Data assimilation from a viewpoint of regularization theory

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Special Semester: Tomography Across the Scales
Workshop 5: Inverse Problems on Large Scales
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This talk is based on joint work with

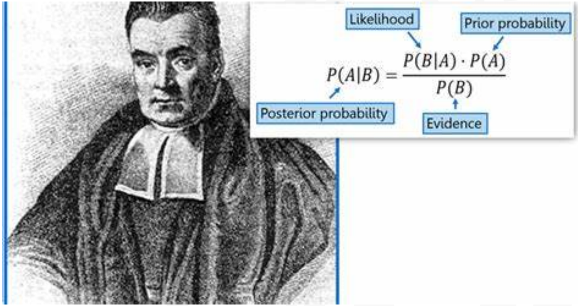
J. Cheng, L. Ding, M. Iglesias, L. Lin, P. Niu, A. Stuart, F. Werner.

Outline

Motivation: history

Bayes' theorem

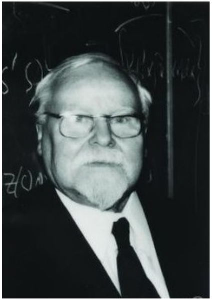
Proposed by Thomas Bayes and summarized by Richard Price (1763)



Motivation: history

Tikhonov–Phillips regularization

Independently by Andrey Tikhonov (1963) and David L. Phillips (1962)



$$T\varphi = g$$

$$\varphi_\alpha^\delta \mapsto \min J_\alpha(\varphi) := \|T\varphi - g^\delta\|^2 + \alpha\|\varphi - \varphi_0\|^2$$

Motivation: history

Kalman filter

By Rudolph Kalman (1960)

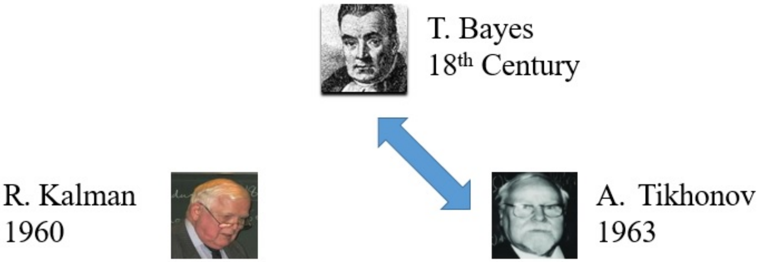
$$v_{j+1} = \Psi(v_j) + \xi_j, \quad j \in \mathbb{Z}^+, \quad v_0 \sim N(m_0, C_0),$$
$$y_{j+1} = h(v_{j+1}) + \eta_{j+1}, \quad j \in \mathbb{Z}^+.$$

1. $v = \{v_j\}_{j \in \mathbb{J}_0}$;
2. $y = \{y_j\}_{j \in \mathbb{J}}$;

find the posterior pdf $\mathbb{P}(v|y)$.

What we know and what we want to know

	Deterministic		Statistical
Steady equations	Regularization	$\Longleftrightarrow$	Bayesian Approach
Dynamic system	??	$\Longleftrightarrow$	Filter based methods



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Filter based methods:

- Kalman Filter, 3DVAR (Online);
- 4DVAR (Offline);
- Kalman-Bucy Filter (Online, continuous dynamic).
- Ensemble Kalman Filter, Particle Filter (Online);

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Filter based methods:

- Kalman Filter, 3DVAR (Online);
- 4DVAR (Offline);
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- Ensemble Kalman Filter, Particle Filter (Online);

Task: deriving optimal error estimates and asymptotic behavior.

Linear Inverse Problems

We consider the linear inverse problem

$$y = Au^{\dagger} + \eta,$$

- A is an injective linear bounded operator acting from a Hilbert space X to Y.
- The exact solution $u^{\dagger} \in X$.
- Noise $\eta \sim \mathcal{N}(0, \gamma^2 I)$.

Artificial Dynamic

Dynamic System

$$u_n = u_{n-1}$$
$$y_n = Au_n + \eta_n,$$

Artificial Dynamic

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Inverse problems

- Steady setting: reconstruct the unknown solution. Method: regularization schemes.
- Artificial dynamic: identify the stationary state and quantify the uncertainty. Method: filter based methods.

M. A. Iglesias, K. J. Law, and A. M. Stuart, *Ensemble Kalman methods for inverse problems*, Inverse Problems **29** (2013), no. 4, 045001.

