Multiscale iterative methods for decomposition, deblurring and denoising of images

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Inverse Problems on Large Scales
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Collaborators

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- W. Li, E. Resmerita and L. Vese, Multiscale hierarchical image decomposition and refinements: Qualitative and quantitative results, *SIAM Journal on Imaging Sciences 14 (2), 844-877* (2021).
- J. Barnett, W. Li, E. Resmerita and L. Vese, Multiscale hierarchical decomposition methods for images corrupted by multiplicative noise, *preprint* (2022).

Introduction

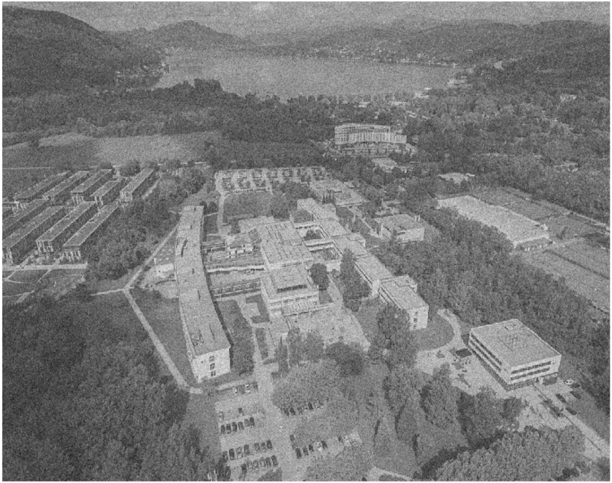
- Let $f \in BV(\Omega)$ be the representation of an image, where
$$BV(\Omega) = \left\{ u \in L^1(\Omega) : TV(u) < \infty \right\}, \quad \Omega \subset \mathbb{R}^2.$$
- Recall the ROF model:
Given the noisy version f^δ of the ground truth image f, find
$$u_0 = \arg \min_{u \in BV(\Omega)} \left\{ \frac{\lambda_0}{2} \| f^\delta - u \|^2 + TV(u) \right\}$$
to approximate f.

Rudin, Osher, Fatemi 1992

Denoising images corrupted by Gaussian noise

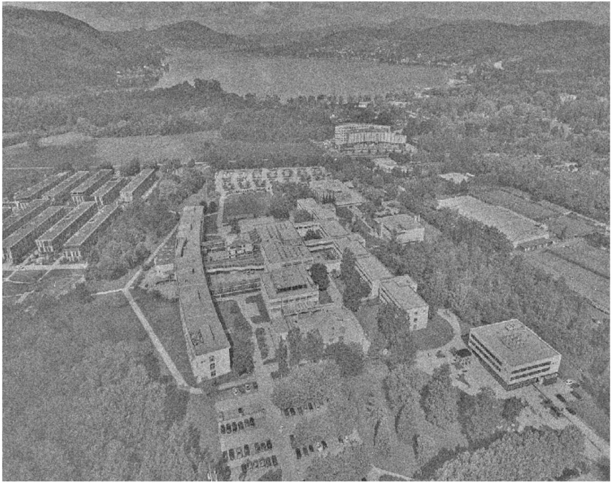
Approximate the ground truth f

when a noisy version f^δ of the ground truth is given:



Step 0 ($\lambda_0 = 0.0001$)

Residual after Step 0



Multiscale hierarchical decomposition method (MHDM)

- The residual might contain also texture, which can be seen as cartoon at finer scales.
- Then increase the parameter λ_0 and perform ROF for the residual as the new data, and so on.

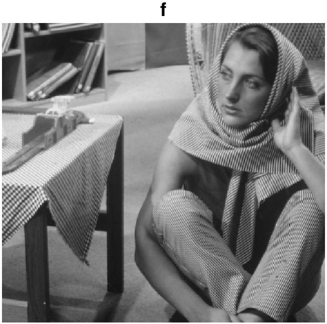
Algorithm Multiscale Hierarchical Decomposition (MHDM)

Start with $\lambda_0 > 0$ and compute
$u_0 \in \arg\min_{u \in BV(\Omega)} \{ \frac{\lambda_0}{2} \| f^\delta - Tu \|^2_{L^2(\Omega)} + TV(u) \}$.
Set $x_0 = u_0, \ v_0 = f^\delta - Tx_0$.
for $k = 1, \dots$ do
Increase $\lambda_k = 2\lambda_{k-1}$ and compute
$u_k \in \arg\min_{u \in BV(\Omega)} \{ \frac{\lambda_k}{2} \| v_{k-1} - Tu \|^2_{L^2(\Omega)} + TV(u) \}$.
Set $x_k = u_0 + \dots + u_k$ and $v_k = f^\delta - Tx_k$.

