On the recovery of coefficients in nonlinear wave equations

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The classical wave equation $u_{tt} - \triangle u = 0$ holds in a homogeneous medium where there is no attenuation of the wave: the solution $u(x, t)$ has purely sinusodial behaviour for all t.

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We also must include the effects of damping: classically this is just $\mathcal{D}u = b\,u_t$.
However, this leads to all solutions $u(x,t)$ having exponential decay in time and this situation often does not correspond to observations – a much slower decay rate is indicated.

The problem of imaging with ultrasound in a lossy media amounts to identification of the space dependent coefficient $\kappa(x)$ for the attenuated Westervelt equation in pressure formulation

$$\big(v - \kappa(x)v^2\big)_{tt} - c(x)^2\triangle v + \mathcal{D}v = r \quad \text{in} \quad \Omega\times(0,T)$$

$$v = 0 \text{ on } \partial\Omega\times(0,T)\ ; \qquad v(0) = v_0(x), \quad v_t(0) = v_1(x) \text{ in } \Omega$$

Here Ω is a bounded subset of $\mathbb{R}^n$ (But we focus on $n = 1$ for the inverse problems).

The typical observations are $g(x,t) = u(x,t), \quad x\in\Sigma,\ t\in(\tau,T)$, $\tau \geq 0$.

[These can be purely time trace ($\Sigma = \{x_0\}$) or purely spatial ($\tau = T$)]

Our interest has continued to extend the scope of the inverse problem to seek further information

- The wave speed $c(x)$ may also be unknown: $\mathcal{L}u = c\triangle u$ or indeed a more general elliptic operator. We may be required to recover both $\kappa(x)$ and $c(x)$.
  Reformulation to include a ‘slowness” term $s = 1/c^2$

$$\big(s(x)u - \tilde{\kappa}(x)u^2\big)_{tt} - \triangle u + \mathcal{D}v = 0$$

- In the literature the Damping operator $\mathcal{D}$ has taken many forms:
  - $b\,\mathcal{L}\partial_t^\alpha$, $b\,\mathcal{L}^\beta\,\partial_t^\alpha$, $\sum_i^N b_i\partial_t^{\alpha_i}$. Here ∂_t^α is a fractional derivative in time of Abel type and Djrbashian-Caputo form. $\mathcal{L}^\beta$ is a “fractional Laplacian” operator.
  - The key point is that fractional space derivatives lead to solutions with only **power law** time decay – courtesy of the modification $e^{-\lambda t} \to E_\alpha(-\lambda t^\alpha)$.
    Critical fact: the Mittag-Leffler function $E_\alpha(-x)$ decays *linearly* for all $x > 0$.
- The nonlinearity may be more complex: $\big(v - \kappa(x)f(v)\big)_{tt}$
  and the inverse problem is to recover f (in addition to . . . .)

Overposed measurements to obtain these unknowns will be

- $g(x,t) = v(x,T)\,,\ x \in \Sigma \subset \Omega\,, \quad t \in (0,T).$ either at single point $\Sigma = \{x_0\}$ or, in the spatially higher dimensional case – on some surface Σ contained in $\overline{\Omega}$.
- Time trace data, $h(x_0,t)$ for a fixed point $x_0 \in \Omega$ or x_0 on $\partial\Omega$, $t > 0$.
    - An important subcase is when only large time measurements are possible $t > T$.

Existence and Uniqueness of the forwards operator

Assume $\Omega \subseteq \mathbb{R}^d$, $d \in \{1, 2, 3\}$, $\partial\Omega \in C^{2,\sigma}$.

Let $U := L^\infty(0, T; H_0^1(\Omega) \cap H^2(\Omega)) \cap H^1(0, T; H_0^1(\Omega)) \cap C([0, T]$.

Let κ, $c(x) \in L^\infty(\Omega)$ and $\alpha \in (0, 1)$.

Suppose the initial conditions $u_0 \in H_0^1(\Omega) \cap H^3(\Omega)$, $u_1 \in H^1(\Omega)$ and the forcing function $r \in (H^1(0, T; L^2(\Omega)))^*$.