Kalman Filter

Prediction step:

$$\begin{array}{rcl} \mathcal{N}(m_{n-1}, C_{n-1}) & \to & \mathcal{N}(\hat{m}_n, \hat{C}_n) \\ \pi(u_{n-1}|y_1, \dots, y_{n-1}) & & \pi(u_n|y_1, \dots, y_{n-1}) \end{array}$$

Kalman Filter

Prediction step:

$$\mathcal{N}(m_{n-1}, C_{n-1}) \quad \rightarrow \quad \mathcal{N}(\hat{m}_n, \hat{C}_n)$$

$$\pi(u_{n-1}|y_1, \ldots, y_{n-1}) \qquad\qquad \pi(u_n|y_1, \ldots, y_{n-1})$$

Analysis step:

$$\mathcal{N}(\hat{m}_n, \hat{C}_n) \quad \rightarrow \quad \mathcal{N}(m_n, C_n)$$

$$\pi(u_n|y_1, \ldots, y_{n-1}) \qquad\qquad \pi(u_n|y_1, \ldots, y_{n-1}, \mathbf{y_n})$$

K. J. Law, A.M. Stuart, and K.C. Zygalakis, *Data Assimilation: A Mathematical Introduction*, Springer, 2015.

Kalman Filter

Dynamic System

$$u_n = Du_{n-1}$$
$$y_n = Hu_n + \eta_n,$$

Kalman Filter

Dynamic System

$$u_n = Du_{n-1}$$
$$y_n = Hu_n + \eta_n,$$

Prediction:

$$\begin{aligned} \hat{m}_n &= Dm_{n-1} \\ \hat{C}_n &= DC_{n-1}D^*. \end{aligned}$$

Analysis:

$$\begin{aligned} m_n &= \hat{m}_n + K_n(y_n - H\hat{m}_n) \\ C_n &= (I - K_nH)\hat{C}_n, \end{aligned}$$

with Kalman gain

$$K_n = \hat{C}_n H^T \left(H\hat{C}_n H^T + \gamma^2 I \right)^{-1}.$$

Kalman Filter for Artificial Dynamic

Kalman filter

Let $D = I$ and $H = A$. Take $\mathcal{N}(m_0, C_0)$ with $C_0 = \frac{\gamma^2}{\alpha}\Sigma_0$ as the initial guess

$$
\begin{aligned}
K_n &= C_{n-1}A^*\left(AC_{n-1}A^* + \gamma^2 I\right)^{-1} \\
m_n &= m_{n-1} + K_n(y_n - Am_{n-1}) \\
C_n &= (I - K_nA)C_{n-1}.
\end{aligned}
$$

3DVAR for Artificial Dynamic

3DVAR

Let $D = I$ and $H = A$. Take $\mathcal{N}(m_0, C_0)$ with $C_0 = \frac{\gamma^2}{\alpha}\Sigma_0$ as the initial guess and fix the Kalman gain

$$
\begin{aligned}
K_n &\equiv \mathcal{K} := C_0 A^* \left(A C_0 A^* + \gamma^2 I\right)^{-1} \\
\zeta_n &= \zeta_{n-1} + \mathcal{K}(y_n - A\zeta_{n-1}) \\
\mathscr{C}_n &\equiv (I - \mathcal{K} A) C_0.
\end{aligned}
$$

Iterative Regularization Methods

Iterative Tikhonov Regularization

$$u_n = \arg\min \frac{1}{2}\|y - Au\|^2 + \frac{\alpha}{2}\|u - u_{n-1}\|^2,$$

or equivalently

$$u_n = u_{n-1} + (A^*A + \alpha I)^{-1} A^*(y - Au_{n-1}).$$

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or equivalently

$$u_n = u_{n-1} + (A^*A + \alpha I)^{-1}A^*(y - Au_{n-1}).$$

- Weighted operator in regularization:

$$(A^*A + \alpha I)^{-1}A^*;$$

- Kalman gain in filter based methods:

$$K_n = C_{n-1}A^*\left(AC_{n-1}A^* + \gamma^2 I\right)^{-1}.$$

[Tikhonov & Arsenin 1977], [Hofmann 1986], [Louis 1989], Tikhonov, Goncharsky, Stepanov & Yagola 1995], [Engl, Hanke & Neubauer 1996], [Nair 2009], [Kabanikhin 2012], [Lu & Pereverzev 2013] ...

