香港城市大學
City University of Hong Kong

Inverse Problems on Large Scales
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RICAM
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Selecting Regularization Parameters for Nuclear Norm Type Minimization Problems

Raymond Chan
Department of Mathematics

KX LI, YW Wen, HW LI
Hunan Normal University, China
Capital Normal University, China

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Outline

- **Introduction**
- **Choosing the parameters for the regularization problems**
- **Applications**
- **Conclusion**

- **Introduction**
- Choosing the parameters for the regularization problems
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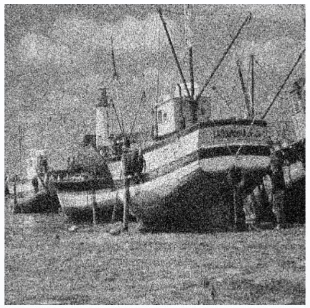
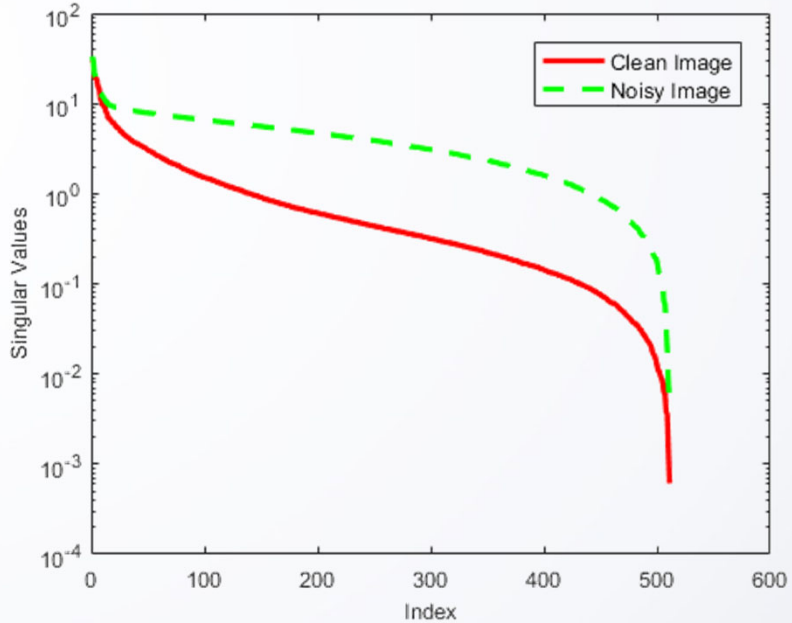
Introduction

Low-rank matrix recovery problems arise in many applications:

- ☐ Image Denoising
- ☐ Image inpainting
- ☐ Matrix completion
- ☐ Background subtraction





The regularized model and constrained model

The noisy observation $Y \in \mathbb{R}^{m \times n}$ can be modeled as

$$Y = X + W,$$

where X is the clean low-rank matrix, and W is the Gaussian white noise with variance τ^2.

- **The constrained model:** $$\min_{\|X-Y\|_F^2 \leq \eta} f(X)$$

- **The regularization model:** $$\min_X f(X) + \frac{1}{2\lambda}\|X - Y\|_F^2$$

$$\eta \longleftrightarrow \lambda$$

The choice of regularization function for low-rank recovery

A rank minimization approach to recovery X is to set

$$f(X) = \text{rank}(X)$$

However, the rank function is nonconvex and NP-hard!

Relaxation:

- Nuclear norm $f(X) = \|X\|_* \equiv \sum_{i=1}^{m} \sigma_{X,i}$
- Truncated nuclear norm $f(X) = \|X\|_{r,*} \equiv \sum_{i=r+1}^{m} \sigma_{X,i}$
- Weighted nuclear norm $f(X) = \|X\|_{\mathbf{w},*} \equiv \sum_{i=1}^{m} w_i \sigma_{X,i}$

where $\sigma_{X,1} \geq \sigma_{X_2} \geq \cdots \geq \sigma_{X,m}$ are singular values of X.

The optimal regularization parameter

- [M. Gavish *et al.* 2014] The optimal **hard threshold** parameter λ in

$$\min_X \operatorname{rank}(X) + \frac{1}{2\lambda}\|X - Y\|_F^2$$

  is chosen as $\lambda = (4/\sqrt{3})\sqrt{n}\tau$ when the noise level τ is known or $\lambda = 0.2858\sigma_{Y,\text{median}}$ when τ is unknown.

- [E. Candes *et al.* 2013] The optimal **soft threshold** parameter λ in

$$\min_X \|X\|_* + \frac{1}{2\lambda}\|X - Y\|_F^2$$

  can be estimated by minimizing

$$\text{SURE}(\lambda) = -mn\tau^2 + \sum_{i=1}^{m} \min\left(\lambda^2, \sigma_{Y,i}^2\right) + 2\tau^2 \operatorname{div}_Y \left(\text{SVT}_\lambda(Y)\right).$$

Other methods: GCV, L-curve, discrepancy principle . . .

Discrepancy principle

- Mathematically, the two minimization models are equivalent:

$$\min_{\|X-Y\|_F^2 \le \eta} f(X) \quad \text{and} \quad \min_X f(X) + \frac{1}{2\lambda}\|X-Y\|_F^2,$$

- Given $\|Y\|_F^2 > \eta > 0$ (exclude $X = 0$), there exists a corresponding optimal $\lambda > 0$ satisfying

$$\|X(\lambda) - Y\|_F^2 = \eta \qquad (*)$$

such that the solution of the former is also a solution of the latter, and vice versa.

- Choosing λ is difficult, while determining η is easier. Can we obtain λ for any given η through solving the nonlinear equation $(*)$?

