Phase and absorption contrast imaging using intensity measurements

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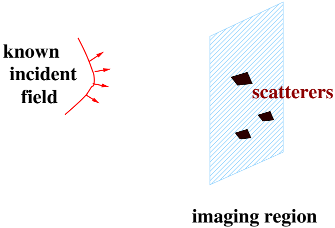
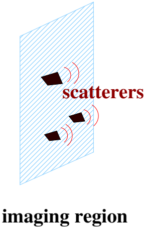
Inverse problems
Wave propagation



- Inverse problems aim to reconstruct a medium characteristics from knowledge of the response of the medium to a known incident field.
- In this talk we seek to reconstruct the transmisivity by recording the medium's response to one or more known excitations.

Inverse problems
Wave propagation



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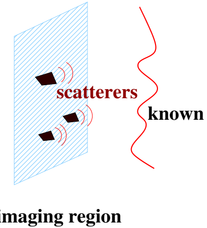

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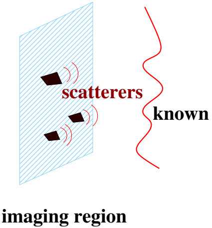

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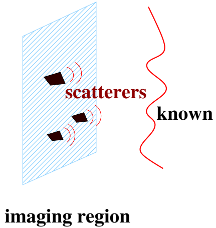

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- Measurements : intensity-only.

Applications

At high frequencies intensities only can be recorded
e.g., CCD's, light detectors can record only intensities

- Optics
- Digital microscopy
- X-ray crystallography

We have developed a computational imaging approach that allows for phase and absorption contrast recovery from intensity measurements.

Multiple illuminations are needed (usual in phase retrieval ; masks).

The keystone for the efficiency of the method is a *robust dimensionality reduction* strategy carried in two steps accounting for both the incoherent (absorption contrast) and coherent contributions (phase contrast) in the data.

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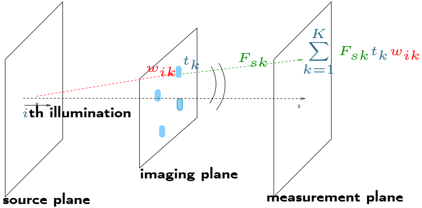
Imaging problem setup

We seek the transmissivity vector

$$\boldsymbol{t} = [t_1, \ldots, t_K]^\intercal = [|t_1|e^{i\varphi_1}, \ldots, |t_k|e^{i\varphi_K}]^\intercal \in \mathbb{C}^K$$

from intensity measurements of the form



$$\begin{aligned} |(\boldsymbol{b}_i)_s|^2 &= \left| \sum_{k=1}^{K} F_{sk} t_k w_{ik} \right|^2 \\ &= \sum_{k=1}^{K} |w_{ik}|^2 |t_k|^2 + \sum_{k=1}^{K} \sum_{\substack{k'=1 \\ k' \neq k}}^{K} F_{sk} F_{sk'}^* w_{ik} w_{ik'}^* t_k t_{k'}^* \end{aligned}$$

$|(\boldsymbol{b}_i)_s|^2$ is the intensity recorded at the s-th transducer when the ith illumination

$$\boldsymbol{w}_i = [w_{i1}, \ldots, w_{iK}]^\intercal \in \mathbb{C}^K$$

impinges on the object plane. F_{sk} is the propagator (DFT) from the object plane to the receiver plane. F and $\boldsymbol{w}_i$ are assumed known.

Imaging problem setup

This problem can be written in matrix form as

$$\mathcal{W}_{incoh}\ \boldsymbol{\chi}_d + \mathcal{W}_{coh}\ \boldsymbol{\chi}_{cross} = \boldsymbol{d}$$

The data are

$$\boldsymbol{d} = [\boldsymbol{d}_1^T, \boldsymbol{d}_2^T, \ldots, \boldsymbol{d}_S^T]^T$$

with $\boldsymbol{d}_s = [|(\boldsymbol{b}_1)_s|^2, |(\boldsymbol{b}_2)_s|^2, \ldots, |(\boldsymbol{b}_N)_s|^2]^T$ the intensities recorded at the detector s for the illuminations $1, 2, \ldots, N$.

The unknown is decomposed into

$$\boldsymbol{\chi}_d = [|t_1|^2, |t_2|^2, \ldots, |t_k|^2]^T$$

and

$$\boldsymbol{\chi}_{cross} = [t_1 t_2^*, t_1 t_3^*, \ldots, t_1 t_K^*, t_2 t_1^*, t_2 t_3^*, \ldots, t_2 t_K^*, t_3 t_1^*, \ldots,]\,,$$

The bottleneck for the inversion is the size of the problem, which is enormous if one wants to form high resolution images. An image with 1000×1000 pixels, amounts to solving a linear system with 10^{12} unknowns !