Tadmor, Nezzar, Vese 2004, 2008

Step 9 ($\lambda_9 = 0.1024$): $x_9 = u_0 + u_1 + \cdots + u_9$

Pictures provided by Tobias Wolf (University of Klagenfurt)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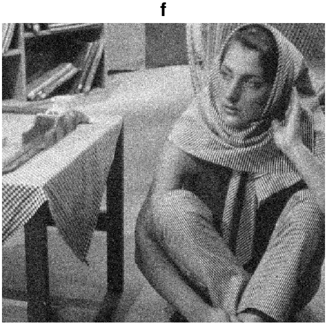
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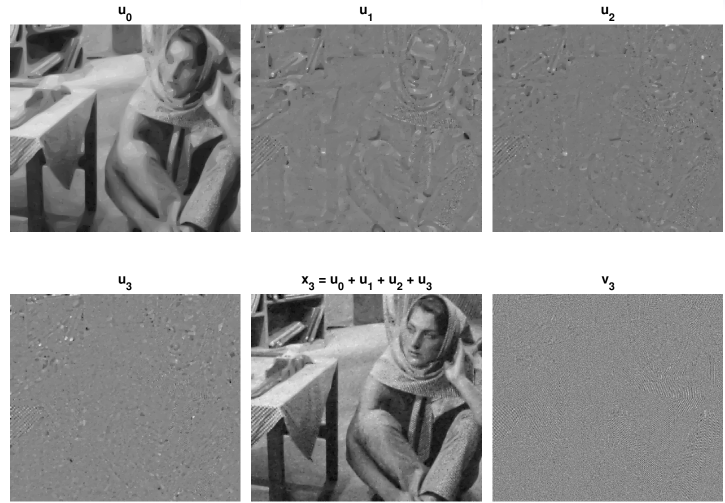


$$f^{\delta} = \left(u_0 + u_1 + u_2 + u_3\right) + v_3$$

Known convergence results for MHDM (exact data)

J stands for TV, but it can be a more general seminorm.

$$u_k \in \arg\min_{u \in BV(\Omega} \left\{ \frac{\lambda_k}{2} \| T(u + x_{k-1}) - f \|^2 + J(u) \right\}, \; k \geq 0.$$

It has been shown that (Tx_k) converges to f as follows,

$$\lim_{k \to \infty} \| T^*(f - Tx_k) \|_* = 0,$$

where T^* is the adjoint of T and $\| \cdot \|_*$ is

$$\|g\|_* = \sup_{\varphi : J(\varphi) \neq 0} \frac{\langle g, \varphi \rangle}{J(\varphi)}, \quad \forall g \in L^2(\Omega).$$

Tadmor, Nezzar, Vese 2004, 2008

$$\lim_{k \to \infty} \|f - Tx_k\| = 0$$

Modin, Nachman, Rondi 2019

Properties of (x_k)

Proposition

The residual $\| Tx_k - f \|^2$ decreases as k increases and the following estimate holds:

$$\| Tx_k - f \|^2 \leq \frac{2J(z)}{(k+1)\lambda_0}, \quad \forall k \geq 0. \tag{1}$$

Consequently, $\lim_{k\to\infty} Tx_k = f$ in the $L^2(\Omega)$ norm.

Corollary (image denoising)

Assume that $T = I$ and $f \in BV(\Omega)$. Then

$$\|x_k - f\| \leq \sqrt{\frac{2J(f)}{(k+1)\lambda_0}} = O(1/\sqrt{k+1}), \quad \forall k \geq 0. \tag{2}$$

Li, Resmerita, Vese 2021

The noisy data case

- Let $f^\delta \in L^2(\Omega)$ be the noisy blurred image. That is,

$$f^\delta = Tz + n$$

where n is additive noise and $\|f - f^\delta\| \le \delta, \quad \delta > 0$.

- We aim at recovering z and its decomposition (ill-posed problem!).
- MHDM: Find $u_k \in BV(\Omega)$ as a minimizer of

$$\frac{\lambda_k}{2}\|T(u + x_{k-1}) - f^\delta\|^2 + J(u).$$

Let $\quad k_*(\delta) := \max\{k \in \mathbb{N} : \|Tx_k - f^\delta\|^2 \ge \tau\delta^2\}, \quad$ for some $\tau > 1$.

Proposition

The following inequality holds

$$\|Tx_k - f^\delta\|^2 \le \delta^2 + \frac{2J(z)}{(k+1)\lambda_0}, \quad \forall k \ge 0$$

and the stopping index defined above is finite.
If $(k_*(\delta))$ is unbounded as $\delta \to 0$, then one has $\lim_{\delta \to 0} Tx_{k_*(\delta)} = f$.

What about convergence results for (x_k)?

