{12} + n_{23}m_{13},\\
0 &= n_{11}m_{31} + n_{12}m_{32} + n_{13}m_{33} + n_{31}m_{11} + n_{32}m_{12} + n_{33}m_{13},\\
0 &= n_{21}m_{31} + n_{22}m_{32} + n_{23}m_{33} + n_{31}m_{21} + n_{32}m_{22} + n_{33}m_{23}.
\end{aligned}
$$

The Variety in $\mathbb{P}^{17}$ IV

Finally, since $N = TR$ we have that NN^T will have the form,

$$NN^T = \lambda \begin{pmatrix} t_2^2 + t_3^2 & -t_1 t_2 & -t_1 t_3 \\ -t_1 t_2 & t_1^2 + t_3^2 & -t_2 t_3 \\ -t_1 t_3 & -t_2 t_3 & t_1^2 + t_2^2 \end{pmatrix}$$

where λ is some constant and the t_is are components of the translation vector. Can be viewed as 2-fold Veronese embedding of $\mathbb{P}^2$ in a $\mathbb{P}^5$. Equations defining this variety are given by asserting that the matrix,

$$(1/2)\,\mathrm{T}r(NN^T)I - NN^T = \lambda \begin{pmatrix} t_1^2 & t_1 t_2 & t_1 t_3 \\ t_1 t_2 & t_2^2 & t_2 t_3 \\ t_1 t_3 & t_2 t_3 & t_3^2 \end{pmatrix}$$

has rank 1. This gives six 2×2 determinants vanishing, so 6 quartic equations in the n_{ij} coordinates.

Closure

Notice, that the map from the Study quadric cannot give a point in $\mathbb{P}^{17}$ with $M = 0$. (This could only happen when $a_0 = a_1 = a_2 = a_3 = 0$, but then the whole map is undefined.)

But the equations for V_6 include the six equations expressing,

$$\mathrm{R}ank\left(\mathrm{T}r(NN^T)I - 2NN^T\right) = 1$$

and these equations are independent of M. So, the intersection of V_6 with the $(17 - 9) = 8$-plane $M = 0$, is the solution to these equations.

From their construction we know that the solution can be parametrised as $N = YP$, where Y is an arbitrary 3×3 antisymmetric matrix and P is and arbitrary 3×3 orthogonal matrix.

Inverse Map

Rational map from V_6 to the Study quadric. Let,

$$A = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix}$$

With $\overline{M} = (M - M^T)$ and$\overline{N} = (N - N^T)$ the map is,

$$\begin{aligned}
a_0 &= -\operatorname{Tr}(\overline{M}^2)\big(6\det(M) + 2\operatorname{Tr}(M)\operatorname{Tr}(MM^T)\big),\\
A &= -2\operatorname{Tr}(\overline{M}^2)\operatorname{Tr}(MM^T)\overline{M},\\
c_0 &= -\operatorname{Tr}(\overline{M}^2)\operatorname{Tr}(MM^T)\operatorname{Tr}(N),\\
C &= -4\operatorname{Tr}(MM^T)\big(\overline{M}^2\,\overline{N} - 2\overline{M}\,\overline{N}\,\overline{M} + \overline{N}\,\overline{M}^2\big) - \\
&\qquad \big(2\operatorname{Tr}(M^3) + 2\operatorname{Tr}(MM^T)\operatorname{Tr}(M) - \\
&\qquad\quad 3\operatorname{Tr}(M^2)\operatorname{Tr}(M) + \operatorname{Tr}(M)^3\big)\operatorname{Tr}(N)\overline{M},
\end{aligned}$$

Exceptional Set

Notice, map is degree 5 in coordinates m_{ij}, n_{kl}. Certainly it is undefined if $M = 0$.

More generally, the map is undefined if $\overline{M} = 0$, that is when M is symmetric.

So, the map is undefined for displacements with rotation angles of 0 or π radians: pure translations and π-screws.

Intersection with 3 Hyperplanes I

To estimate the degree of V_6 we can look at the intersection with three hyperplanes. In this special case the intersection breaks-up into a number of components which can be easily identified.

Consider the problem of requiring a single line to be simultaneously reciprocal to three screws. The line will be reciprocal to any screw in the 3-system generated by the screws. In particular, it will be reciprocal to the regulus of lines in the 3-system. Suppose this system has a circular symmetry so the regulus is from a hyperboloid of revolution.

