Seismic imaging with generalized Radon transforms: stability of the Bolker Condition

Todd Quinto

Department of Mathematics
Tufts University

Joint work with Christine Grathwohl, Peer Kunstmann, and Andreas Rieder
(Karlsruhe Institute of Technology)

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An inverse problem for the acoustic wave equation

$u(t; \mathbf{x}, \mathbf{x_s})$ acoustic potential at $\mathbf{x} \in \mathbb{R}^d$, $d \in \{2, 3\}$, at time $t \geqslant 0$

$$\frac{1}{v_p^2} \partial_t^2 u - \Delta_{\mathbf{x}} u = \delta(\mathbf{x} - \mathbf{x_s})\delta(t)$$

$v_p = v_p(\mathbf{x})$ speed of sound, $\mathbf{x_s}$ excitation (source) point.

Seismic imaging of backscattered (reflected) fields $u(t; \mathbf{x_r}, \mathbf{x_s})$, $t \in [0, T_{\max}]$, $(\mathbf{x_r}, \mathbf{x_s}) \in \mathcal{R} \times \mathcal{S}$ where $\mathcal{R}/\mathcal{S}$ sets of receiver/source points, and $T_{\max}$ observation period.

Simplifying ansatz

Consider the ansatz

$$\frac{1}{v_p^2(\mathbf{x})} = \frac{1 + n(\mathbf{x})}{v^2(\mathbf{x})},$$

with v_p the actual sound speed (high frequency content n) and $v = v(\mathbf{x})$ smooth and known background velocity, $v_p \sim v$.

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Relation between velocity and travel time:

$$|\nabla_{\mathbf{x}}\tau| = v^{-1} \qquad \tau(\mathbf{x_s}, \mathbf{x_s}) = 0.$$

In general, you know v and need to find τ

Determining n using the Ansatz

To determine n, we use a reduction from the solution to the wave equation to an expression for n (e.g. SYMES 1998):

$$Fn(t; \mathbf{x_r}, \mathbf{x_s}) = \int \frac{n(\mathbf{x})}{v^2(\mathbf{x})} A(\mathbf{x}, \mathbf{x_s}) A(\mathbf{x}, \mathbf{x_r}) \delta\big(t - \tau(\mathbf{x_s}, \mathbf{x}) - \tau(\mathbf{x}, \mathbf{x_r})\big) \mathrm{d}\mathbf{x}$$

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Note: F is a Radon transform that, for each t, integrates n over reflection isochrones (surfaces of constant travel time t from $\mathbf{x_s}$ to $\mathbf{x}$ to $\mathbf{x_r}$): $t = \tau(\mathbf{x_s},\mathbf{x}) + \tau(\mathbf{x},\mathbf{x_r})$.

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Under certain conditions F is an Fourier integral operator.

Microlocal and other properties of F and reconstruction methods in various geometric settings have been studied by many authors, such as:

BEYLKIN 1985, RAKESH 1988, NOLAN/SYMES 1997, ... etc.

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A series of works with Christine Grathwohl, Peer Kunstmann, and Andreas Rieder (working with geophysicists at KIT)

We now talk about some of our recent results!

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- *Common offset data acquisition geometry:* Let $\alpha \geqslant 0$ be the common offset. Then, sources and receivers on the surface are parameterized by $s \in \mathbb{R}$ and

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Some geophysicists approximate known background velocity in layered media by an affine speed.

The 2D situation for *affine* velocity $v(x_1, x_2) = b + ax_2$ for a and b positive

Travel time: $\tau((r,0)(x_1,x_2)) = \frac{1}{a}\,\mathrm{acosh}\left(1 + \frac{a^2}{2b}\,\frac{(x_1-r)^2+x_2^2}{b+ax_2}\right)$

See Lesson 41 in [Slotnick, Lessons in Seismic Computing, 1959].

The 2D situation for *affine* velocity $v(x_1, x_2) = b + ax_2$ for a and b positive and offset $\alpha \geqslant 0$.

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Domains: $X = \{\mathbf{x} \in \mathbb{R}^2 : x_2 > x_{\min}\}$ and $Y = S \times]t_{\min}, \infty[$

where S is an open subset of $\mathbb{R}$ and

$$x_{\min} := \frac{b}{a}\left(\sqrt{1 + \frac{a^2\alpha^2}{b^2}} - 1\right), \quad t_{\min} := \frac{2}{a}\,\mathrm{asinh}\left(\frac{a\,\alpha}{b}\right),$$