Imaging problem setup

This problem is nonlinear and there is much interest in finding algorithms that give the true global solution effectively.

Iterative projection methods

R.W. Gerchberg and W.O. Saxton, *A practical algorithm for the determination of phase from image and diffraction plane pictures*, Optik 35, 237-246 (1972).

J.R. Fienup, *Reconstruction of an object from the modulus of its Fourier transform*, Optics Letters 3, 27-29 (1978).

simple to implement & very flexible in practice

do not always converge to the true solution unless good prior information is available.

Imaging problem setup

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Quadratic methods seek for the matrix unknown tt^* using nuclear norm minimization

A. Chai, M. Moscoso and G. Papanicolaou, *Array imaging using intensity-only measurements*, Inverse Problems 27 (2011), 015005.

E. J. Candès, Y. C. Eldar, T. Strohmer, and V. Voroninski, *Phase Retrieval via Matrix Completion*, SIAM J. on Imaging Sci. 6 (2013), 199-225.

convex problem $\rightsquigarrow$ convergence to the true solution

computational complexity limits the usefulness of this approach

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The noise collector and dimension reduction

We propose the following robust dimensionality reduction strategy.

Instead of solving the problem with K^2 unknowns we reduce its dimensionality constructing linear problems for only $O(K)$ unknowns & absorb the error, that is the *contribution of the unmodeled unknowns* using a *Noise Collector*.

We solve linear systems of the form

$$\mathcal{A}\chi + \mathcal{C}\eta = \boldsymbol{d}$$

- $\mathcal{A}$ a matrix with $O(K)$ subsampled columns of $[\mathcal{W}_{incoh}|\mathcal{W}_{coh}]$.
- χ is a sparse vector that represents the object
- η is a vector with no physical meaning that absorbs the noise
- $\mathcal{C}$ is a *Noise Collector* matrix.

The Noise collector

The Noise Collector is a method that allows us to find the sparse solution of

$$\mathcal{A}\boldsymbol{\chi} = \boldsymbol{d}(= \boldsymbol{d}_0 + \boldsymbol{e})$$

from highly incomplete ($1 \ll \mathcal{N} < \mathcal{K}$) and noisy data $\boldsymbol{d} \in \mathbb{C}^{\mathcal{N}}$ (noise $\boldsymbol{e} \in \mathbb{C}^{\mathcal{N}}$).
Main result : The support of $\boldsymbol{\chi}_\tau$ found as

$$(\boldsymbol{\chi}_\tau, \boldsymbol{\eta}_\tau) = \arg\min_{\boldsymbol{\chi},\boldsymbol{\eta}} \left(\tau\|\boldsymbol{\chi}\|_{\ell_1} + \|\boldsymbol{\eta}\|_{\ell_1}\right), \quad \text{subject to } \mathcal{A}\boldsymbol{\chi} + \mathcal{C}\boldsymbol{\eta} = \boldsymbol{d}$$

is exact when the noise is not too large.

$\mathcal{C}$ is the *Noise Collector* matrix $\mathcal{C} \in \mathbb{C}^{\mathcal{N}\times\Sigma}$, $\Sigma = \mathcal{N}^\beta$, for $\beta > 1$ and τ is an $O(1)$ no-phantom weight that is independent of the dimension of the problem and the level of noise in the data.

This minimization problem can be understood as a relaxation of

$$\boldsymbol{\chi}_* = \arg\min_{\boldsymbol{\chi}} \|\boldsymbol{\chi}\|_{\ell_1}, \text{ subject to } \mathcal{A}\boldsymbol{\chi} = \boldsymbol{d},$$

It works by absorbing *all* the noise, and possibly some signal, in $\mathcal{C}\boldsymbol{\eta}_\tau$.

$\boldsymbol{\eta}$ does not correspond to a physical quantity. It is introduced to provide an appropriate linear combination of the columns of $\mathcal{C}$ that produces a good approximation to the noise vector $\boldsymbol{e}$.

The Noise collector

- The columns of $\mathcal{C}$ are chosen independently and at random on the unit sphere $\mathbb{S}^{\mathcal{N}-1}$ so that we could approximate well a typical noise vector.