To be definite, let us assume that the fixed lines are,

$$\ell_1 = \begin{pmatrix} (1/\sqrt{2})\sin\alpha \\ (1/\sqrt{2})\sin\alpha \\ \cos\alpha \\ -(1/\sqrt{2})\rho\sin\alpha \\ -(1/\sqrt{2})\rho\cos\alpha \\ \rho\sin\alpha \end{pmatrix}, \ell_2 = \begin{pmatrix} 0 \\ \sin\alpha \\ \cos\alpha \\ 0 \\ -\rho\cos\alpha \\ \rho\sin\alpha \end{pmatrix}, \ell_3 = \begin{pmatrix} -(1/\sqrt{2})\sin\alpha \\ (1/\sqrt{2})\sin\alpha \\ \cos\alpha \\ (1/\sqrt{2})\rho\sin\alpha \\ -(1/\sqrt{2})\rho\cos\alpha \\ \rho\sin\alpha \end{pmatrix}.$$

Here, ρ and α are constant parameters.

Intersection with 3 Hyperplanes II

The moving line will be initially set to,

$$\ell = \begin{pmatrix} 0 \\ -\sin\alpha \\ \cos\alpha \\ 0 \\ -\rho\cos\alpha \\ -\rho\sin\alpha \end{pmatrix}.$$

The three linear equations in the variables m_{ij}, n_{kl} are thus,

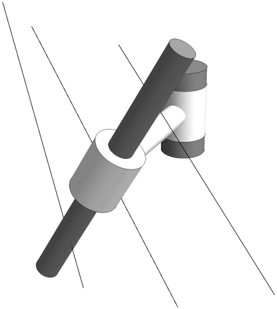
$$\ell_1^T \begin{pmatrix} N & M \\ M & 0 \end{pmatrix} \ell = 0,\ \ell_2^T \begin{pmatrix} N & M \\ M & 0 \end{pmatrix} \ell = 0,\ \ell_3^T \begin{pmatrix} N & M \\ M & 0 \end{pmatrix} \ell = 0$$

Easy to check they are linearly independent since ℓ_1, ℓ_2 and ℓ_3 are.

RC Mechanism

The equations are easy to solve since all the lines in the opposite regulus will be reciprocal to every line in the original regulus.

So the displacements of ℓ keeping it reciprocal to ℓ_1, ℓ_2 and ℓ_3 can be generated by an RC mechanism. The first R joint sweeps the axis (ℓ) of the C joint along the regulus. The C joint then gives a self motion of ℓ.



More Solution

There are more solutions though.

Suppose that before displacing ℓ we subject it to a half-turn about the common perpendicular to the axes of the R and C joints. This displacement cannot be achieved by the RC mechanism, but following it by the displacements allowed by the mechanism give another 3-dimensional set of solutions.

We must also consider non-physical solutions in the closure of $SE(3)$. Recall, this is where $M = 0$, so the equations to solve reduce to,

$$\boldsymbol{\omega}_1^T N\boldsymbol{\omega} = 0, \quad \boldsymbol{\omega}_2^T N\boldsymbol{\omega} = 0, \quad \boldsymbol{\omega}_3^T N\boldsymbol{\omega} = 0$$

where $N = YP$.

Even More Solutions

All three equations can be satisfied if the anti-symmetric matrix Y corresponds to the vector $P\boldsymbol{\omega}$. That is, if $Y = P\Omega P^T$, so that $N = P\Omega$. Then, $P\Omega\boldsymbol{\omega} = P(\boldsymbol{\omega} \times \boldsymbol{\omega}) = 0$.

If P is any rotation matrix we get a 3-dimensional set of solutions. Call this set X.
Note, X is a projection of the Veronese 2-fold embedding of $\mathbb{P}^3$ in $\mathbb{P}^9$.

To find the degree of X we can intersect it with 3 arbitrary hyperplanes. These can be written,

$$\mathbf{w}_1^T P\Omega\mathbf{u}_1 = 0, \quad \mathbf{w}_2^T P\Omega\mathbf{u}_2 = 0, \quad \mathbf{w}_3^T P\Omega\mathbf{u}_3 = 0$$

where $\mathbf{w}_i$ and $\mathbf{u}_j$ are arbitrary 3-vectors.

Degree of X

Since Ω is fixed we need to solve for the rotation matrix P. Each of these equations can be interpreted as requiring the rotations P, to place $\mathbf{w}_i$ on the plane normal to $\boldsymbol{\omega} \times \mathbf{u}_i$.

Solutions for one equation can be parametrised by any rotation about $\mathbf{w}_i$ followed by a rotation about the normal $\boldsymbol{\omega} \times \mathbf{u}_i$, preceded by a rotation that takes $\mathbf{w}_i$ from its initial position to the plane.

These rotations comprise a spherical RR dyad. Intersecting two such dyads will produce a spherical 4-bar mechanism. Using quaternions to represent the rotations, can see that the motion of the coupler bar of the 4-bar is the intersection of a pair of quadric surfaces in $\mathbb{P}^3$, hence an elliptic quartic curve.