Determining an upper bound for η

Theorem 1. [J. L. Mead *et al.*, 2013] *Assume that the error matrix* $Y - X = W \sim \mathcal{N}(0, \tau^2 I)$. *Then* $\|W\|_F^2$ *is* χ^2*-distributed with variance* τ^2 *and* mn *degrees of freedom. Moreover, we have*

$$\mathbb{E}\|W\|_F^2 = mn\tau^2.$$

Theorem 2. [N. Halko *et al.*, 2010] *Given a Gaussian matrix* $W \sim \mathcal{N}(0, \tau^2 I)$, *we have*

$$\mathbb{P}\{\|W\|_F \geq \mathbb{E}\|W\|_F + \tau t\} \leq \mathrm{e}^{-t^2/2}.$$

⇒ $\mathbb{P}\{\|W\|_F \leq \mathbb{E}\|W\|_F + t\tau\} \geq 1 - \mathrm{e}^{-t^2/2}$. When $t \geq 4$, the noise is bounded with probability greater than 0.9996.

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The upper bound η

- Set the upper bound to

$$\eta = cmn\tau^2,$$

$\eta \longrightarrow \lambda$

where $c \simeq 1$ can be adjusted appropriately to suit the applications.

- If the noise level is not given, τ can be estimated using the median rule [D. L. Donoho *et al.*, 1994]:

$$\tau = \frac{\text{median}(\tilde{Y}_{H,H})}{0.6745}$$

where $\tilde{Y}_{H,H}$ is the high-high coefficient of Y at the finest wavelet transform level.

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Nuclear Norm Minimization (NNM)

NNM: [J. F. Cai *et al.* 2010] $\min_{X} \|X\|_* + \frac{1}{2\lambda}\|Y - X\|_F^2.$

Lemma 1. *Let* $Y = \sum_{i=1}^{m} \sigma_{Y,i} u_i v_i^T$ *be the SVD of the matrix* Y*. Then the singular value thresholding* (SVT) *operator*

$$\mathrm{SVT}_{\lambda}(Y) \equiv \sum_{i=1}^{m} (\sigma_{Y,i} - \lambda)_+ u_i v_i^T$$

is the minimizer of NNM.

How to find λ when given η?

Nuclear norm minimization

Theorem 3. *Let*

$$\widehat{X}(\lambda) = \arg\min_{X} \|X\|_* + \frac{1}{2\lambda}\|Y - X\|_F^2,$$

i.e., $\widehat{X}(\lambda) = \mathrm{SVT}_\lambda(Y)$, *then the nonlinear function*

$$\psi(\lambda) \equiv \|Y - \widehat{X}(\lambda)\|_F^2$$

is a positive, continuous, and strictly increasing function of λ *in* $[0, \sigma_{Y,1}]$.

Proof:

$$\psi(\lambda) = j\lambda^2 + \sum_{i=j+1}^{m} \sigma_{Y,i}^2, \quad \lambda \in (\sigma_{Y,j+1}, \sigma_{Y,j}], \quad j = 1, \ldots, m.$$

Nuclear norm minimization

Theorem 4. *Let* $\widehat{X}(\lambda) = \mathrm{SVT}_\lambda(Y)$, *then the nonlinear equation* $\psi(\lambda) \equiv \|Y - \widehat{X}(\lambda)\|_F^2 = \eta$ *has a unique solution* $\lambda > 0$ *for any* η *satisfying* $0 < \eta < \|Y\|_F^2$.

Proof:

- Define a new sequence $b_j = j\sigma_{Y,j}^2 + \sum_{i=j+1}^m \sigma_{Y,i}^2, j = 1, ..., m$.
- $\{b_j\}$ is non-increasing: $b_1 \geq b_2 \geq \ldots \geq b_m \geq b_{m+1} = 0$.
- $\exists$ an k s.t. $b_{k+1} < \eta \leq b_k$ and recall $\psi(\lambda) = k\lambda^2 + \sum_{i=k+1}^m \sigma_{Y,i}^2$:

$$\lambda = \sqrt{(\eta - \sum_{i=k+1}^{m} \sigma_{Y,i}^2)/k}$$

Truncated nuclear norm minimization (TNNM)

TNNM: [Y. Hu *et al.*, 2012; T. H. Oh *et al.*, 2013]

$$\min_X \|X\|_r + \frac{1}{2\lambda}\|Y - X\|_F^2,$$

where $\|X\|_r = \sum_{i=r+1}^{m} \sigma_{X,i}$ is the truncated nuclear norm (TNN).

Lemma 2. *For any* $\lambda > 0$*, a global solution of TNNM with a target rank* $r < m$ *is given by the partial singular value thresholding* (PSVT) *operator*

$$\mathrm{PSVT}_{r,\lambda}(Y) \equiv \sum_{i=1}^{r} \sigma_{Y,i} u_i v_i^T + \sum_{i=r+1}^{m} (\sigma_{Y,i} - \lambda)_+ u_i v_i^T.$$

Truncated nuclear norm minimization

- Denote $Y_r = \sum_{i=1}^{r} \sigma_{Y,i} u_i v_i^T$ and $Z = Y - Y_r$, then

$$\min_X \|X\|_r + \frac{1}{2\lambda}\|Y - X\|_F^2 \quad \Rightarrow \quad \min_X \|X\|_* + \frac{1}{2\lambda}\|Z - X\|_F^2.$$

- The solution of TNNM is given by

$$X = Y_r + \mathrm{SVT}_\lambda(Z).$$

Therefore, although $\|X\|_r$ is non-convex, we can solve it easily with the help of convex nuclear norm optimization.

Generalized weighted nuclear norm minimization (GWNNM)

GWNNM: [Huang *et al.*, 2019]:

$$\min_X \|X\|_{\mathbf{w},*} + \frac{1}{2\lambda}\|Y - X\|_F^2$$

where $\|X\|_{\mathbf{w},*} = \sum_{i=1}^{m} w_i \sigma_{X,i}$ is the generalized weighted nuclear norm, with the weights

$$w_i = (\sigma_{Y,i} + \epsilon)^{p-1}, i = 1, ..., m,$$

$0 \leq p < 1$ and ϵ is a sufficiently small positive number.

