A level-set approach for problem of crack detection from electrical measurements

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Workshop 5 – Inverse Problems on Large Scales

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Outline

1. Introduction
2. Modelling the parameter space
3. Numerical experiments
4. Tikhonov regularization

Collaborators (current project):
E. Hafemann (UFSC), A. De Cezaro (Rio Grande), A. Osses (Santiago)

A. DeCezaro, E. Haffeman, A. Osses, A.L., *A regularization method based on level-sets for the problem of crack detection from electrical measurements,* Inv. Probl., submitted

Previous Collaborators:
O. Scherzer(2004/05), X.-C. Tai(2009/13), U. Ascher(2010), O. Dorn(2003)

Outline

- Inverse problem: to determine the position and shape of a crack in a bounded domain $\Omega \subset \mathbb{R}^2$ from electrical measurements on the boundary $\partial\Omega$.

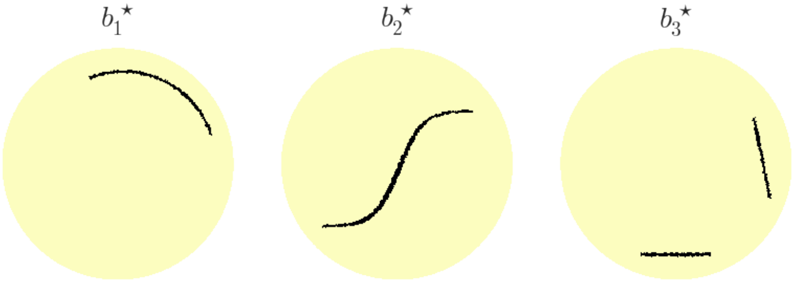

Figure: Tipical crack scenarios $b_k^\star$, $k = 1, 2, 3$.

- Based on the level-set approach in [2] and on the regularization strategy in [3], we propose a Tikhonov type method for stabilizing the inverse problem.

D. Alvarez, O. Dorn, N. Irishina, M. Moscoso, *Crack reconstruction using a level-set strategy,* Journal of Computational Physics 228 (2009)

A. De Cezaro, A. Leitão, X.-C. Tai, *On multiple level-set regularization methods for inverse problems,* Inverse Problems 25 (2009)

- An iterative method of multiple level-set type is derived from the optimality conditions for the Tikhonov functional, and a relation between this method and the iterated Tikhonov method is establised.

- We assume that the domain Ω has Lipschitz boundary and represents the specimen under investigation.
- A set of currents profiles $\{\eta_j\}_{j=1}^N$ are applied at $\partial\Omega$, for with, we have access to measurements of the corresponding potentials $\{u_j\}_{j=1}^N$ on $\partial\Omega$.
- The corresponding electric potential u_j satisfies

$$\nabla \cdot (b(x)\nabla u_j(x)) = 0\,,\; x \in \Omega\,, \qquad b(x)(u_j(x))_\nu = \eta_j(x)\,,\; x \in \partial\Omega. \quad (1)$$

with $\int_{\partial\Omega} \eta_j = 0$, for $j = 1, \cdots, N$

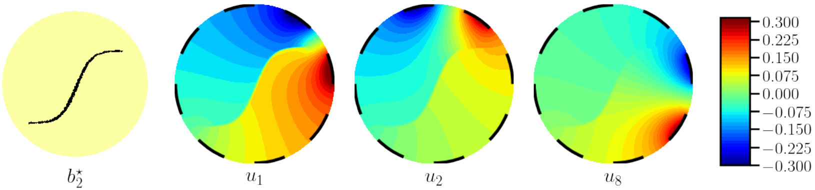

Figure: Typical NtD experiment. (LEFT) Crack $b_2^\star$. (RIGHT) Solutions u_1, u_2 and u_8 of (1) for the Neumann data η_1, η_2 and η_8 respectively.

- Insulating cracks with finite conductivity contrast between the interior and the exterior of the crack are modeled by the conductivity coefficient $b(x)$.
  - — $b(x) = b_i$, if x is inside the crack;
  - — $b(x) = b_e$, if x is outside the crack;
  - — $b_e >> b_i$;
  - — $\beta > 0$ fixed crack thickness.

- This corresponds to the assumption that cracks can be modeled as a thin structure with small thickness along a curve contained in Ω.

- The inverse problem we are concerned with consists in identifying the coefficient function $b(x)$ from a finite number N of experiments, where the current profiles η_j, $j = 1, \ldots, N$, are chosen in an appropriate way and the corresponding measurements $\gamma_j := u_j|_{\partial\Omega}$ are available.