Statistical Inverse Problems

Spectral methods of the form

$$u_\gamma^\delta = q_\gamma(A^*A)A^*y^\delta$$

Variational ones of the form

$$u_\gamma^\delta \in \arg\min_{u\in X}\left[\|Au\|_Y^2 - 2\left\langle Au, y^\delta\right\rangle_{Y\times Y^*} + \gamma R(u)\right]$$

Table 1. Rates for deterministic and stochastic model

	Deterministic	Stochastic
Direct	ε^2	$\varepsilon^{\frac{4\alpha}{2\alpha+1}}$
Polynomial	$\varepsilon^{\frac{4\alpha}{2\alpha+2\beta}}$	$\varepsilon^{\frac{4\alpha}{2\alpha+2\beta+1}}$
Exponential	$\varepsilon^{\frac{4\alpha}{2\alpha+2\beta}}$	$\varepsilon^{\frac{4\alpha}{2\alpha+2\beta}}$

L. Cavalier, *Ch.1 Inverse Problems in Statistics*, P. Alquier et al. (eds.), Inverse Problems and High-Dimensional Estimation, Lecture Notes in Statistics 203, Springer-Verlag Berlin Heidelberg 2011.

Data Model in Artificial Dynamics

Data model 1: updating observation

$$u_n = u_{n-1}$$
$$y_n = Au_n + \eta_n.$$

Here each η_n is i.i.d Gaussian noise drawn from $\mathcal{N}(0, \gamma^2 I)$.

Data Model in Artificial Dynamics

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Decreased uncertainty

Auxiliary element (the law of large number)

$$\bar{y} = \frac{1}{n}\sum_{j=1}^{n} y_j = Au^\dagger + \frac{1}{n}\sum_{j=1}^{n} \eta_j.$$

with $\bar{\eta} = \frac{1}{n}\sum_{j=1}^{n} \eta_j$ whose distribution is $\mathcal{N}(0, \frac{\gamma^2}{n} I)$.

B. T. Knapik; A. W. Van Der Vaart and J. H. Van Zanten, *Bayesian inverse problems with Gaussian priors*, The Annals of Statistics 39 (2011), 2626–2657.

Data Model in Artificial Dynamics

Data model 2: fixed observation

$$u_n = u_{n-1}$$
$$y_n = Au_n + \eta.$$

Here η is i.i.d Gaussian noise drawn from $\mathcal{N}(0, \gamma^2 I)$.

Data Model in Artificial Dynamics

Data model 2: fixed observation

$$u_n = u_{n-1}$$
$$y_n = Au_n + \eta.$$

Here η is i.i.d Gaussian noise drawn from $\mathcal{N}(0, \gamma^2 I)$.

Nonlinear problems extension:

- Deterministic: Levenberg–Marquardt Methods;
- Statistical: Extended Kalman Filter Method;
  Ensemble Kalman Filter Method.

M. A. Iglesias, K. J. Law, and A. M. Stuart, *Ensemble Kalman methods for inverse problems*, Inverse Problems **29** (2013), no. 4, 045001.

Outline

General Filter Properties

General formulae:

$$\text{Updated mean: } m_n = (I - K_n A) m_{n-1} + K_n y_n;$$

$$\text{True solution: } u^\dagger = (I - K_n A) u^\dagger + K_n A u^\dagger.$$

General Filter Properties

Bias–Variance Decomposition:

Total error $e_n := m_n - u^\dagger$:

$$e_n = \prod_{j=1}^{n}(I - K_jA)e_0 + \sum_{j=1}^{n-1}\left(\prod_{i=n-j}^{n-1}(I - K_{i+1}A)\right)K_{n-j}\eta_{n-j} + K_n\eta_n$$
$$:= J_1 + J_2$$

with an initial guess m_0 and

$$e_0 = m_0 - u^\dagger;$$
$$J_1 = \prod_{j=1}^{n}(I - K_jA)e_0 \qquad \text{(Bias)};$$
$$J_2 = \sum_{j=1}^{n-1}\left(\prod_{i=n-j}^{n-1}(I - K_{i+1}A)\right)K_{n-j}\eta_{n-j} + K_n\eta_n \qquad \text{(Variance)}.$$