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Theorem . *There exists a unique solution $u \in U$ of the Westervelt equation for some fixed τ, $0 < \tau < T$.*

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Theorem . *Take any fixed $\overline{\tau} < 1$, $\underline{\tau} \geq 0$, $M > 0$ and $f \in C^{0,1}([-M,M])$ such that $-\underline{\tau} \leq f \leq \overline{\tau}$ on $[-M,M]$. Then there exists a unique solution $u \in U$ of the $f(u)$ version of the general nonlinear Westervelt equation.*

The inverse problems represented by the (generalised) Westervelt equation are challenging on (at least) three counts.

- First, the underlying equation is nonlinear and the nonlinearity is in the highest order term.
- Second, the unknown coefficient $\kappa(x)$ is directly coupled to this term.
- Third, time-trace data $g(t)$ is in the “orthogonal” time direction and is well known to lead to severe ill-conditioning of the inversion of the map: data $(t) \mapsto$ unknown (x)

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We consider the maps $G : \kappa \to u$ where u solves the Westervelt equation and its linearisation $z = G'(\kappa)\delta\kappa$

$$(1-2\kappa u)z_{tt} + c^2\mathcal{L}z + Dz - 4\kappa u_t\, z_t - 2\kappa u_{tt}\, z = 2\delta\kappa(u\, u_{tt} + u_t^2) \quad \text{in } \Omega \times (0,T)$$

$$z(0) = 0, \qquad z_t(0) = 0 \quad \text{in } \Omega$$

for a given κ and $\delta\kappa$. Here $\mathcal{L} = -\frac{c^2(x)}{c_0^2}\triangle$ subject to homogeneous conditions on $\partial\Omega$, $c(x) \in L^{\infty}(\Omega)$. We denote the spectrum of $\mathcal{L}$ by $\{\lambda_n\}$.

We now look at the Laplace transformed solutions of the linearised equation:

$$\hat{w}(\lambda, s) = \frac{1}{\omega(\lambda, s)} \quad \text{with} \quad \omega(\lambda, s) = s^2 + b\lambda^\beta s^\alpha + c^2\lambda$$

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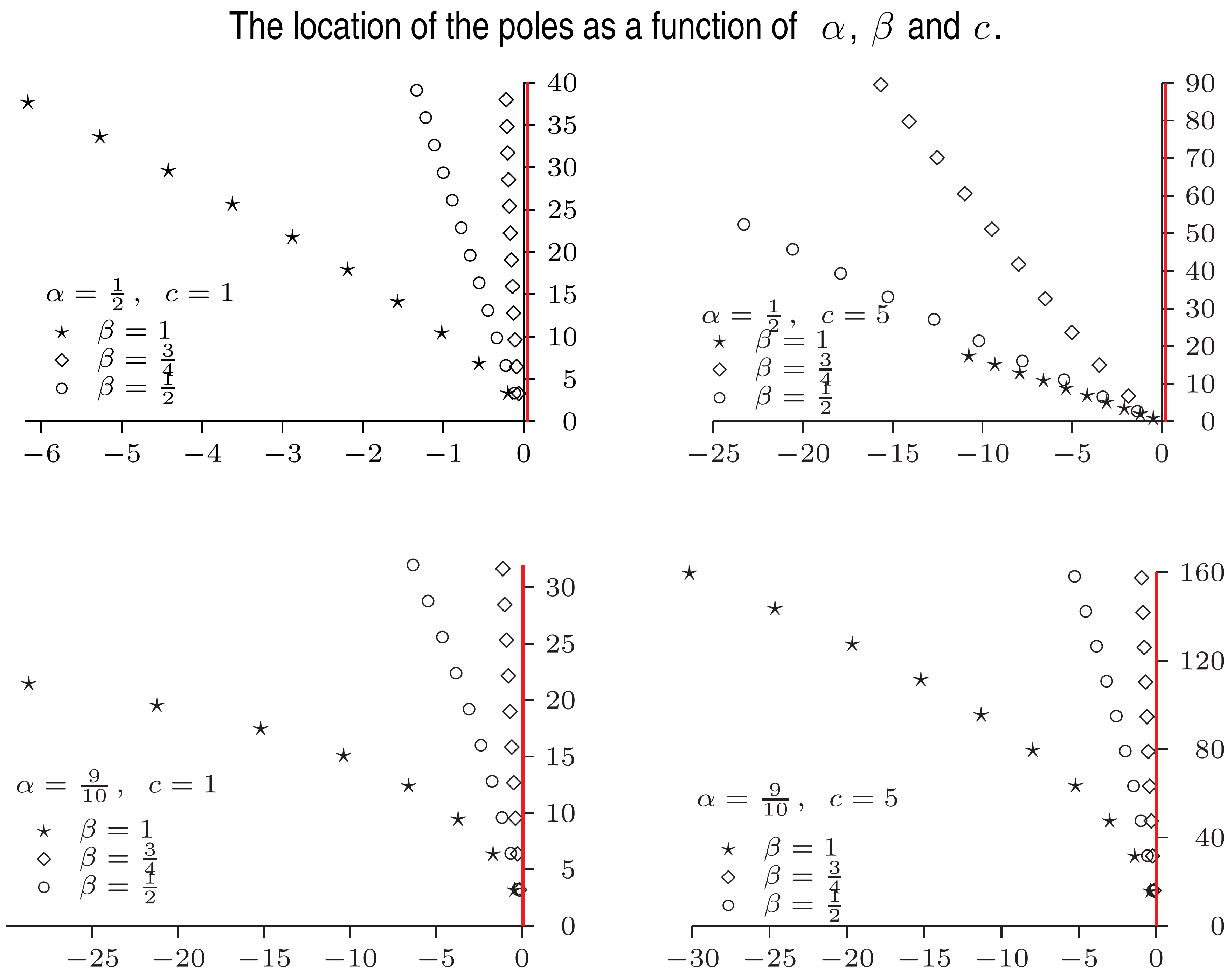
Lemma: For fixed λ, the function $\omega(\lambda, s)$ has precisely two complex-conjugate roots lying in the left hand complex plane and for $\lambda \neq \tilde{\lambda}$ the roots differ and the poles are single.