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- The main result is obtained under the assumption that the columns of $\mathcal{A}$ are incoherent,

$$|\langle \boldsymbol{a}_i, \boldsymbol{a}_j \rangle| \leqslant \frac{1}{3M} \text{ for all } i \text{ and } j,$$

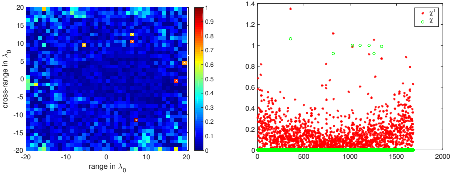
  and that the noise is not too large

$$\max\left(1, \|\boldsymbol{e}\|_{\ell_2}\right) \leqslant c_1 \frac{\|\boldsymbol{d}_0\|^2_{\ell_2}}{\|\boldsymbol{\chi}\|_{\ell_1}} \sqrt{\frac{\mathcal{N}}{\ln \mathcal{N}}},$$

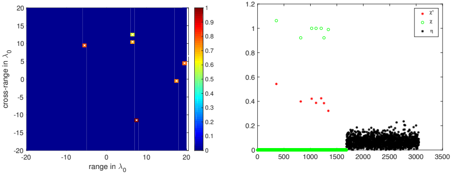
The Noise collector

Noise Collector at work

ℓ_1 reconstruction without the Noise Collector



ℓ_1 reconstruction with the Noise Collector



$N = 1369$ measurements. $K = 1681$ pixels in the images. 100% noise.

The Noise Collector allows for exact support recovery !

The algorithm

The algorithm has three steps

(1) In the first step, we seek the strong absorbing objects. We set $\mathcal{A} = \mathcal{W}_{incoh}$, and solve

$$\mathcal{A}\boldsymbol{\chi} + \mathcal{C}\boldsymbol{\eta} = \boldsymbol{d} \, (= \mathcal{W}_{incoh}\boldsymbol{\chi}_d + \mathcal{W}_{coh}\boldsymbol{\chi}_{cross}),$$

for $\boldsymbol{\chi} = \boldsymbol{\chi}_d = [|t_1|^2, |t_2|^2, \ldots, |t_k|^2]^T$.

$\mathcal{C}\boldsymbol{\eta}$ absorbs the contributions of χ_{cross} to the intensities which are treated in this step as noise. The model is not exact so only the strong absorbers are detected.

Looking at $|(\boldsymbol{b}_i)_s|^2 = \underbrace{\sum_{k=1}^{K} |w_{ik}|^2 |t_k|^2}_{\text{indep of s}} + \sum_{k=1}^{m} \sum_{\substack{k'=1 \\ k' \neq k}}^{K} F_{sk} F^*_{sk'} w_{ik} w^*_{ik'} t_k t^*_{k'}$

first term is independent of $s \Rightarrow$ use total intensity as data.

Consider m *strong absorbers* $|t_i| = O(1)$, $i = 1, \ldots, m$ *and* n *weak (phase contrast)* $|t_j| = O(\varepsilon)$, $j = 1, \ldots, n$. *During the first step we only recover* $|t_i|^2$, $i = 1, \ldots, m$ *because the contribution from* $|t_j|^2 = O(\varepsilon^2)$ $j = 1, \ldots, n$ *is lost in the noise.*

The algorithm

(2) In the second step :
- We first remove from the data the contributions already found ($O(1)$ *contributions*) what remains is

$$|(\boldsymbol{b}_i)_s|^2 = \underbrace{\sum_{k=1}^{n} |w_{ik}|^2 |t_k|^2}_{O(\varepsilon^2)} + \underbrace{\sum_{k=1}^{m} \sum_{\substack{k'=1 \\ k' \neq k}}^{m} F_{sk} F^*_{sk'} w_{ik} w^*_{ik'} t_k t^*_{k'}}_{O(1)}$$
$$+ \underbrace{\sum_{k=1}^{m} \sum_{\substack{k'=1 \\ k' \neq k}}^{n} F_{sk} F^*_{sk'} w_{ik} w^*_{ik'} t_k t^*_{k'}}_{O(\varepsilon)} + \underbrace{\sum_{k=1}^{n} \sum_{\substack{k'=1 \\ k' \neq k}}^{n} F_{sk} F^*_{sk'} w_{ik} w^*_{ik'} t_k t^*_{k'}}_{O(\varepsilon^2)}$$

- Then for every pixel $i = 1, \ldots, m$ detected during the first step we seek for its interactions $t_i^* t_j$ with all the other $K-1$ pixels in the object plane, $j = 1, \ldots, K$, $j \neq i$ ($O(1)$ *and* $O(\varepsilon)$ *contributions*).