Mean Squared Error

Mean squared error:

$$\mathbb{E}\|m_n - u^\dagger\|^2 = \|J_1\|^2 + \mathbb{E}\|J_2\|^2.$$

Mean Squared Error

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Key formula:

Kalman filter:

$$\prod_{j=1}^{n}(I - K_j A) = C_n C_0^{-1} = (C_0^{-1} + nA^*A/\gamma^2)^{-1} C_0^{-1}$$
$$= C_0^{\frac{1}{2}} \gamma^2 (\gamma^2 I + n C_0^{\frac{1}{2}} A^* A C_0^{\frac{1}{2}})^{-1} C_0^{-\frac{1}{2}}.$$

3DVAR:

$$(I - \mathscr{K} A)^n = C_0^{\frac{1}{2}} \gamma^2 (\gamma^2 I + C_0^{\frac{1}{2}} A^* A C_0^{\frac{1}{2}})^{-n} C_0^{-\frac{1}{2}}.$$

Main Assumptions

Assumption 1

1. the initial variance $C_0 = \frac{\gamma^2}{\alpha}\Sigma_0$ and $\mathscr{R}(\Sigma_0^{1/2}) \subset \mathscr{D}(A)$, where α is a positive constant and Σ_0 is positive self-adjoint, and Σ_0^{-1} is a densely defined unbounded self-adjoint strictly positive operator;
2. the forward operator A satisfies

$$C^{-1}\|\Sigma_0^{\frac{a}{2}} x\| \leq \|Ax\| \leq C\|\Sigma_0^{\frac{a}{2}} x\|$$

   on X for some constants $a > 0$ and $1 \leq C < \infty$;
3. the initial mean satisfies $m_0 - u^\dagger \in \mathscr{D}(\Sigma_0^{-\frac{s}{2}})$ (or $\zeta_0 - u^\dagger \in \mathscr{D}(\Sigma_0^{-\frac{s}{2}})$) with $0 \leq s \leq a+2$;
4. the operator Σ_0 in item 1. is trace-class on X.

H. W. Engl; M. Hanke and A. Neubauer, *Regularization of inverse problems*. Kluwer Academic Publishers Group, Dordrecht, 1996.

Main Assumptions

Assumption 2

1. Let the initial variance $C_0 = \frac{\gamma^2}{\alpha}\Sigma_0$. The operators Σ_0 and A^*A have the same eigenfunctions $\{e_i\}$ with their eigenvalues $\{\lambda_i\}$ and $\{\kappa_i^2\}$ satisfying

$$\lambda_i = i^{-1-2\varepsilon}, \qquad C^{-1}i^{-p} \le \kappa_i \le Ci^{-p}$$

   for some $\varepsilon > 0$, $p > 0$ and $C \ge 1$.

2. By choosing the initial mean $m_0 = 0$, the true solution $u^\dagger$ with its coordinates $\{u^{\dagger,i}\}$ in the basis $\{e_i\}$ obeys $\sum_{i=1}^{\infty}(u^{\dagger,i})^2 i^{2\beta} < \infty$.

Assumptions 1 and 2 are identical if $a = \frac{2p}{1+2\varepsilon}$ and $s = \frac{2\beta}{1+2\varepsilon}$.

B. T. Knapik; A. W. Van Der Vaart and J. H. Van Zanten, *Bayesian inverse problems with Gaussian priors*, The Annals of Statistics 39 (2011), 2626–2657.

Asymptotic Analysis

Setup revisit:

Two data models: Updating or fixed observation;

Two assumptions: General operators or SVD systems;

Asymptotic Analysis

Setup revisit:

Two data models: Updating or fixed observation;

Two assumptions: General operators or SVD systems;

Initial values revisit:

Initial mean: $m_0 = 0$;

Initial variance: $C_0 = \frac{\gamma^2}{\alpha}\Sigma_0$ with noise variance $\gamma^2 I$ and an operator Σ_0. The parameter α is a tuning parameter.

Kalman Filter: Data Model 1

Kalman filter (Assumption 1)

Let Assumption 1 hold. Then the Kalman filter method yields a bias-variance decomposition of the mean squared error

$$\mathbb{E}\|m_n - u^\dagger\|^2 \leq C\left(\frac{\alpha}{n}\right)^{\frac{s}{a+1}} + \frac{\gamma^2}{\alpha}\mathrm{tr}(\Sigma_0)$$

for the Data Model 1. Setting $\alpha = N^{\frac{s}{s+a+1}}$ and stopping the iteration when $n = N$ then gives

$$\mathbb{E}\|m_N - u^\dagger\|^2 \leq \left(C + \gamma^2\mathrm{tr}(\Sigma_0)\right)N^{-\frac{s}{s+a+1}}.$$

Asymptotical boundness of Kalman filter (for any fixed α and γ).