- Notation:

— $D_F := \{b \in L^\infty(\Omega); \overline{b} \geq b(x) \geq \underline{b} > 0, \text{ a.e. in } \Omega\}$

— $X := \{b \in L^\infty(\Omega); b(x) \geq \underline{b} > 0, \text{ a.e. in } \Omega\}$

— $Y := H^{1/2}(\partial\Omega)$

- If $\eta_j \in H^{-1/2}(\partial\Omega)$, the Neumann BVP in (1) has a unique solution

$$u_j \in H^1_* := \{u \in H^1(\Omega)\,;\, \int_{\partial\Omega} u_j = 0\}.$$

- The crack detection problem can be written in terms of the system of nonlinear operator equations

$$\begin{aligned} F_j : D_F \subset X &\rightarrow Y \\ b &\mapsto F_j(b) = u_j|_{\partial\Omega} =: \gamma_j \end{aligned} \tag{2}$$

- Literature overview (far from being complete):

— A. Friedman, M. Vogelius, Determining cracks by boundary measurements, Indiana University Mathematics Journal 38 (1989)

— M. Bruhl, M. Hanke, M. Pidcock, Crack detection using electrostatic measurements, ESAIM: Mathematical Modelling and Num. Anal. 35 (2001)

— Y. Boukari, H. Haddar, The factorization method applied to cracks with impedance boundary conditions, Inv. Probl. & Imaging 7 (2013)

— J. Guo, X. Zhu, The factorization method for cracks in EIT, Comp. Appl. Math. 40 (2021)

— A. Hauptmann, M. Ikehata, H. Itou, S. Siltanen, Revealing cracks inside conductive bodies by electric surface measurements, Inverse Problems 35 (2018)

— W-K. Won-Kwang Park, Performance analysis of multi-frequency topological derivative for reconstructing perfectly conducting cracks, Journal of Comput. Physics 335 (2017)

- Main results:

  i) **Modelling the parameter space:**
  The parameters $b \in D_F$ are represented using pairs of level-set functions $(\varphi, \psi) \in H^1(\Omega)^2$, i.e., $b = P(\varphi, \psi)$ where P is a discontinuous operator;

  ii) **Tikhonov regularization approach:**
  The multiple level-set approach in *i*) is used to define a Tikhonov functional based on *TV*-H^1 regularization;

  iii) **Iterative method:**
  The optimality conditions for this Tikhonov functional allow the derivation of an iterative multiple level-set type method for solving the crack identification problem.

Outline

- Solve the abstract operator equation

$$F(u) = y, \quad \|y^\delta - y\|_Y \le \delta,$$

$F : \mathbb{D} \subset X \to Y$ is a Fréchet diff. mapping X Banach, Y Hilbert space.

- Assumption: the solution u of the Inv.Probl. above is a simple function defined on a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, and assuming at most N different values,

- Ansatz: A solution u can be represented in the form

$$\begin{aligned} u = \; & c_1 H(\phi^1) H(\phi^2) + c_2 H(\phi^1)(1 - H(\phi^2)) + \\ & c_3 (1 - H(\phi^1)) H(\phi^2) + c_4 (1 - H(\phi^1))(1 - H(\phi^2)) \; =: \; P(\phi^1, \phi^2), \end{aligned}$$

("color level-set" or "multiple level-sel"; [Tai/Chan'04], [Chan/Vese'02] + Tai, Dorn, Ascher, van den Doel, A.L.).

- Rewrite the inverse problem in the form:

$$F(P(\phi^1, \phi^2)) = y,$$

and solve it in terms of (ϕ^1, ϕ^2), the level-set functions.

- Choose a Tikhonov functional:

– The least square approach leads to the Santosa model

$$\mathcal{F}_\alpha(\phi^1,\phi^2) := \|F(P(\phi^1,\phi^2)) - y^\delta\|_Y^2 ,$$

– The ROF approach leads to the Chan-Vese model

$$\mathcal{F}_\alpha(\phi^1,\phi^2) := \|F(P(\phi^1,\phi^2)) - y^\delta\|_Y^2 + \alpha \sum_{j=1}^{2} |H(\phi^j)|_{\mathrm{BV}} ,$$

(in general one cannot guarantee the existence of a minimizers)

– The BV-H^1 approach leads to the models in [Scherzer/AL'05] and [Tai/AL'09]

$$\mathcal{G}_\alpha(\phi^1,\phi^2) := \|F(P(\phi^1,\phi^2)) - y^\delta\|_Y^2 + \alpha \sum_{j=1}^{2} \left\{ \beta |H(\phi^j)|_{\mathrm{BV}} + \|\phi^j - \phi_0^j\|_{H^1(\Omega)}^2 \right\} .$$

– (PLAY VIDEO) Example: Inverse Potential Problem in 2D.