Generalized weighted nuclear norm minimization

Lemma 3. [S. Gu *et al.*, CVPR, 2014] *For any* $\lambda > 0$, *if* $\{w_i\}_{i=1}^m$ *satisfy* $0 \leq w_1 \leq w_2 \leq ... \leq w_m$, *then a global solution of GWNNM is given:*

$$\mathrm{WSVT}_\lambda(Y) \equiv \sum_{i=1}^{m} (\sigma_{Y,i} - \lambda w_i)_+ u_i v_i^T.$$

Theorem 5. *Let* $\widehat{X}_{\mathbf{w}}(\lambda) = \arg\min_X \|X\|_{\mathbf{w},*} + \frac{1}{2\lambda}\|Y - X\|_F^2$, *then*

$$\phi(\lambda) = \|Y - \widehat{X}_{\mathbf{w}}(\lambda)\|_F^2$$

is a positive, continuous, and strictly increasing function of λ *in* $[0, \sigma_{Y,1}]$.

Generalized weighted nuclear norm minimization

Theorem 6. *Let* $\widehat{X}_{\mathbf{w}}(\lambda) = \mathrm{WSVT}_{\lambda}(Y)$*, then* $\|Y - \widehat{X}_{\mathbf{w}}(\lambda)\|_F^2 = \eta$ *has a unique solution* $\lambda > 0$ *for any* η *satisfying* $0 < \eta < \|Y\|_F^2$.

Proof:

- $\phi(\lambda) = \lambda^2 \sum_{i=1}^{k} w_i^2 + \sum_{i=k+1}^{m} \sigma_{Y,i}^2$.
- Define the sequence $b_j = (\frac{\sigma_{Y,j}}{w_j})^2 \sum_{i=1}^{j} w_i^2 + \sum_{i=j+1}^{m} \sigma_{Y,j}^2, j = 1, ..., m$.
- $\{b_j\}$ is non-increasing and there exists a k such that $b_{k+1} < \eta \le b_k$.

$$\lambda = \sqrt{(\eta - \sum_{i=k+1}^{m} \sigma_{Y,i}^2) / \sum_{i=1}^{k} w_i^2}$$

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Randomized method

- SVD cost for $Y \in \mathbb{R}^{m\times n}$ is $O(mn\min(m,n))$ and is expensive.

- Let $Q_\ell \in \mathbb{R}^{n\times\ell}$, $\ell \le m$, be an orthogonal matrix. Then for all $A \in \mathbb{R}^{m\times\ell}$ and $X = AQ_\ell^\top$, we have

$$\|Y - X\|_F^2 = \|YQ_\ell - A\|_F^2 + a.$$

  Here $a = \|Y\|_F^2 - \|YQ_\ell\|_F^2$.

- Hence we can consider the reduced problem:

$$\min_A f(A) + \frac{1}{2\lambda}\|YQ_\ell - A\|_F^2.$$

  The size of the matrix YQ_ℓ ($\in \mathbb{R}^{m\times\ell}$) is smaller than that of Y ($\in \mathbb{R}^{m\times n}$).

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Simulations on synthetic data

- The s-rank matrix $X_\dagger = MN^\top \in \mathbb{R}^{m\times n}$ where $M \in \mathbb{R}^{m\times s}$ and $N \in \mathbb{R}^{n\times s}$ are uniform-random matrices.
- $s = \lfloor \rho \cdot \min(m, n) \rfloor$ with ρ from 1% to 40%.
- A noise matrix $W \sim \mathcal{N}(0, \tau^2 I)$ is added to $X_\dagger$ to get the observed matrix Y, i.e., $Y = X_\dagger + W$.
- We recover the matrix $X_\dagger$ from its noisy observation matrix Y.
- The bound $\eta = mn\tau^2$.
- For the randomized algorithms, $\ell = s + 5$.

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Methods to compare

- [M. Gavish *et al.* 2014] The optimal **hard threshold** parameter λ in

$$\min_X \operatorname{rank}(X) + \frac{1}{2\lambda}\|X - Y\|_F^2$$

where $\lambda = (4/\sqrt{3})\sqrt{n}\tau$.

- [E. Candes *et al.* 2013] The optimal **soft threshold** parameter λ in

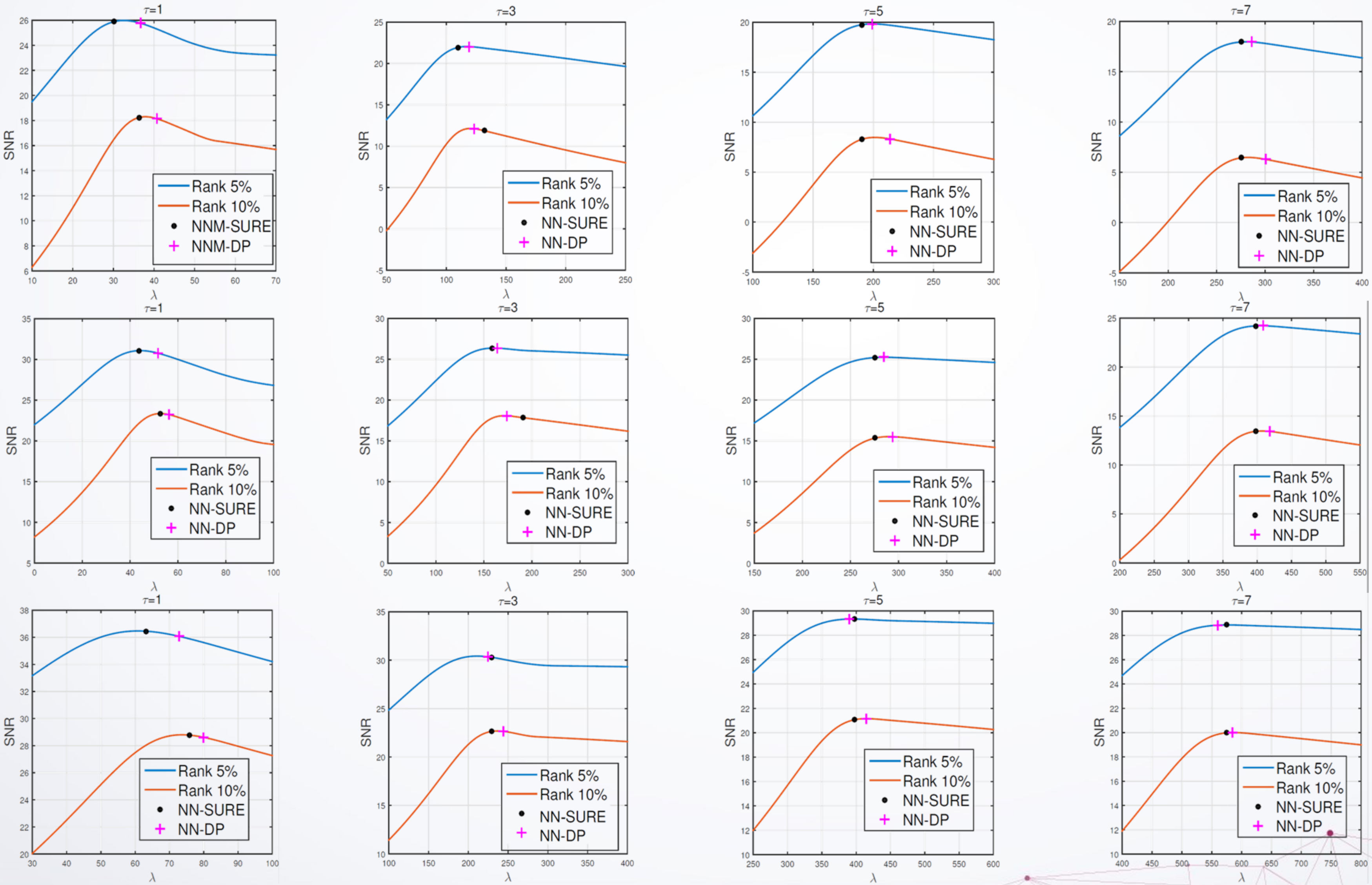
$$\min_X \|X\|_* + \frac{1}{2\lambda}\|X - Y\|_F^2$$