Intersecting with another quadric—the third equation, will give 8 solutions in general. So the degree of X is 8.
Get another set of solutions when P is a reflection, the product of any rotation and a fixed reflection. Clearly this component has the same degree as X.

Homology

Homology is a generalisation of the degree of a variety. In $\mathbb{P}^n$ a variety of dimension k will have a homology class $d\alpha_k$ where d is an integer and α_k is the homology class of a k-dimensional plane. In this case d corresponds to the degree of the variety.

Bézout says that intersecting varieties with homology $d\alpha_k$ and $e\alpha_l$ gives another variety with homology,

$$d\alpha_k \cap e\alpha_l = (de)\alpha_{k+l-n}$$

in general.

If $k + l - n < 0$ then, in general, the varieties do not meet and the homology is zero. The class of a point is α_0 and a variety with class $k\alpha_0$ consists of k points counted with multiplicity.

Homology in Segre Varieties

For a Segre variety $\mathbb{P}^m \times \mathbb{P}^n$ homology in dimension k given by classes,

$$\alpha_i \beta_j, \quad \text{where} \quad i + j = k.$$

α_i is a class in the first factor and β_j is from the second factor, so $0 \leq i \leq m$ and $0 \leq j \leq n$.

The intersection of a pair of classes given by the intersection of the classes from each factor,

$$\alpha_i \beta_j \cap \alpha_p \beta_q = (\alpha_i \cap \alpha_p)(\beta_j \cap \beta_q) = \alpha_{i+p-m} \beta_{j+q-n}.$$

Intersection of $\mathbb{P}^m \times \mathbb{P}^n$, with a hypersurface defined by a polynomial separately homogeneous of degree d in the coordinates of the first factor and homogeneous of degree e in the second factor has class,

$$d\alpha_{m-1}\beta_n + e\alpha_m\beta_{n-1}$$

RC Mechanism again

Returning to the RC mechanism, rotations about R-joint can be parametrised by rotation matrix with entries that are quadratic in homogeneous parameters, $c_1 = \cos\theta_1/2$, $s_1 = \sin\theta_1/2$. Similarly, rotations about the C-joint are quadratic in $c_2 = \cos\theta_2/2$, $s_2 = \sin\theta_2/2$. Translations along the C-joint are linear in homogeneous parameters.

If we intersect the 3-dimensional variety parametrised the RC-linkage with an arbitrary hyperplane, we get a polynomial with degree 2 in $c_, s_1$, degree 2 in c_2, s_2 and linear in the parameters of the translation. So has homology,

$$2\alpha_0\beta_1\gamma_1 + 2\alpha_1\beta_0\gamma_1 + \alpha_1\beta_1\gamma_0$$

RC Mechanism Degree in V_6

To find the degree of the variety we intersect with 3 arbitrary hyperplanes,

$$
\begin{aligned}
&(2\alpha_0\beta_1\gamma_1 + 2\alpha_1\beta_0\gamma_1 + \alpha_1\beta_1\gamma_0)^3 \\
&= (2\alpha_0\beta_1\gamma_1 + 2\alpha_1\beta_0\gamma_1 + \alpha_1\beta_1\gamma_0)^2 \cap (2\alpha_0\beta_1\gamma_1 + 2\alpha_1\beta_0\gamma_1 + \alpha_1\beta_1\gamma_0) \\
&= (8\alpha_0\beta_0\gamma_1 + 4\alpha_1\beta_1\gamma_1 + 4\alpha_1\beta_0\gamma_0) \cap (2\alpha_0\beta_1\gamma_1 + 2\alpha_1\beta_2\gamma_1 + \alpha_1\beta_1\gamma_0) \\
&= 8\alpha_0\beta_0\gamma_0 + 8\alpha_0\beta_0\gamma_0 + 8\alpha_0\beta_0\gamma_0 \\
&= 24\alpha_0\beta_0\gamma_0
\end{aligned}
$$

That is, we expect 24 complex solutions counted with multiplicity.

Degree of V_6

Finally, we can estimate the degree of V_6. The intersection of V_6 with the three hyperplane given by restricting a line to the three special linear line complexes breaks into (at least) ~~three~~ four components, two with degree 24 corresponding to the displacements of an RC-linkage and right translate of this component. The other two components lie~~s~~ in the non-physical region of the variety and each has degree 8. So the total degree is,

$$24 + 24 + 8 + 8 = \cancel{56} \quad 64$$

Conclusions

- Lots of assumptions here — needs checking!
- Important point is representations of $SE(3)$ give different compactifications of the group and these should be important for kinematic problems.
- Also would be useful to look at flag varieties, and other homogeneous spaces. E.g. pointed planes (Study's quirls).
- For 3-dimensional varieties of displacements, knowing the bi-degree in the Study quadric would also be helpful.
- Can compute homology by solving simpler kinematic problems.

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THANK YOU