Multiple level-set representation in [2] for cracks:

• A level-set function $\varphi : \Omega \to \mathbb{R}$ is chosen such that its zero level-set $\Gamma_\varphi := \{x \in \Omega;\ \varphi(x) = 0\}$ defines a connected curve within Ω; the cracks are located 'along' Γ_φ.

• Another level-set function $\psi : \Omega \to \mathbb{R}$ is chosen such the cracks are contained in the set $B := \{x \in \Omega;\ \psi(x) < 0\}$.

— The intersections of the level-set curve $\Gamma_\psi := \{x \in \Omega;\ \psi(x) = 0\}$ with Γ_φ coincide with the endpoints of the cracks.

— The position of the cracks corresponds to the set

$$S = S(\varphi, \psi) := \Gamma_\varphi \cap B\,.$$

- We consider the cracks to have small fixed thickness $\beta > 0$ and conductivity $b_i > 0$ much smaller than the background value $b_e > 0$ (the three constants are known).

- The position of the cracks is represented by the set

$$S_\beta = S_\beta(\varphi, \psi) := \{x \in \Omega;\ 0 < \varphi(x) < \beta\} \cap \{x \in \Omega;\ \psi(x) < 0\}.$$

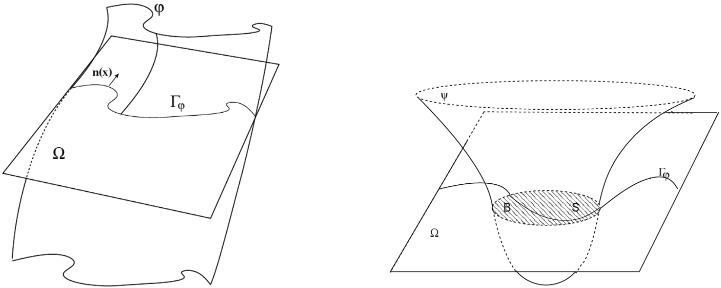

Figure: Multiple level-set representation for cracks.

- The conductivity distribution $b(x)$ in (1) is modeled by

$$b(x) \;=\; b_e + (b_i - b_e)\,\chi_{S_\beta}(x)\,, \tag{3}$$

where χ_{S_β} is the indicator function of the set S_β.

(multiple level-set representation of the parameter b)

- Following [3] we introduce the Heaviside projector

$$(H(\phi))(x) \;:=\; \begin{cases} 1, & \text{if } \phi(x) > 0 \\ 0, & \text{if } \phi(x) \le 0 \end{cases},$$

and the translation $(H_\beta(\phi))(x) \;:=\; H(\phi(x) - \beta)$.

- The conductivity distribution $b(x)$ can be written in the form

$$b \;=\; (b_i - b_e)\,[H(\varphi) - H_\beta(\varphi)]\; H(\psi) + b_e \;=:\; P(\varphi, \psi)\,. \tag{4}$$

- As already observed in [3], the operator H maps $H^1(\Omega)$ into the space

$$\mathcal{V}_{0,1} \ := \ \{w \in L^\infty(\Omega) \,|\, w = \chi_S \,,\ S \subset \Omega \text{ measurable}, \ \mathcal{H}^1(\partial S) < \infty\}\,, \qquad (5)$$

($\mathcal{H}^1(S)$ denotes the one-dimensional Hausdorff-measure of the set S)

- The operator P in (4) maps $H^1(\Omega) \times H^1(\Omega)$ into the admissible class

$$\mathcal{V} \ := \ \{w \in L^\infty(\Omega) \,|\, w = b_e + (b_i - b_e)\chi_S \,,\ S \subset \Omega \text{ measurable}, \ \mathcal{H}^1(\partial S) < \infty\}\,,$$

- Within this framework, the inverse problem (2) can be written in the form of the system of operator equations

$$F_j(P(\varphi, \psi)) \ = \ \gamma_j^\delta \,,\ j = 1, \dots, N. \qquad (6)$$

(once a solution (φ, ψ) of (6) is obtained, a corresponding solution of (2) is given by $b = P(\varphi, \psi)$)

A Thikhonov approach:

- We follow [3] and introduce the Tikhonov functional

$$\mathcal{G}_\alpha(\varphi,\psi) := \sum_{j=1}^{N} \|F_j(P(\varphi,\psi)) - \gamma_j^\delta\|_Y^2 + \alpha\Big\{ |H(\varphi)|_{\mathrm{BV}} + |H_\beta(\varphi)|_{\mathrm{BV}} + |H(\psi)|_{\mathrm{BV}} + \|\varphi-\varphi_0\|_{H^1(\Omega)}^2 + \|\psi-\psi_0\|_{H^1(\Omega)}^2 \Big\}, \quad (7)$$

based on TV–H^1 penalization.