- The tight MHDM: Find a solution u_k of

$$\min_{u\in BV(\Omega)} \left\{ \frac{\lambda_k}{2} \| T(u + x_{k-1}) - f \|^2 + \lambda_k a_k J(u + x_{k-1}) + J(u) \right\}, \; x_{k-1} = \sum_{j=0}^{k-1} u_j.$$

Proposition

If $\sum_{k=0}^{\infty} a_k < \infty$ and $\limsup_{k\to\infty} \frac{2^k}{\lambda_k a_k} = 0$, then there is a subsequence of (x_k) which converges w.r.t. $d(u,v) = \|u - v\|_{L^1} + |J(u) - J(v)|$ to a J minimizing solution of $Tu = f$.

Modin, Nachman, Rondi 2019

- A refinement of the tight MHDM: u_k solves

$$\min_{u\in BV(\Omega)} \left\{ \frac{\lambda_k}{2} \| T(u + x_{k-1}) - f \|^2 + \lambda_k a_k J(u + x_{k-1}) + R_k(u) \right\}, \quad k \geq 0,$$

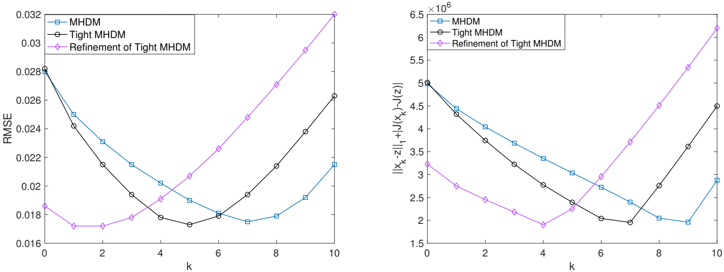
Similar convergence results can be shown for this MHDM extension.

Li, Resmerita, Vese 2021

MHDM versions for denoising-deblurring

$$\text{MHDM}(RMSE = 0.0304, d(x_1, z) = 6.06 \times 10^6)$$
$$\text{Tight MHDM}(RMSE = 0.0298, d(x_1, z) = 6.03 \times 10^6)$$
$$\text{Refined Tight MHDM}(RMSE = 0.0293, d(x_1, z) = 5.53 \times 10^6)$$

MHDM versions for denoising-deblurring: Errors



Errors of deblurring-denoising results of the degraded pollen image.

MHDM for denoising images corrupted by multiplicative noise

- Approximate the true image z from $f^{\delta} = z \cdot \eta$, where η multiplicative noise.
- Natural noise in radar, synthetic aperture radar (SAR), ultrasound etc.
- More difficult than the additive noise, since the multiplicative noise is data dependent.

How about $\log(f^{\delta}) = \log(z) + \log(\eta)$, then applying ROF?
The mean of the reconstruction would be much smaller than that of the original image! Aubert and Aujol 2008

GOAL: Adapt summed-MHDM to multiplicative noise.

Consider several existing models

- 1994 Rudin and Osher: Multiscale denoising proposed by Tadmor, Nezzar and Vese (TNV) in 2004, explored in 2008,

$$\min_u \left\{ TV(u) + \lambda \int_\Omega \left(\frac{f^\delta}{u} - 1 \right)^2 \right\}$$

- 2008 Aubert and Aujol (AA): Gamma noise model

$$\min_u \left\{ TV(u) + \lambda \int \left(\log(u) + \frac{f^\delta}{u} \right) \right\}$$

- 2008 Shi and Osher (SO): Gamma noise convex model

$$\min_w \left\{ TV(w) + \lambda \int \left(af^\delta e^{-w} + \frac{b}{2}(f^\delta)^2 e^{-2w} + cw \right) \right\} \quad u = e^w.$$

Multiplicative MHDM framework

Model name	Minimization functional
AA MHDM	$\lambda_k \int_\Omega \left(\frac{f^\delta}{ux_{k-1}} + \log(ux_{k-1}) - \log(f^\delta) - 1 \right) + TV(u)$
SO MHDM	$\lambda_k \int_{\tilde{\Omega}} \left(f^\delta e^{-(y_{k-1}+w)} + y_{k-1} + w - \log(f^\delta) - 1 \right) + TV(w)$
AA-log MHDM	$\lambda_k \int_\Omega \left(\frac{f}{ux_{k-1}} + \log(ux_{k-1}) - \log(f^\delta) - 1 \right) + TV(\log(u))$
TNV	$\lambda_k \int_\Omega \left(\frac{f^\delta}{ux_{k-1}} - 1 \right)^2 + TV(u)$
TNV-log	$\lambda_k \int_\Omega \left(\frac{f^\delta}{ux_{k-1}} - 1 \right)^2 + TV(\log(u))$

Barnett, Li, Resmerita, Vese 2022

Theoretical results for the denoising problem

- All these MHDM models are well-defined.
- The residual $H(f^\delta, x_k)$ is decreasing.
- For the log-models:

$$H(f^\delta, x_k) \leq \delta^2 + \frac{2TV(\log(z))}{(k+1)\lambda_0}.$$

Moreover, $(x_{k^*(\delta)})$ converges a.e to z on a subsequence, with

$$k^*(\delta) := \max\{k \in \mathbb{N} : H(f^\delta, x_k) \geq \tau\delta^2\}, \quad \text{for some}\, \tau > 1.$$

- Similar results hold for tight and refined versions.