is given by minimizing

$$\text{SURE}(\lambda) = -mn\tau^2 + \sum_{i=1}^{m} \min\left(\lambda^2, \sigma_{Y,i}^2\right) + 2\tau^2 \operatorname{div}_Y\left(\text{SVT}_\lambda(Y)\right)$$

over 101 equi-spaced λ's in $\log[10^{-1}, 10^7]$.

Regularization parameter selection (NNM-DP VS. NNM-SURE)



$m = n = 500$

$m = n = 1000$

$m = n = 2000$

SNR(dB) for different methods with the standard algorithm

		$f(X)$	Rank(X)	$\|X\|_*$		$\|X\|_r (r=1)$		$\|X\|_{\mathbf{w},*}$	
$m=n$	τ	ρ	HardT	NN-SURE	NN-DP	TNN-SURE	TNN-DP	GWNN-SURE	GWNN-DP
500	3	1%	14.47	11.78	*11.96*	***14.47***	***14.47***	*13.49*	13.05
		5%	**22.99**	21.84	*21.94*	***22.99***	***22.99***	*22.82*	*22.82*
		10%	26.25	25.69	*25.72*	26.25	***26.29***	*26.22*	26.21
		20%	29.43	*29.30*	*29.30*	***29.59***	***29.59***	29.51	*29.54*
		40%	32.48	32.72	*32.75*	32.84	***32.89***	*32.84*	*32.84*
	5	1%	**11.35**	*8.53*	8.51	***11.35***	***11.35***	*9.96*	9.31
		5%	**21.80**	19.71	*19.75*	***21.80***	21.79	*21.30*	21.26
		10%	**25.57**	24.16	*24.20*	***25.57***	25.53	*25.34*	25.32
		20%	**29.06**	28.26	*28.27*	***29.06***	29.03	*28.97*	28.94
		40%	32.29	*31.92*	31.91	***32.31***	***32.31***	*32.27*	32.26
Average SNR			24.57	23.39	*23.43*	***24.62***	***24.62***	*24.27*	24.16
Average time (sec)			0.054	1.124	0.059	1.178	0.060	1.144	0.058
1000	3	1%	19.22	17.69	*17.94*	19.22	***19.25***	18.90	*18.97*
		5%	26.46	*26.34*	*26.34*	26.64	***26.65***	26.61	*26.62*
		10%	29.51	29.71	*29.72*	29.87	***29.88***	29.84	*29.85*
		20%	32.54	*33.14*	33.10	***33.20***	33.18	*33.16*	33.15
		40%	35.39	*36.67*	36.59	***36.71***	36.63	*36.65*	36.59
	5	1%	**17.76**	15.34	*15.48*	***17.76***	***17.76***	*16.98*	16.90
		5%	**26.08**	25.22	*25.25*	***26.08***	26.05	*25.97*	25.95
		10%	**29.32**	*28.89*	28.88	***29.32***	29.31	*29.30*	29.26
		20%	32.44	*32.29*	*32.29*	***32.52***	32.51	*32.47*	32.46
		40%	35.51	35.57	*35.63*	35.68	***35.73***	35.67	*35.69*
Average SNR			28.42	28.09	*28.12*	***28.70***	***28.70***	*28.56*	28.54
Average time (sec)			0.240	5.646	0.261	5.992	0.267	5.895	0.249