- The H^1–terms act simultaneously as a control on the size of the norm and as a regularization on the space $H^1(\Omega)$.

- The BV-seminorm terms are well know for penalizing the length of the Hausdorff measure of the boundary of the sets $\{x : \varphi(x) > 0\}$, $\{x : \varphi(x) > \beta\}$ and $\{x : \psi(x) > 0\}$.

Outline

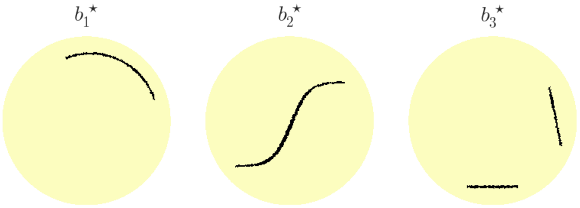

Figure: Exact cracks $b_k^\star$, $k = 1,2,3$ used in the numerical experiments.

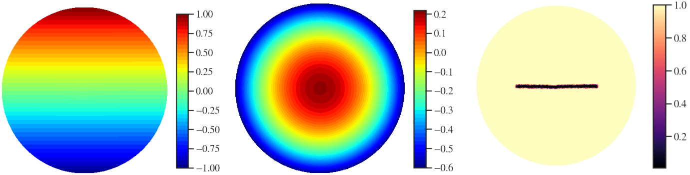

Figure: Initial guess for the level-set method: (LEFT) φ_0, (CENTER) ψ_0, (RIGHT) corresponding crack $P_\varepsilon(\varphi_0, \psi_0)$.

Initial guess for the level-set method

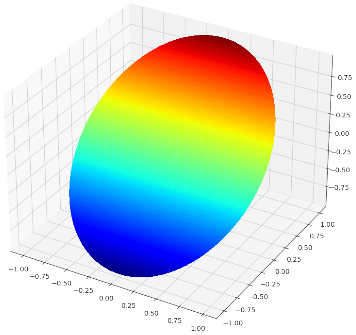

Figure: Initial guess for the level-set method: (LEFT) φ_0, (RIGHT) ψ_0.

Crack scenario $b_3^\star$ with $\delta = 1\%$

Figure: Crack scenario $b_3^\star$ with $\delta = 1\%$. Evolution of $b_k = P_\varepsilon(\varphi_k, \psi_k)$ for $0 \leq k \leq 1500$.

Crack scenario $b_3^\star$ with $\delta = 1\%$

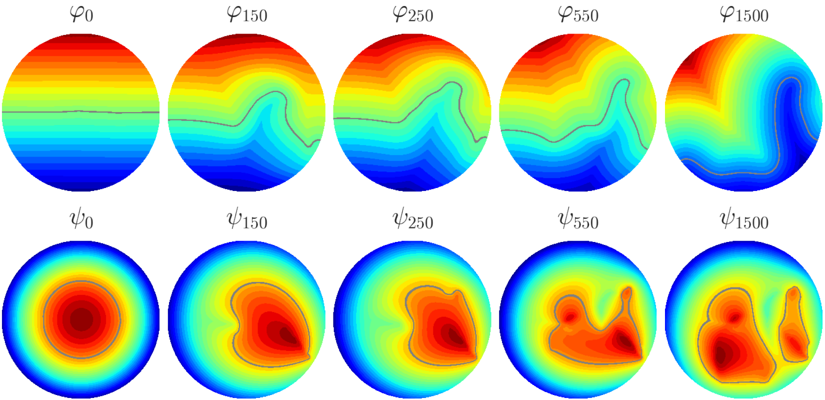

Figure: Crack scenario $b_3^\star$ with $\delta = 1\%$. Evolution of the level-set functions φ_k and ψ_k for $0 \leq k \leq 1500$.

Crack scenarios $b_1^\star, b_2^\star, b_3^\star$ with $\delta = 1\%$ & $\delta = 20\%$

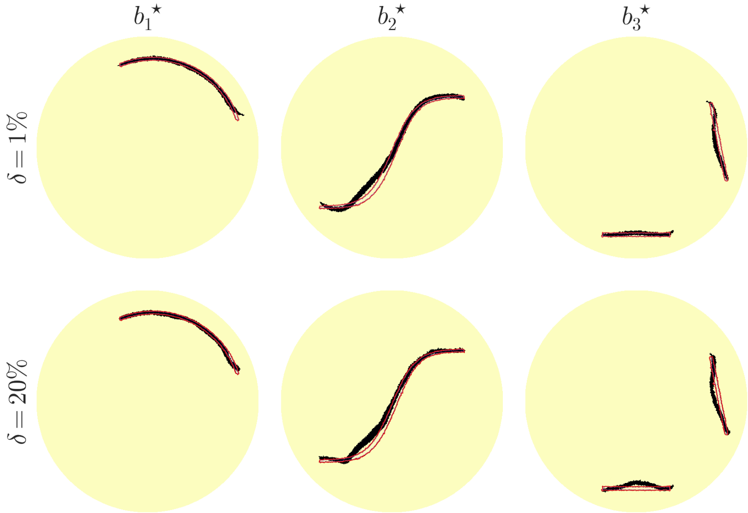

Figure: Reconstructions: Crack scenarios $b_1^\star$, $b_2^\star$, $b_3^\star$ divided by columns. (TOP ROW) noise level $\delta = 1\%$. (BOTOM ROW) noise level $\delta = 20\%$.