Numerical tests for denoising

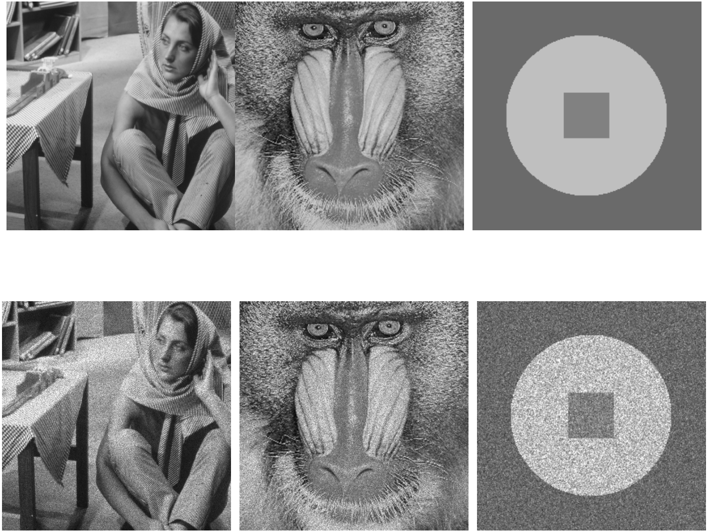


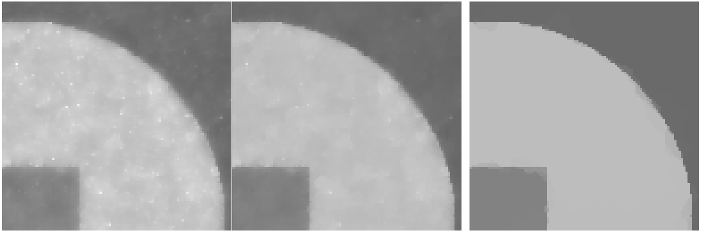
Table: SNR values (higher is better) from various denoising recoveries at the minimizing indices k_{min} and the stopping criteria $k^*(\delta)$. Bold entries are the maximum of their respective columns.

SNR at	Barbara k_{min}	Barbara k^*	Mandril k_{min}	Mandril k^*	Geometry k_{min}	Geometry k^*
SO MHDM (EL)	19.70	19.08	20.38	20.06	28.21	24.82
SO Tight (EL)	19.71	18.98	20.37	19.89	29.46	27.50
SO Refined (EL)	**19.88**	19.33	**20.73**	20.32	30.11	29.66
SO MHDM (ADMM)	19.31	19.12	19.93	19.93	**34.67**	31.53
SO Tight (ADMM)	19.74	**19.36**	20.46	20.19	34.60	**34.60**
AA MHDM	19.53	19.14	20.21	20.05	28.21	24.77
AA-log MHDM	19.65	19.04	20.35	20.02	28.21	24.83
AA-log Tight	19.67	18.93	20.37	19.79	30.22	26.11
AA-log Refined	19.64	19.21	20.55	**20.41**	27.12	13.81
TNV-log	19.53	17.97	20.37	19.02	25.70	23.17
TNV-log Tight	19.54	17.35	20.42	17.89	26.39	21.02
TNV	19.73		20.40		25.94	
Dong	19.13		20.22		31.32	

Numerical tests for denoising

Dong, AA-log refined and SO refined (EL)

Numerical tests for denoising



TNV, AA-log tight and SO tight (ADMM)

Table: SNR values (higher is better) for restoring noisy-blurred images at the k_{min}. Bold entries are the maximums of their respective columns.

SNR at k_{min}	Barbara	Mandril	Geometry
AA MHDM	17.69	18.65	26.45
AA-log MHDM	17.69	18.64	26.66
AA-log Tight	**17.74**	**18.66**	**28.76**
AA-log Refined	17.54	18.47	27.12
TNV-log	17.30	18.13	24.17
TNV-log Tight	17.35	18.16	25.17
TNV	17.32	18.17	24.23
Dong	17.36	17.24	28.32

Summary and further work

- Error estimates for the multiscale hierarchical decomposition method (MHDM) and for its extensions (tight, refined).
- Early stopping criteria have been provided in the case of noisy data.
- MHDM schemes have been proposed, analyzed and tested for images corrupted by multiplicative noise.

To investigate: MHDM theory for blurred noisy images involving multiplicative noise, for color images, etc.

The 3rd Alps-Adriatic Workshop on Inverse Problems

July 3-7, 2023, University of Klagenfurt