In this case $\mathcal{A} = (\mathcal{W}_{coh})_{sub}$, where $(\mathcal{W}_{coh})_{sub}$ contains the $m\,(K-1)$ columns that correspond to the interactions between the m detected objects in the first step and all the other pixels in the image.

Since we are neglecting the $O(\varepsilon^2)$ *contributions*, the system is not exact.

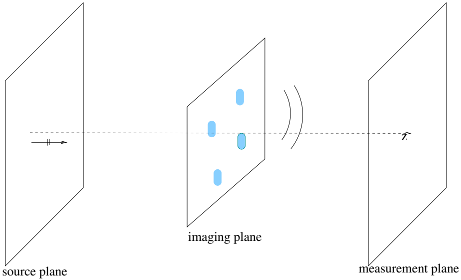
The algorithm

(3) The third step is optional. It is used to obtain more precise quantitative images.

Once the strong and weak absorbing objects are found, we solve the full problem but restricted to the recovered support.

This is now a small problem that can be solved using an ℓ_2 minimization method that gives very accurate results.

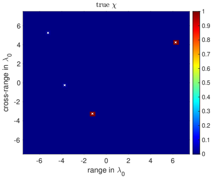
Setup : transmission problem with multiple illuminations



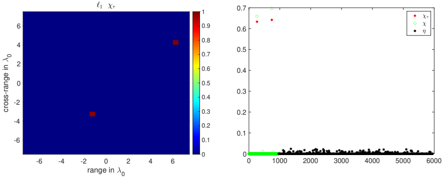
- Wavelength $\lambda = 500$ nm
- Source plane : 21×21 evenly distributed sources on 8mm×8mm at $z = -8$mm.
- $N = 300$ different illumination patterns are used.
- Imaging region 31×31 pixels centered at the origin. Thin object. pixel size $\lambda/2 = 250$nm.
- Measurements sampled on 5×5 receivers located on a 8mm×8mm aperture at $z = +8$mm.

Results

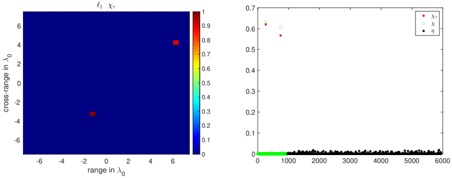
true $|t_k|^2$



ℓ_1 no noise



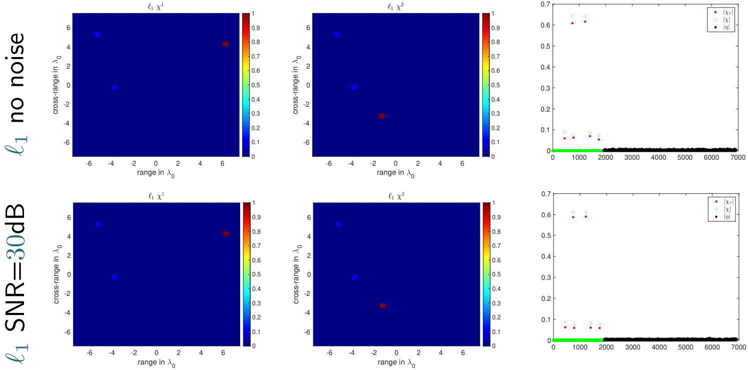
ℓ_1 SNR$=30$ dB



First step. Imaging $M = 4$ absorbers using the total power received.
$N = 300$ illumination patterns.

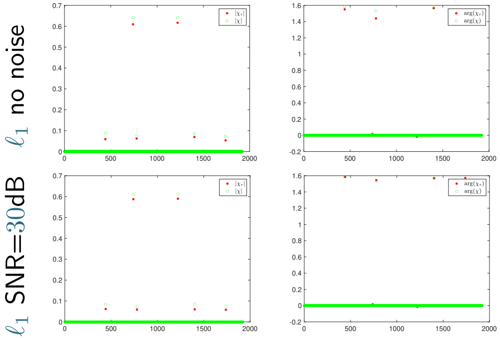
Top : true $|t_k|^2$- two strong (red squares) and two weak (white crosses) absorbers.
Bottom : reconstructions without noise and with additive noise SNR 30dB.

Results



Second step. Imaging $M = 4$ scatterers using intensity measurements over the receiving array. Top row : noiseless data. Bottom row : SNR $= 30$dB.