Crack scenarios $b_1^\star, b_2^\star, b_3^\star$ with $\delta = 1\%$ & $\delta = 20\%$

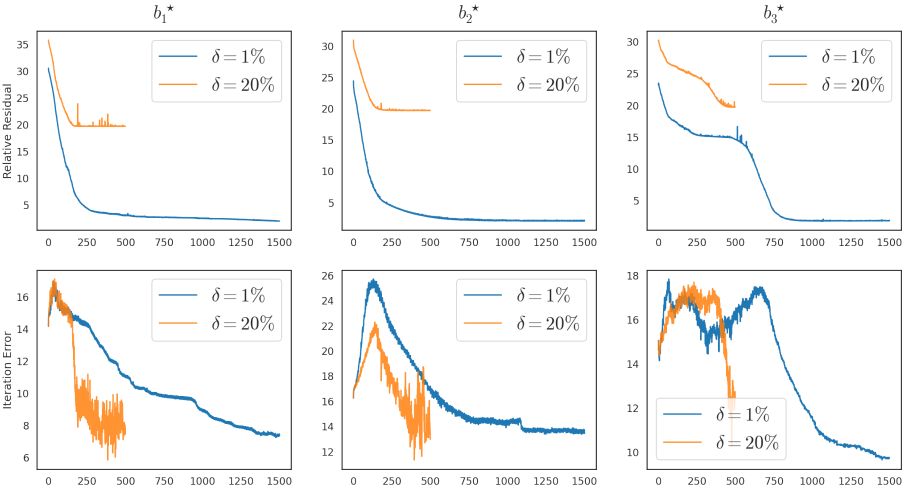

Figure: Residual/Error: Crack scenarios $b_1^\star$, $b_2^\star$, $b_3^\star$ divided by columns. (TOP ROW) Relative residual. (BOTTOM ROW) Relative iteration error.

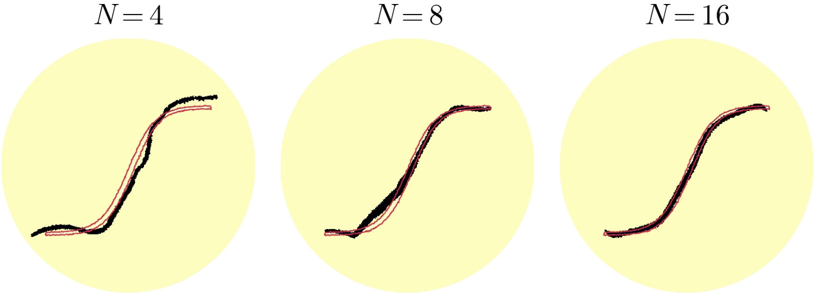

Figure: Crack scenario $b_2^\star$ revisited: Noise level $\delta = 20\%$. Reconstruction results for distinct values of N.

- Current patterns:

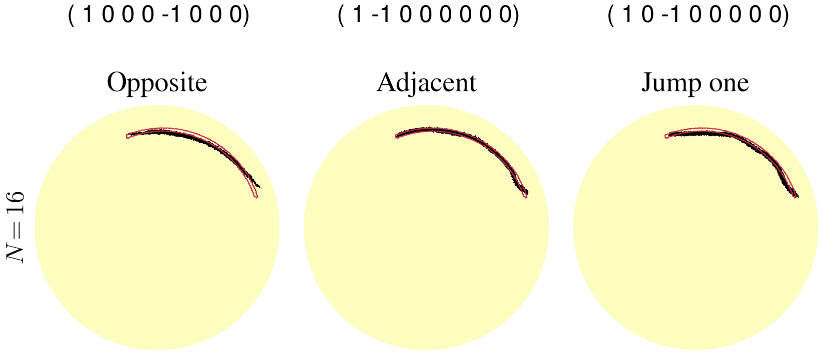

Figure: Crack scenario $b_1^\star$ revisited: $\delta = 20\%$ and $N = 16$. Reconstruction results for distinct current patterns.