The key point:

- ♠ We are going to convert time values into $\omega(s)$ values for s real and positive.
- ♠ Computing the poles/residues from this information is analytic continuation.
- ♠ The further the poles lie from the measured values the greater the ill-conditioning.



The location of the poles as a function of α, β and c.
$\alpha = \frac{1}{2}$, $c = 1$
$\beta = 1$
$\beta = \frac{3}{4}$
$\beta = \frac{1}{2}$
$\alpha = \frac{1}{2}$, $c = 5$
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Newton's method

We define the iterate κ_{k+1} implicitly by the linearised problem

$$F'(\kappa_k)(\kappa_{k+1} - \kappa_k) = g - F(\kappa_k)\,,$$

or its frozen version

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Theorem. The linearised derivative of the map F taking κ to the time trace $u(:,t)$ on the measurement surface at $\kappa = 0$ is injective.

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The problem is severely ill-conditioned and will rely on a regularized least squares variant

$$\kappa_{k+1} = \text{argmin}_{\tilde{\kappa} \in \mathcal{D}} \| F(\kappa_k) + F'(\kappa_0)(\tilde{\kappa} - \kappa_k) - g \|_Y + \gamma \| \tilde{\kappa} - \kappa_0 \|_X$$

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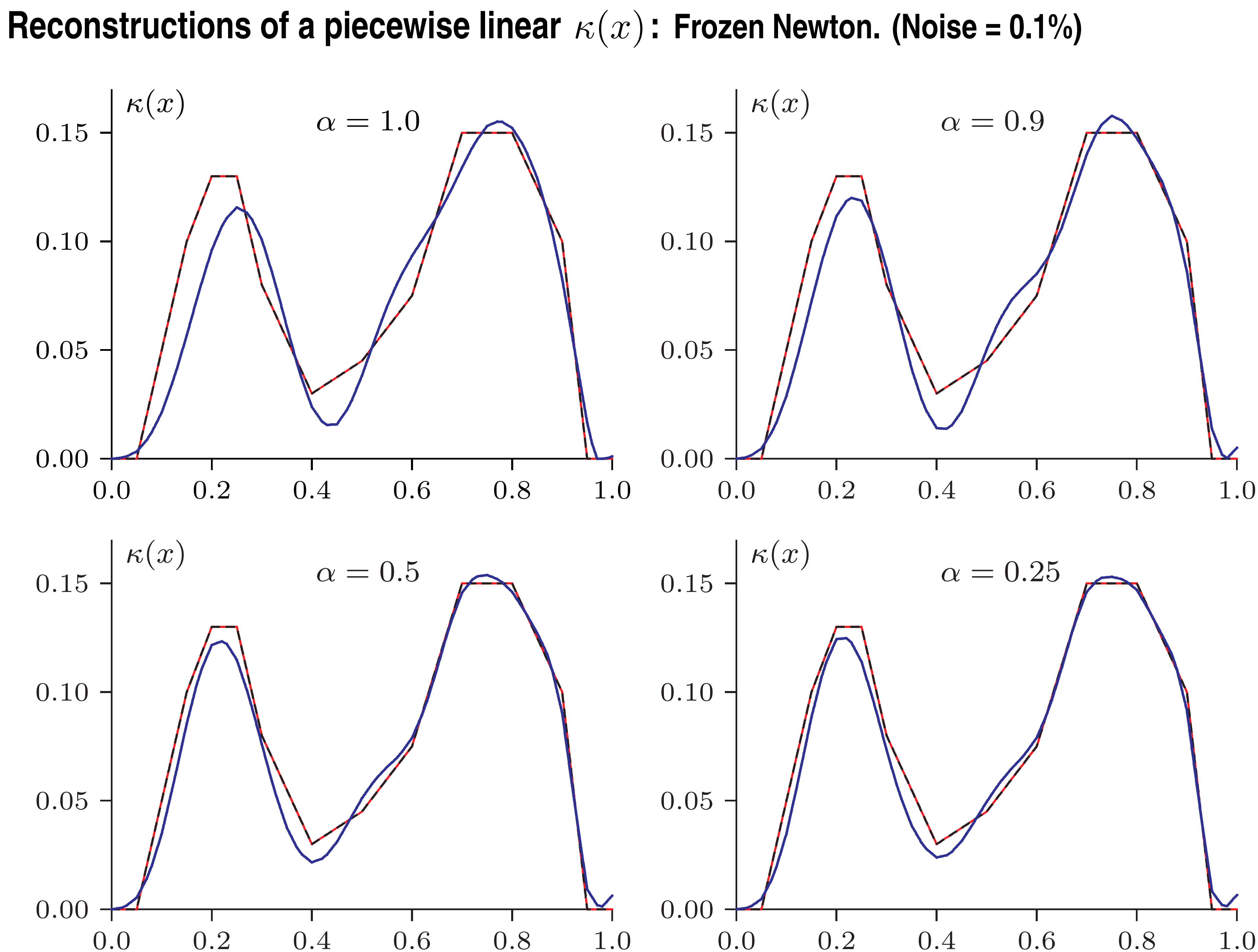
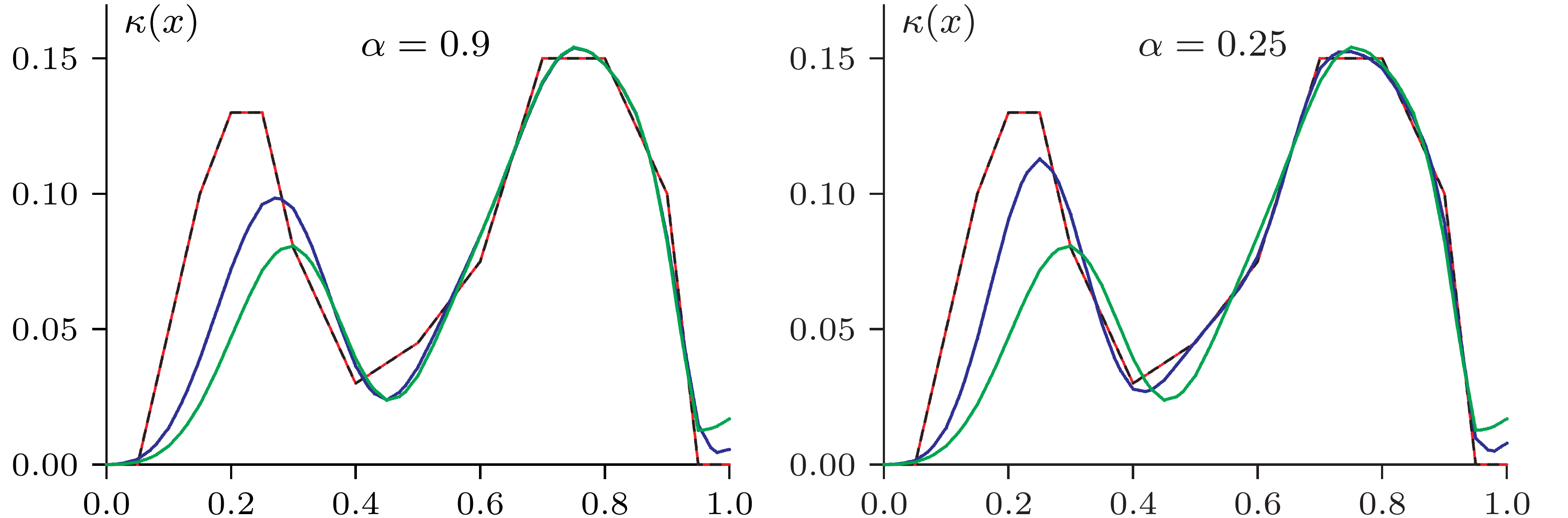
In the pictures to follow boundary conditions were $u(0) = 0$ and $u'(1) = 0$.