SNR(dB) for different methods with the randomized algorithm

		$f(X)$	Rank(X)	$\|X\|_*$		$\|X\|_r(r=1)$		$\|X\|_{\mathbf{w},*}$	
$m=n$	τ	ρ	HardT	NN-SURE	NN-DP	TNN-SURE	TNN-DP	GWNN-SURE	GWNN-DP
500	3	1%	**14.47**	11.78	*12.28*	***14.47***	***14.47***	*13.62*	13.15
		5%	**22.99**	21.75	*22.02*	***22.99***	22.97	22.75	*22.81*
		10%	**26.25**	25.52	*25.72*	***26.25***	***26.25***	26.15	*26.18*
		20%	29.43	29.05	*29.29*	29.43	***29.57***	29.39	*29.52*
		40%	32.48	32.19	*32.74*	32.48	***32.88***	32.47	*32.84*
	5	1%	**11.35**	8.19	*8.79*	***11.35***	***11.35***	*9.97*	9.45
		5%	**21.80**	19.39	*19.90*	***21.80***	21.79	*21.30*	21.29
		10%	**25.57**	23.90	*24.25*	***25.57***	25.52	*25.34*	25.32
		20%	**29.06**	28.04	*28.27*	***29.06***	29.03	28.93	*28.94*
		40%	32.29	31.72	*31.91*	32.29	***32.31***	32.25	*32.26*
Average SNR			24.57	23.15	*23.52*	24.57	***24.61***	*24.22*	24.18
Average time (sec)			0.005	0.116	0.006	0.123	0.007	0.117	0.006
1000	3	1%	**19.22**	17.71	*18.11*	***19.22***	19.18	18.90	*18.95*
		5%	26.46	26.00	*26.25*	26.46	***26.51***	26.43	*26.47*
		10%	29.51	29.32	*29.64*	29.51	***29.78***	29.50	*29.75*
		20%	32.54	32.53	*33.07*	32.54	***33.14***	32.54	*33.12*
		40%	35.56	35.76	*36.59*	35.82	***36.62***	35.89	*36.59*
	5	1%	**17.76**	15.56	*15.80*	***17.76***	17.75	*17.19*	16.99
		5%	**26.08**	25.04	*25.30*	***26.08***	26.01	*25.97*	25.92
		10%	**29.32**	28.72	*28.88*	***29.32***	29.28	*29.27*	29.23
		20%	32.44	32.12	*32.28*	32.44	***32.49***	32.41	*32.45*
		40%	35.51	35.35	*35.63*	35.51	***35.73***	35.51	*35.68*
Average SNR			28.44	27.81	*28.16*	28.47	***28.65***	28.36	*28.52*
Average time (sec)			0.020	0.366	0.022	0.394	0.023	0.393	0.021

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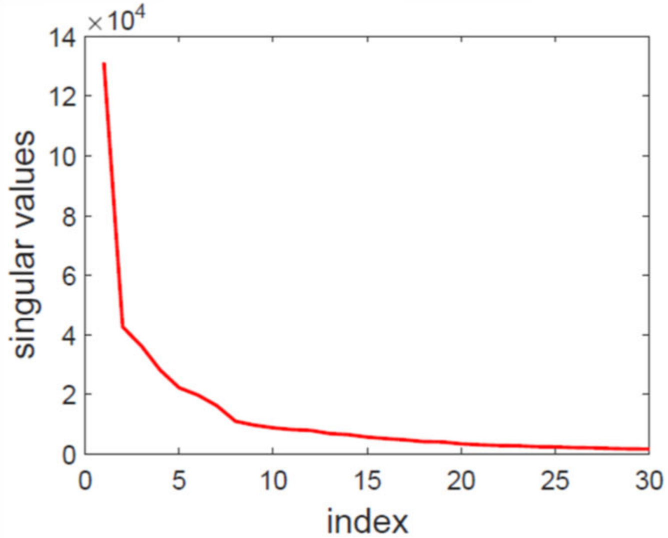
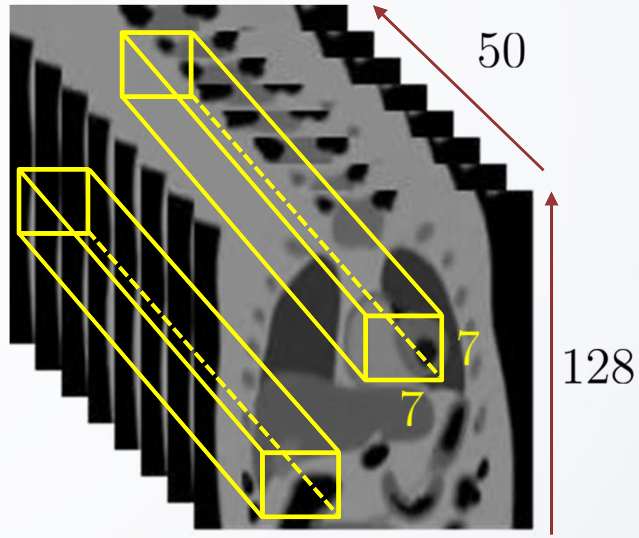
Simulations on PINCAT numerical phantom

The PINCAT phantom simulates a first-pass myocardial perfusion real-time MRI series. There are 50 images of size 128×128 taken at 50 time-steps.