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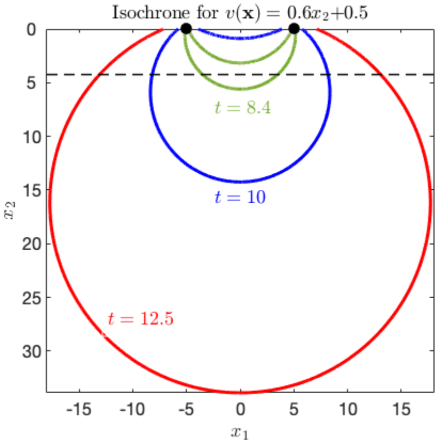
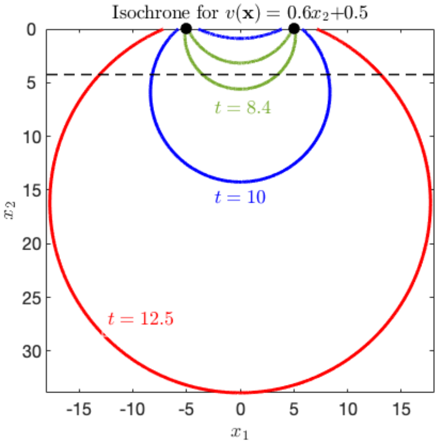
- For points with $x_2 < x_{\min}$ the operator F is not well-behaved. . .



Isochrones $\mathcal{L}_{0,t}$ for $t \in \{8.4, 10, 12.5\}$ for wave speed $v(\mathbf{x}) = 0.5 + 0.6\, x_2$.



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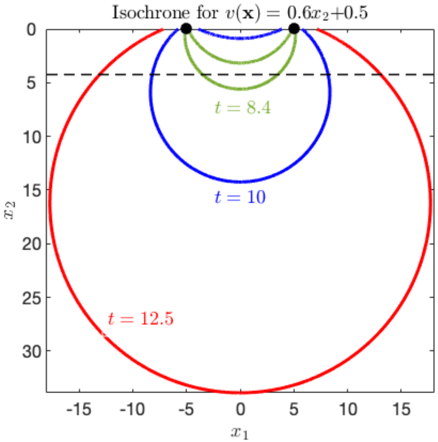

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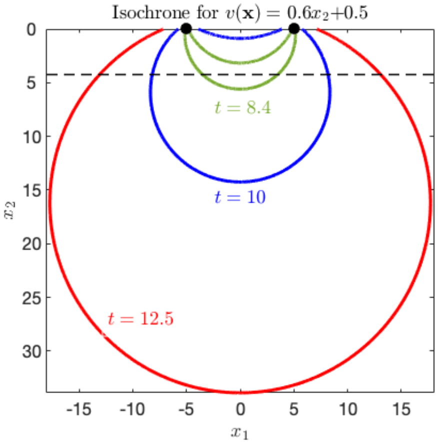

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Warning: Points with $x_2 < x_{\text{min}}$ can intersect the "top" of small isochrones, causing added artifacts in the normal operator (mirror points).

F_α as a FIO with affine wave speed and $\alpha \geqslant 0$

$$\varphi_\alpha(s, \mathbf{x}) = \tau((s-\alpha, 0), \mathbf{x}) + \tau(\mathbf{x}, (s+\alpha, 0)),$$

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Moral: When Bolker holds, the reconstruction operator images (maybe only some) singularities of f but **does not add artifacts.**

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Conjecture: For *some?* positive offsets, $\alpha > 0$, and affine wave speed, we conjecture that F_α satisfies the Bolker assumption.

The answer from [KQR 2023]:

Close enough phase functions give similar operators: Let F_j, $j = 0, 1$ be FIO from $\mathcal{D}'(X)$ to $\mathcal{E}'(S')$ with nondegenerate phase functions
$\Phi_j = \omega(t - \tau_j(\mathbf{x_s}(s), \mathbf{x}) - \tau_j(\mathbf{x}, \mathbf{x_r}(s))) = \omega(t - \varphi_j(s, \mathbf{x}))$.

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Proof: If α is close to zero, then phase function ϕ_α is close to ϕ_0.

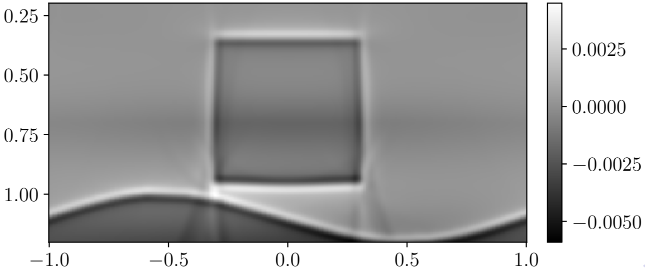
Using the symbol to get a reconstruction operator

For affine velocity, we develop a reconstruction operator (like [GKQR 2018])

$$\Lambda(f) = \Delta F_0^\dagger \psi F_0 f$$

where ψ is a cutoff so $F_0^\dagger$ and F_0 can be composed and $F_0^\dagger$ is a backprojection with weight of $F_0^\dagger$ adjusted to provide a more uniform symbol in depth and so a reconstruction with intensity less dependent on depth.

Here is a reconstruction where the data are generated using the wave equation, not Radon data. Unperturbed sound speed is $v(x_2) = 1.0 + 0.5x_2$.



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Thank you for your attention!