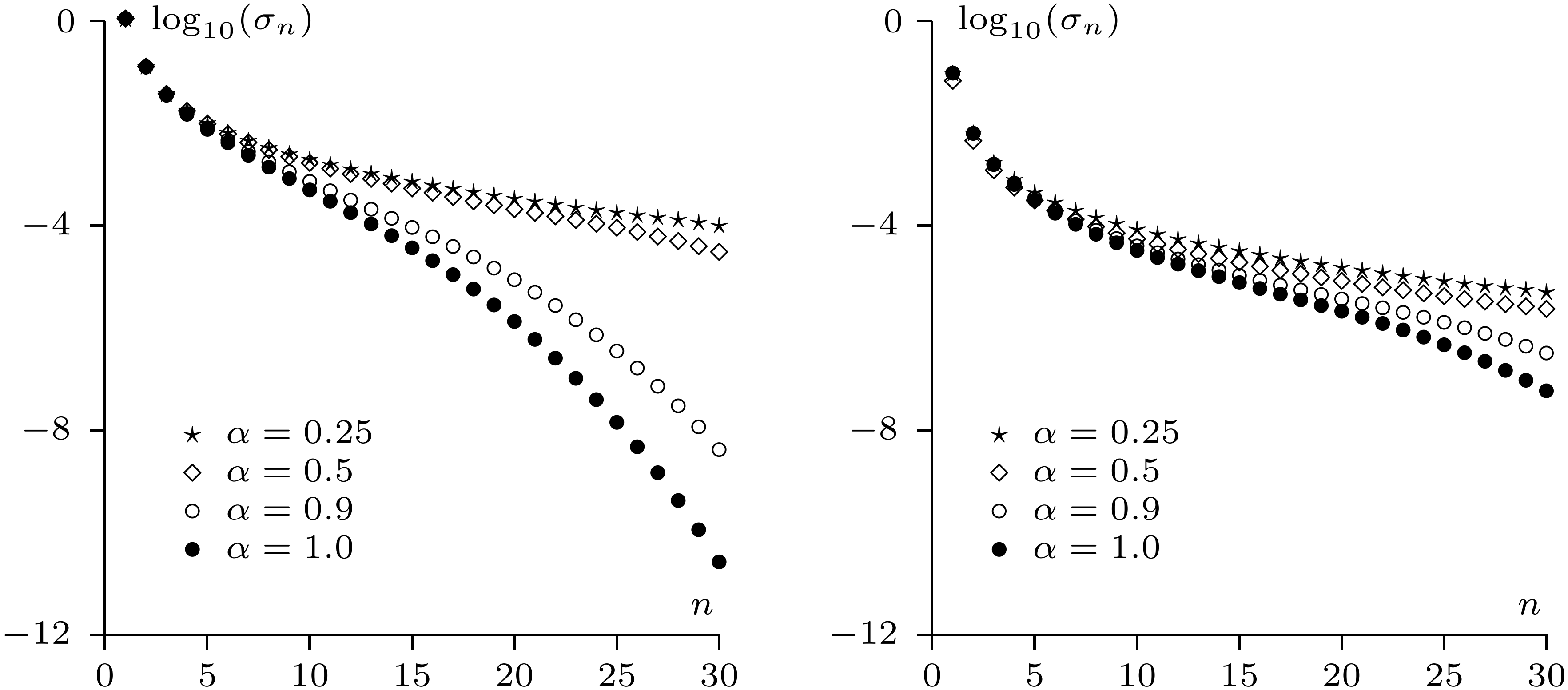
Due to the coupling $u(x,:)\kappa(x)$, we “lose” small u-value information – which means at the left hand $(x = 0)$ endpoint.