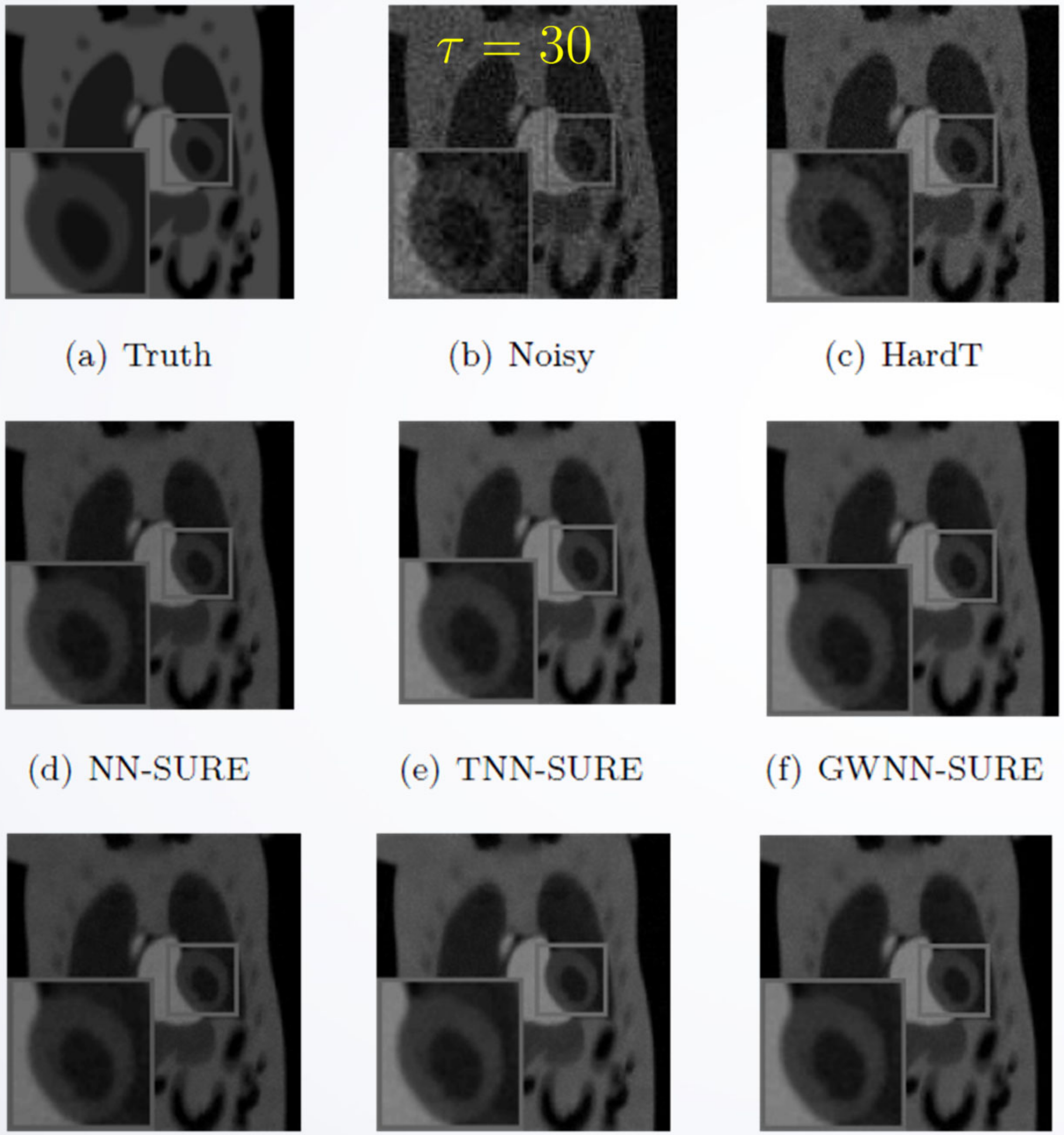


(a) Global Casorati matrix

$128^2 \times 50$ matrix



Denoised with a locally low-rank method with sliding windows [J. Trzasko, A. Manduca, and E. Borisch, 2011].