Outline

(A1) $\Omega \subseteq \mathbb{R}^2$ is bounded with piecewise C^1 boundary $\partial\Omega$.

(A2) System (6) has a solution, i.e. there exists $b^* \in \mathcal{U}$ s.t. $F_j(b^*) = \gamma_j$, $j = 1, \ldots, N$.

There exist functions φ^*, $\psi^* \in H^1(\Omega)$ satisfying $P(\varphi^*, \psi^*) = b^*$, with $|\nabla\varphi^*| \neq 0$, $|\nabla\psi^*| \neq 0$ in a neighborhood of $\{\varphi^* \in [-\beta/2, \beta/2]\}$, $\{\psi^* = 0\}$ respectively.

It holds $H(\varphi^* + \beta/2) = z^1$, $H(\varphi^* - \beta/2) = z^2$, $H(\psi^*) = z^3$, for some z^1, z^2, $z^3 \in \mathcal{V}_{0,1}$.

- Continuous approximations to the operators P and H. Given $\varepsilon > 0$, define

$$P_\varepsilon(\varphi, \psi) \;:=\; (b_i - b_e)[H_\varepsilon(\varphi) - H_{\beta,\varepsilon}(\varphi)]\, H_\varepsilon(\psi) + b_e\ , \qquad (8)$$

and

$$H_\varepsilon(\phi) := \begin{cases} 0, & \text{if } \phi < -\varepsilon \\ 1 + \frac{\phi}{\varepsilon}, & \text{if } \phi \in [-\varepsilon, 0] \\ 1, & \text{if } \phi > 0. \end{cases}$$

(the operators $H_{\beta,\varepsilon}$ are defined analogously)

Definition (Generalized minimizers)

a) A tuple of functions $(z^1, z^2, z^3, \varphi, \psi) \in (L^\infty(\Omega))^3 \times (H^1(\Omega))^2$ *is called* **admissible** *if there exist sequences* $\{\varphi_k\}_{k\in\mathbb{N}}$, $\{\psi_k\}_{k\in\mathbb{N}}$ *in* $H^1(\Omega)$, *and a sequence* $\{\varepsilon_k\}_{k\in\mathbb{N}}$ *of positive numbers converging to zero such that*

$$\lim_{k\to\infty} \|\varphi_k - \varphi\|_{L^2(\Omega)} = 0\,, \quad \lim_{k\to\infty} \|\psi_k - \psi\|_{L^2(\Omega)} = 0\,,$$

$$\lim_{k\to\infty} \|H_{\varepsilon_k}(\varphi_k) - z^1\|_{L^1} = \lim_{k\to\infty} \|H_{\beta,\varepsilon_k}(\varphi_k) - z^2\|_{L^1} = \lim_{k\to\infty} \|H_{\varepsilon_k}(\psi_k) - z^3\|_{L^1} = 0.$$

b) A minimizer of $\hat{\mathcal{G}}_\alpha$ *is considered to be any admissible tuple of the form* $(z^1, z^2, z^3, \varphi, \psi)$ *minimizing*

$$\hat{\mathcal{G}}_\alpha(z^1, z^2, z^3, \varphi, \psi) := \sum_{j=1}^{N} \|F_j(q(z^1, z^2, z^3)) - \gamma_j^\delta\|_Y^2 + \alpha\rho(z^1, z^2, z^3, \varphi, \psi) \tag{9}$$

over all admissible tuples.

Definition (continuation)

Here the functional ρ *is defined by*

$$\rho(z^1,z^2,z^3,\varphi,\psi) := \inf\Big\{ \liminf_{k\to\infty}\Big(\mu_1|H_{\varepsilon_k}(\varphi_k+\tfrac{\beta}{2})|_{\mathrm{BV}}+\mu_2|H_{\varepsilon_k}(\varphi_k-\tfrac{\beta}{2})|_{\mathrm{BV}} + \mu_3|H_{\varepsilon_k}(\psi_k)|_{\mathrm{BV}}+\mu_4\|\varphi_k-\varphi_0\|^2_{H^1}+\mu_5\|\psi_k-\psi_0\|^2_{H^1}\Big)\Big\}, \quad (10)$$

where the infimum is taken with respect to all sequences $\{\varepsilon_k\}$ *and* $\{(\varphi_k,\psi_k)\}$ *satisfying (a).*

c) A generalized minimizer of $\mathcal{G}_\alpha(\varphi,\psi)$ *is a minimizer of* $\hat{\mathcal{G}}_\alpha(z^1,z^2,z^3,\varphi,\psi)$ *on the set of admissible tuples.*

Lemma (Closedness of the set of admissible tuples)