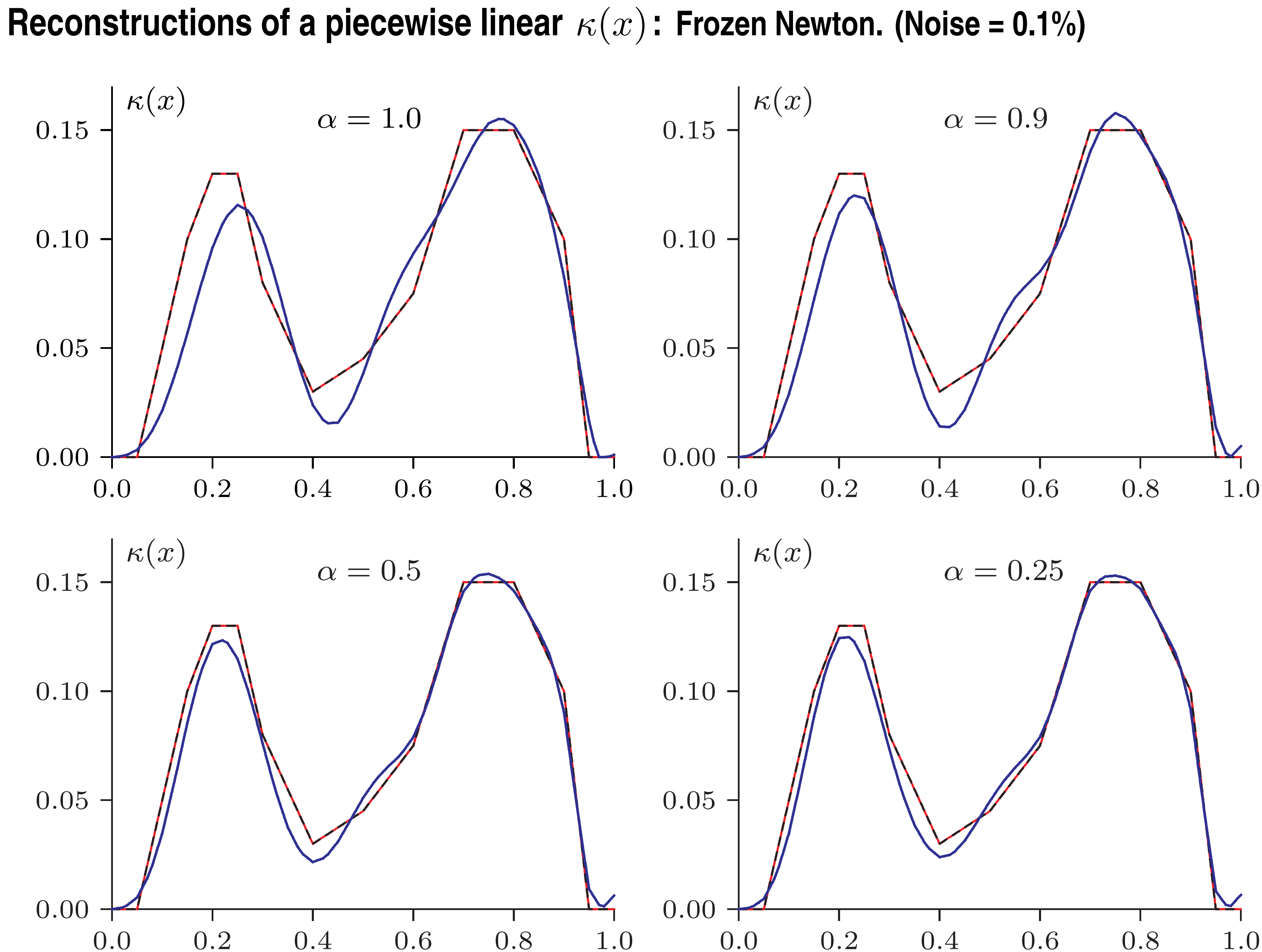
Reconstructions of a piecewise linear $\kappa(x)$: Frozen Newton. (Noise = 0.1%)
$\kappa(x)$
$\alpha = 1.0$
$\alpha = 0.9$
$\alpha = 0.5$
$\alpha = 0.25$
0.00
0.05
0.10
0.15
0.0
0.2
0.4
0.6
0.8
1.0