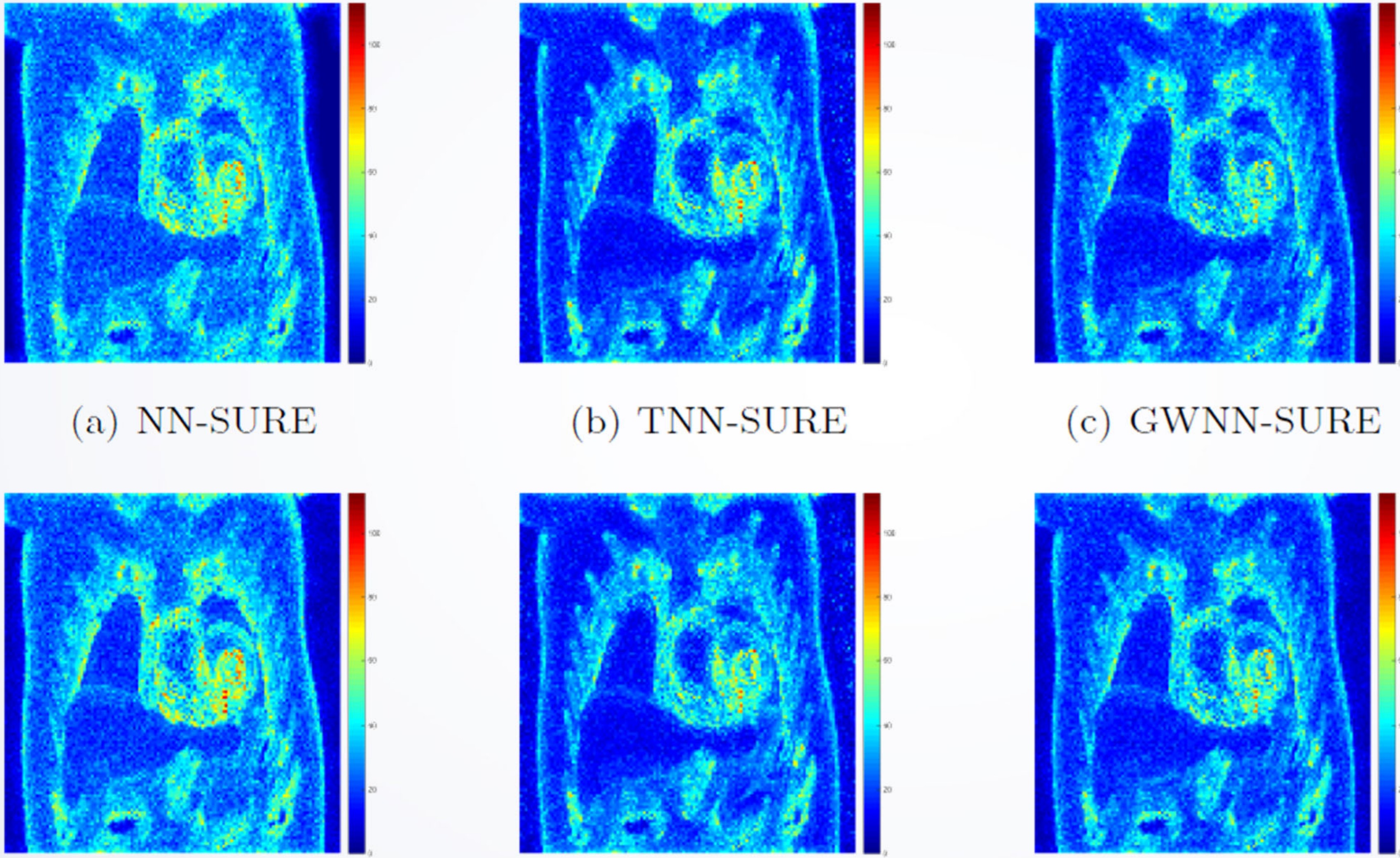
(a) Truth (b) Noisy (c) HardT

(d) NN-SURE (e) TNN-SURE (f) GWNN-SURE

(g) NN-DP (h) TNN-DP (i) GWNN-DP

Need to solve 16,384 low-rank problems of size $7^2 \times 50$

Images at $T = 10$ with an enlarged portion cropped



(a) NN-SURE (b) TNN-SURE (c) GWNN-SURE

(d) NN-DP (e) TNN-DP (f) GWNN-DP

The worst-case error through time of different methods

SNRs of different methods for the PINCAT Numerical Phantom

τ	Noisy	HardT	NN-SURE	NN-DP	TNN-SURE	TNN-DP	GWNN-SURE	GWNN-DP
5	27.50	33.19	*33.76*	33.71	34.34	*34.42*	***35.03***	35.02
10	21.48	27.60	*28.79*	28.73	29.64	*29.69*	***30.13***	***30.13***
15	17.96	24.33	*25.90*	25.85	26.89	*27.01*	***27.33***	27.31
20	15.46	22.01	*23.88*	23.83	25.09	*25.14*	***25.34***	25.31
25	13.52	20.21	22.23	*22.26*	*23.73*	23.70	***23.81***	23.76
30	11.94	18.74	*21.02*	20.98	***22.54***	22.53	***22.54***	22.48
35	10.60	17.49	19.84	*19.90*	***21.59***	21.55	*21.46*	21.40
40	9.44	16.41	*18.97*	18.96	***20.77***	20.70	*20.52*	20.46
45	8.42	15.45	18.11	*18.13*	***19.99***	19.94	*19.68*	19.62
50	7.50	14.59	17.29	*17.39*	***19.34***	19.27	*18.92*	18.88
Average	14.38	21.00	*22.98*	22.97	24.39	*24.40*	***24.48***	24.44
Average Time	–	19.01	733.34	20.12	761.91	20.06	754.24	20.81

Need to solve 16,384 low-rank problems of size $7^2 \times 50$

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Conclusion

- We considered the regularized models for various nuclear norm type regularizers.
- We derived simple formulas for the λ for these models in terms of the noise level τ.
- Our approach is competitive to the SURE methods in terms of noise removal but is much faster in terms of the CPU time.

References and Related Work

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- Y. Wen and R. Chan, Using Generalized Cross Validation to Select Regularization Parameter for Total Variation Regularization Problems, *Inverse Problems and Imaging*, 7 (2018) 1103–1120.
- K.X. Li, H.W. Li, R.H. Chan, and Y.W. Wen, Selecting Regularization Parameters for Nuclear Norm Type Minimization Problems, accepted by *SIAM Journal on Scientific Computing.*

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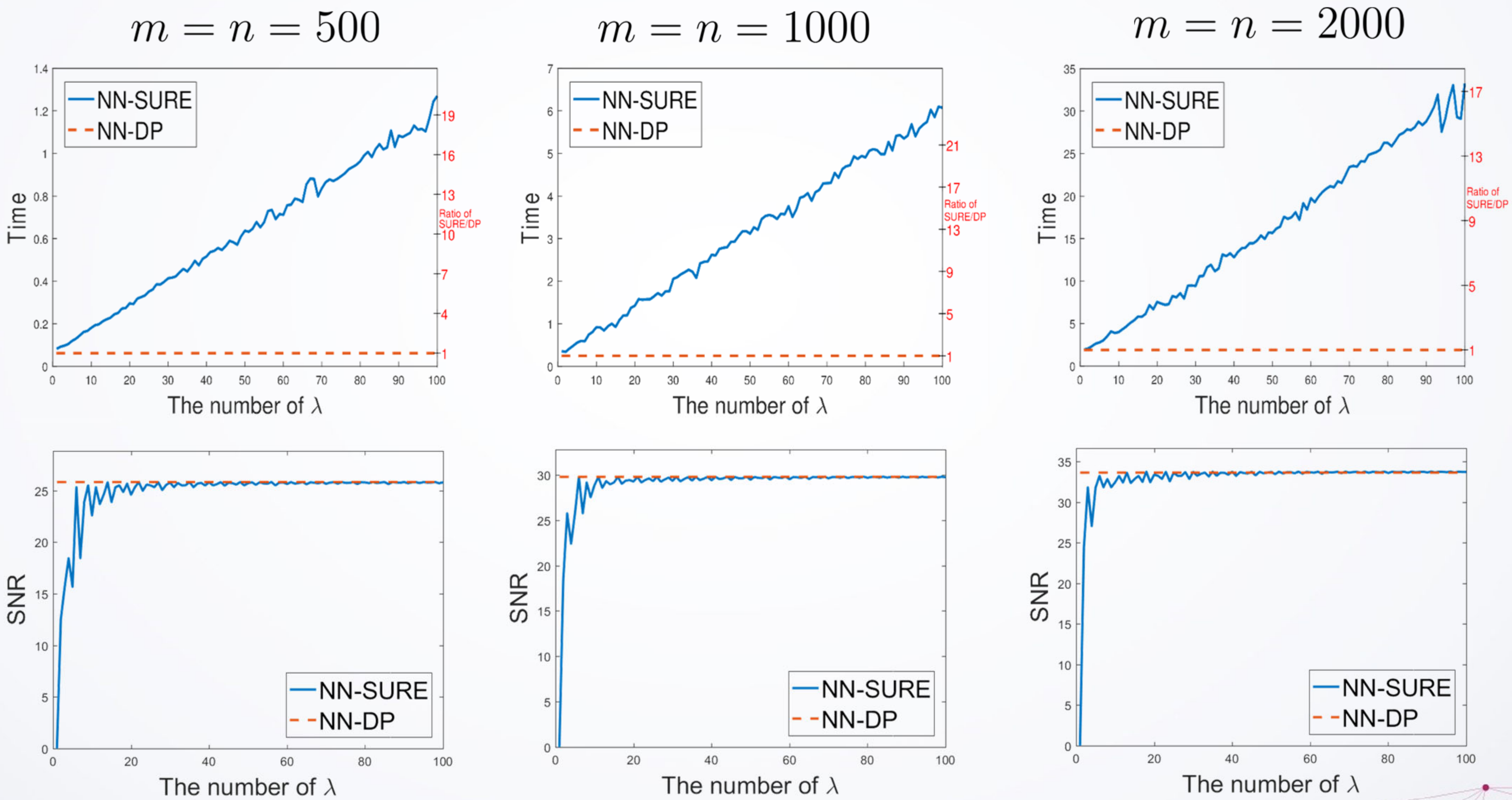
Thank you!