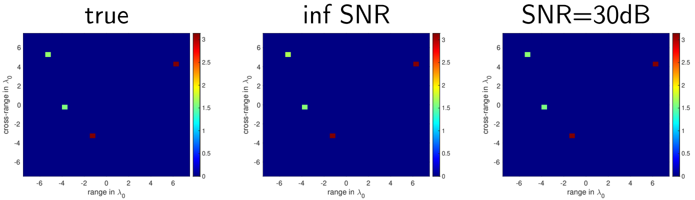
Results



Second step. Top row : noiseless data. Bottom row : SNR $= 30$dB.

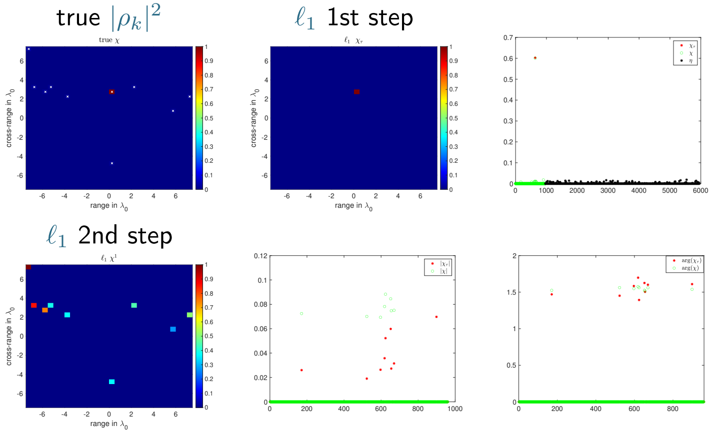
Left column shows the recovered amplitudes ; right column the recovered phases.

Results



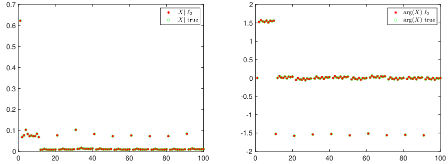
Phase maps. The left plot is the true phase distribution. Middle and right plots are the recovered phase distributions without noise and with noise, respectively.

Results (more challenging case)

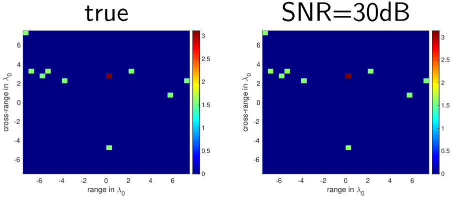


First step and second steps for $M = 10$ scatterers SNR$= 30$dB.

Results (more challenging case)



Third step for the full unknown $X = \rho\rho^*$ restricted to the recovered support.



True and recovered phase maps for the $M = 10$ absorbers.

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Concluding remarks

- We presented a two (*three*) step algorithm for phase retrieval based on a *robust dimensionality reduction* strategy carried in two steps accounting for both the incoherent (absorption contrast) and coherent contributions (phase contrast) in the data.
- The algorithm is efficient because its cost is linear in the number of pixels !
- It guarantees exact recovery if the image is sparse with respect to a given basis.
- May be used, without any modification, for partially coherent data. This is very important for, for example, phase-contrast X-ray imaging because fully coherent sources of X-rays are very hard to be obtained.

Concluding remarks

- More on the Noise Collector and its theoretical analysis in

M. Moscoso, A. Novikov, G. Papanicolaou, CT, *Imaging with highly incomplete and corrupted data*, Inverse Problems, 36(3), p. 035010, 2020. https://doi.org/10.1088/1361-6420/ab5a21

M. Moscoso, A. Novikov, G. Papanicolaou, CT, *The Noise Collector for sparse recovery in high dimensions*, Proceedings of the National Academy of Sciences, 117 (21), p. 11226-11232, 2020. https://doi.org/10.1073/pnas.1913995117

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- More on the Noise Collector for quadratic (cross-correlation) measurements

M. Moscoso, A. Novikov, G. Papanicolaou, CT, *Fast signal recovery from quadratic measurements*, IEEE Transactions on Signal Processing, vol. 69, pp. 2042–2055, 2021. doi:10.1109/TSP.2021.3067140 (deterministic case)

M. Moscoso, A. Novikov, G. Papanicolaou, CT, *Quantitative phase and absorption contrast imaging*, IEEE Transactions on Computational Imaging, vol. 8, pp. 784-794, 2022. doi:10.1109/TCI.2022.3204401.

M. Moscoso, A. Novikov, G. Papanicolaou, CT, *The random case : coming soon*