Reconstructions of $\kappa(x)$ for $\alpha = 0.25,\ 0.9$ Noise = 0.5% (blue) and 1% (green)



Singular values of F for various α values: left; $c = 1$, right; $c = 5$



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Other work, recent and in planning.

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We have at least 4 terms of physical interest that could be recovered:

The imaging constant $\kappa(x)$

The wave speed $c(x)$

The nonlinear term $f(v)$

The damping operator $\mathcal{D}$ – in its many possible forms.

$$\big(v - \kappa(x) f(v)\big)_{tt} - c(x)^2 \triangle v + \mathcal{D} v = r$$

Other work, recent and in planning.

- Given $\kappa(x)$ we can to show the unique recovery of the nonlinear term f in $\big(1 - \kappa f(u)\big)$.

  Both a Newton-type scheme and one based on Picard iteration are feasible. In the latter case convergence is very slow but can be significantly improved by using Anderson acceleration.

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- If all else is known in the equation we can recover a multi-term fractional order damping term $\mathbb{D}u=\sum_{j=i}^{N} b_i \partial_t^{\alpha_i}$. In fact, this can be done solely from very large time measurements.

  Key observations are: for large times the nonlinearity in u has essentially vanished and we have a linear wave equation with damping $\mathbb{D}$. After taking a Laplace transform, we have

$$\text{known}(s)=\sum_{n=1}^{\infty}\frac{1}{s^2+c^2\lambda_n+\sum_{k=1}^{N} b_k\, s^{\alpha_k}\,\lambda_n^{\beta_k}}. \qquad (\beta_k=0)$$

  Values of s can be obtained for s sufficiently small by using a Tauberian theorem that converts large t into small s. Then solve the finite dimensional (nonlinear) equation for $\{b_i,\alpha_i,N\}$.

  We hope to extend this to include recovery of terms involving space fractional β_k exponents.

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- The simultaneous recovery of $c(x)$ and $\kappa(x)$. We have made progress. An important observation is that large time measurements are useful for recovering $c(x)$ but not for $\kappa(x)$: this gives a partial decoupling of the two terms. The degree of ill-conditioning is very high.

For background on nonlinear acoustic PDE see

[0] Barbara Kaltenbacher. Mathematics of Nonlinear Acoustics. Evolution Equations and Control Theory (EECT), 4:447–491, 2015.

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and other mentioned results by BK-BR

[2] Determining damping terms in fractional wave equations, *Inverse Problems* 2022.

[3] On an inverse problem of nonlinear imaging with fractional damping. *Math. Comp.*, **91** (2021), no. 333, 245–276.

[4] Determining the nonlinearity in an acoustic wave equation. *Math. Methods Appl. Sci.* **45**, (2022), 3554–3573.

[5] Some inverse problems for wave equations with fractional derivative attenuation. *Inverse Problems*, **37**, (2021), 045002, 28 pp.

A soon-to-appear book on inverse problems involving fractional operators

Barbara Kaltenbacher and William Rundell, *Inverse problems for Fractional Differential Equations,* Graduate Studies in Mathematics, American Math Society, 2023.