Let $(z_k^1,z_k^2,z_k^3,\varphi_k,\psi_k)$ *be a sequence of admissible tuples converging in* $(L^1(\Omega))^3\times(L^2(\Omega))^2$ *to some* $(z^1,z^2,z^3,\varphi,\psi)\in(L^\infty(\Omega))^3\times(H^1(\Omega))^2$*. Then* $(z^1,z^2,z^3,\varphi,\psi)$ *is an admissible tuple.*

Lemma (coercivity and l.s.c. of ρ on the set of admissible tuples)

For each admissible quintuple $(z^1, z^2, z^3, \varphi, \psi)$*, we have*

$$\sum_{i=1}^{3} \mu_i |z^i|_{\mathrm{BV}} + \mu_4 \|\varphi - \varphi_0\|_{H^1}^2 + \mu_5 \|\psi - \psi_0\|_{H^1}^2 \;\leq\; \rho(z^1, z^2, z^3, \varphi, \psi)\,. \qquad (11)$$

Moreover, given a sequence $\{(z_k^1, z_k^2, z_k^3, \varphi_k, \psi_k)\}_{k\in\mathbb{N}}$ *of admissible tuples such that* $z_k^i \to z^i$ *in* $L^1(\Omega)$*,* $\varphi_k \rightharpoonup \varphi$ *in* $H^1(\Omega)$*,* $\psi_k \rightharpoonup \psi$ *in* $H^1(\Omega)$*, where* $(z^1, z^2, z^3, \varphi, \psi)$ *is some admissible tuple, then*

$$\rho(z^1, z^2, z^3, \varphi, \psi) \;\leq\; \liminf_{k\in\mathbb{N}} \rho(z_k^1, z_k^2, z_k^3, \varphi_k, \psi_k)$$

(i.e., ρ *is weak-lower semi-continuous)*

Proposition (Regularity property of the operators F_j)

Let the boundary data in the BVP (1) *satisfy* $\eta_j \in (W^{1-1/q,q}(\partial\Omega))'$, *for* $q = p/(p-1)$, *for any* $p \in (2, p_0)$.
Then, the operators $F_j : D(F) \subset L^1(\Omega) \to Y$ *are continuous on* $D(F)$ *with respect to the* $L^1(\Omega)$*-topology.*

Theorem (Well-Posedness of the functionals $\mathcal{G}_\alpha$)

The functional $\mathcal{G}_\alpha$ *in* (7) *attains generalized minimizers on the set of admissible tuples.*

Sketch of the proof. First, notice that the set of admissible tuples is not empty. Given a minimizing sequence of admissible quintuples for $\hat{\mathcal{G}}_\alpha$, it follows from the coercivity of ρ, the Sobolev compact embedding (of H^1 in L^2) and the compact embedding of BV into L^1, that this minimizing sequence converges to some tuple which is admissible (due to Lemma 1).
From the weak lower semi-continuity of ρ together with the cont. of F_j and the cont. of q, we conclude that the limit tuple is a minimizer of $\hat{\mathcal{G}}_\alpha$. □

- Main convergence and stability results.

Theorem (Convergence for exact data)

Assume that we have exact data, i.e. $\gamma_j^\delta = \gamma_j$. *For every* $\alpha > 0$, *let* $(z^1_\alpha, z^2_\alpha, z^3_\alpha, \varphi_\alpha, \psi_\alpha)$ *denote a minimizer of* $\hat{\mathcal{G}}_\alpha$ *on the set of admissible tuples. Then, for every sequence of positive numbers* $\{\alpha_k\}_{k\in\mathbb{N}}$ *converging to zero there exists a subsequence, denoted again by* $\{\alpha_k\}_{l\in\mathbb{N}}$, *such that* $(z^1_{\alpha_k}, z^2_{\alpha_k}, z^3_{\alpha_k}, \phi^1_{\alpha_k}, \phi^2_{\alpha_k})$ *is strongly convergent in* $(L^1(\Omega))^3 \times (L^2(\Omega))^2$. *Moreover, the limit is a solution of* (6).

Theorem (Convergence for noisy data)

Let $\alpha = \alpha(\delta)$ *be a function satisfying* $\lim_{\delta\to 0} \alpha(\delta) = 0$ *and* $\lim_{\delta\to 0} \delta^2 \alpha(\delta)^{-1} = 0$. *Moreover, let* $\{\delta_k\}_{k\in\mathbb{N}}$ *be a sequence of positive numbers converging to zero and* $\gamma^{\delta_k} \in Y$ *be corresponding noisy data satisfying* (**??**). *Then, there exist a subsequence (denoted again by* $\{\delta_k\}$*) and a sequence* $\{\alpha_k := \alpha(\delta_k)\}$ *such that* $(z^1_{\alpha_k}, z^2_{\alpha_k}, z^3_{\alpha_k}, \varphi_{\alpha_k}, \psi_{\alpha_k})$ *converges in* $(L^1(\Omega))^3 \times (L^2(\Omega))^2$ *to solution of* (6).