Timing and SNR w.r.t. number of λ's tried

$\tau = 3$

$\rho = 0.1$



40 λ's may be enough, yet our gain in speed is around 7-10 times.

How to find Q_ℓ in the randomized algorithm

How to comput matrix Q_ℓ? [N. Halko *et al.*, 2010]

- ☐ Draw a Gaussian random matrix $\Omega \in \mathbb{R}^{m\times\ell}$;
- ☐ Form the matrix $B = (Y^\top Y)^q Y^\top \Omega \in \mathbb{R}^{n\times\ell}$ ($q = 2$ is enough).
- ☐ Construct a matrix $Q_\ell \in \mathbb{R}^{n\times\ell}$ whose columns form an orthonormal basis for the range of B, e.g., the QR factorization $B = Q_\ell R$.

Rank of individual blocks in the MRI images

c)

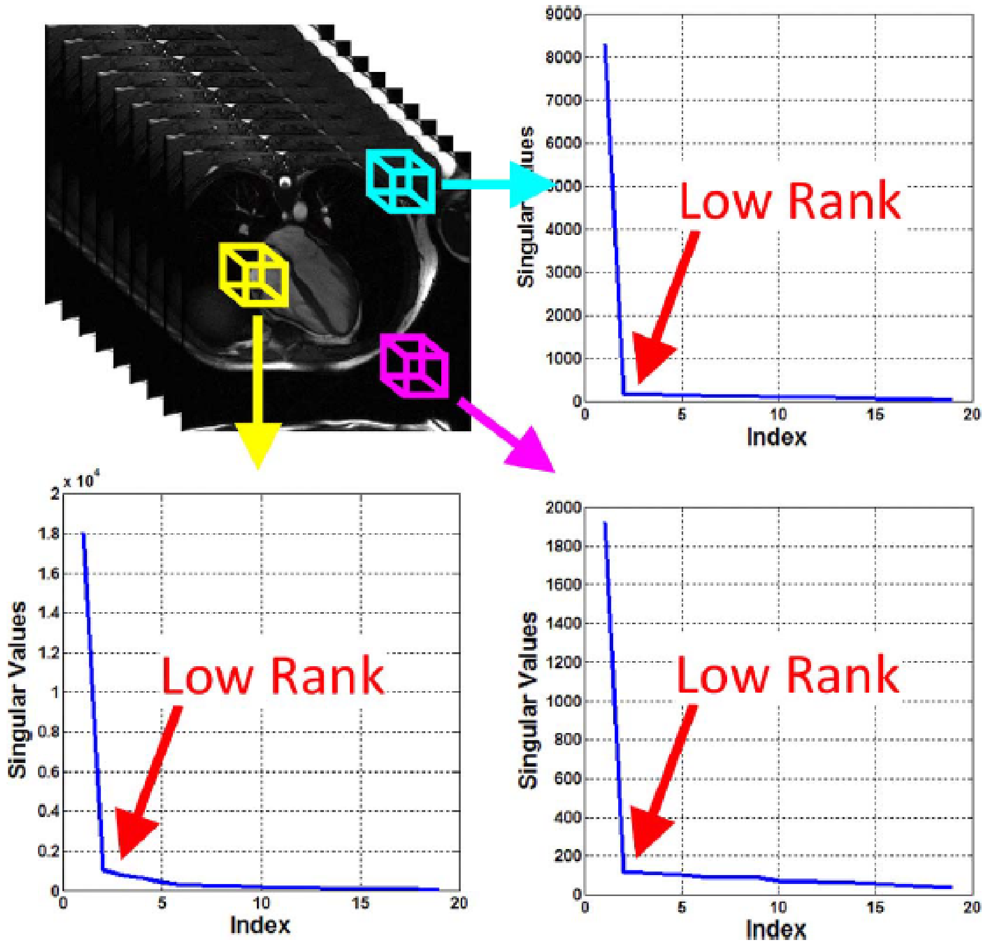


Fig. 2. Singular values of the Casorati matrices formed from three different $8 \times 8 \times 19$ block sets extracted from a cine cardiac sequence, including: a static region (cyan); a dynamic region (yellow); and background noise (magenta).

The block-wise operation in the MRI experiment is as follows:

1. We first pad the $128 \times 128 \times 50$ tensor $\mathbf{g}$, where only the bottom and right sides are filled. ($\mathbf{u}$ = padarray($\mathbf{g}$, [6 6 0], 'circular', 'post')). The dimension of $\mathbf{u}$ is $(128+6) \times (128+6) \times 50$.

2. We start to process each block, that is, transform the block with dimension $7 \times 7 \times 50$ into $7^2 \times 50$ and perform denoising processing. The step size is 1, and then perform the same processing on the next block.

3. Add the blocks together to get $\mathbf{v}$ of dimension $(128+6) \times (128+6) \times 50$, and we do the following for the pixels on the left and above to obtain $128 \times 128 \times 50$ $\mathbf{h}$:

```
h             =  v(1:128,1:128,:)
h(1:6,:,:)    =  h(1:6,:,:) + v(129:end,1:128,:)
h(:,1:6,:)    =  h(:,1:6,:) + v(1:128,129:end,:)
h(1:6,1:6,:)  =  h(1:6,1:6,:) + v(129:end,129:end,:)
```

4. In this way, each pixel value of $\mathbf{h}$ undergoes 7^2 operations, and we can take the average to get the finally denoised results ($\mathbf{h} = \mathbf{h}/7^2$).