- Consider the smoothed Tikhonov functional

$$\mathcal{G}_{\varepsilon,\alpha}(\varphi,\psi) := \sum_{j=1}^{N} \| F_j(P_\varepsilon(\varphi,\psi)) - y_j^\delta \|_Y^2 + \alpha \Big\{ |H_\varepsilon(\varphi)|_{\mathrm{BV}} + |H_{\beta,\varepsilon}(\varphi)|_{\mathrm{BV}} + |H_\varepsilon(\psi)|_{\mathrm{BV}} + \|\varphi - \varphi_0\|_{H^1(\Omega)}^2 + \|\psi - \psi_0\|_{H^1(\Omega)}^2 \Big\}, \quad (12)$$

where $P_\varepsilon(\varphi,\psi) := q(H_\varepsilon(\varphi), H_{\beta,\varepsilon}(\varphi), H_\varepsilon(\psi))$.

Lemma (Well posedness of $\mathcal{G}_{\varepsilon,\alpha}$)

Given α, $\varepsilon > 0$ and $\varphi_0, \psi_0 \in H^1(\Omega)$, the functional $\mathcal{G}_{\varepsilon,\alpha}$ in (12) attains a minimizer on $(H^1(\Omega))^2$.

Theorem (Relation between minimizers of $\mathcal{G}_\alpha$ and $\mathcal{G}_{\varepsilon,\alpha}$)

Let $\alpha > 0$ be given. For each $\varepsilon > 0$ denote by $(\varphi_{\varepsilon_k,\alpha}, \psi_{\varepsilon_k,\alpha})$ a minimizer of $\mathcal{G}_{\varepsilon,\alpha}$. There exists a sequence of positive numbers $\{\varepsilon_k\}$ converging to zero such that $(H_{\varepsilon_k}(\varphi_{\varepsilon_k,\alpha}), H_{\varepsilon_k}(\psi\varepsilon_k,\alpha), H_{\beta,\varepsilon_k}(\psi\varepsilon_k,\alpha), \varphi_{\varepsilon_k,\alpha}, \psi_{\varepsilon_k,\alpha})$ converges strongly in $(L^1(\Omega))^3 \times (L^2(\Omega))^2$ and the limit minimizes $\hat{\mathcal{G}}_\alpha$ in the set of admissible 4-tuples.

- Conditions of optimality for $\mathcal{G}_{\varepsilon,\alpha}$

$$\alpha(\Delta - I)(\varphi_{k+1} - \varphi_k) = R^1_{\varepsilon,\alpha}(\varphi_k, \psi_k)\,, \text{ in } \Omega \tag{13a}$$
$$(\varphi_{k+1} - \varphi_k)_\nu = 0\,, \text{ at } \partial\Omega \tag{13b}$$
$$\alpha(\Delta - I)(\psi_{k+1} - \psi_k) = R^2_{\varepsilon,\alpha}(\varphi_k, \psi_k)\,, \text{ in } \Omega \tag{13c}$$
$$(\psi_{k+1} - \psi_k)_\nu = 0\,, \text{ at } \partial\Omega \tag{13d}$$

where

$$R^1_{\varepsilon,\alpha}(\varphi,\psi) = \Theta^1_\varepsilon\; F_j'(P_\varepsilon(\varphi,\psi))^*\;(F_j(P_\varepsilon(\varphi,\psi)) - y_j^\delta) + \; + \textit{terms related to } |\nabla H_\varepsilon(\varphi)|_{BV}\,,\; |\nabla H_{\beta,\varepsilon}(\varphi)|_{BV}\,, \tag{14a}$$
$$R^2_{\varepsilon,\alpha}(\varphi,\psi) = \Theta^2_\varepsilon\; F_j'(P_\varepsilon(\varphi,\psi))^*\;(F_j(P_\varepsilon(\varphi,\psi)) - y_j^\delta) + \; + \textit{terms related to } |\nabla H_\varepsilon(\psi)|_{BV}\,, \tag{14b}$$

and

$$\Theta^1_\varepsilon(\varphi,\psi) = (b_i - b_e)\; H_\varepsilon(\psi)\;[H_\varepsilon'(\varphi) - H_{\beta,\varepsilon}'(\varphi)]\,, \tag{15a}$$
$$\Theta^2_\varepsilon(\varphi,\psi) = (b_i - b_e)\;[H_\varepsilon(\varphi) - H_{\beta,\varepsilon}(\varphi)]\; H_\varepsilon'(\psi)\,. \tag{15b}$$

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The end

Thank You !!!