3DVAR: Data Model 1

3DVAR (Assumption 1)

Let Assumption 1 hold. Then 3DVAR filter yields a bias-variance decomposition of the mean squared error

$$\mathbb{E}\|\zeta_n - u^\dagger\|^2 \le C\left(\frac{\alpha}{n}\right)^{\frac{s}{a+1}} + C\frac{\gamma^2 \ln n}{\alpha}\mathrm{tr}(\Sigma_0)$$

for the Data Model 1. Setting $\alpha = N^{\frac{s}{s+a+1}}$ and stopping the iteration when $n = N$ then gives

$$\mathbb{E}\|\zeta_N - u^\dagger\|^2 \le C\left(1 + \gamma^2 \mathrm{tr}(\Sigma_0)\right) N^{-\frac{s}{s+a+1}} \ln N.$$

Blow up of 3DVAR (for any fixed α and γ).

Kalman Filter: Data Model 1

Kalman filter (Assumption 2)

Let Assumption 2 hold. Then the Kalman filter method yields a bias-variance decomposition of the mean squared error

$$\mathbb{E}\|m_n - u^\dagger\|^2 \leq C\left(\frac{\alpha}{n}\right)^{\frac{2\beta}{1+2\varepsilon+2p}} + \gamma^2 n^{-\frac{2\varepsilon}{1+2\varepsilon+2p}} \alpha^{-\frac{1+2p}{1+2\varepsilon+2p}}$$

for the Data Model 1. Setting $\alpha = N^{\frac{2(\beta-\varepsilon)}{1+2\beta+2p}}$ and stopping the iteration when $n = N$ then gives following minimax convergence rate:

$$\mathbb{E}\|m_N - u^\dagger\|^2 \leq CN^{-\frac{2\beta}{1+2\beta+2p}}.$$

Unconditional convergence of Kalman filter (for any fixed α and γ).

Kalman Filter: Data Model 2

Kalman Filter: Assumption 1

Let Assumption 1 hold. Then the Kalman filter method yields a bias-variance decomposition of the mean squared error

$$\mathbb{E}\|m_n - u^\dagger\|^2 \leq C\left(\frac{\alpha}{n}\right)^{\frac{s}{a+1}} + \frac{n\gamma^2}{\alpha}\mathrm{tr}(\Sigma_0)$$

for Data Model 2. Fix $\alpha = 1$ and assume that the noise variance $\gamma^2 = N^{-\frac{a+s+1}{a+1}}$. If the iteration is stopped at $n = N$ then following convergence rate is valid:

$$\mathbb{E}\|m_N - u^\dagger\|^2 \leq (C + \mathrm{tr}(\Sigma_0))\, N^{-\frac{s}{a+1}}.$$

Blow up of Kalman filter (for any fixed α and γ).

Outline

Linear Inverse Problems

We consider the linear inverse problem

$$y = Au^{\dagger} + \eta,$$

- A is an injective linear bounded operator acting from a Hilbert space X to Y.
- The exact solution $u^{\dagger} \in X$.
- Noise $\eta \sim \mathcal{N}(0, \varepsilon^2 I)$.

Towards a continuous analog

Artificial dynamical system (Data model 1)

$$u_n = u_{n-1}$$
$$y_n = Au_n + \eta_n$$

with $\eta_n \sim \mathcal{N}(0, \tau^{-1}\Sigma)$.

The continuous analog

Denote $z_1, ..., z_n$ be equidistant (approximate) samples of a random process z such that $y_n = \left(\frac{z_n - z_{n-1}}{\tau}\right)$ with a time step $\tau > 0$, let $\tau \to 0$ and

$$\mathrm{d}u = 0, \quad u(0) = u^\dagger;$$
$$\mathrm{d}z = Au\mathrm{d}t + \sqrt{\Sigma}\mathrm{d}W, \quad z(0) = 0.$$

Consistency check

Original steady problem

$$y^\delta = Au^\dagger + \delta\eta$$

with a noise level δ and we assume $\delta \to \mathbf{0}$.

Consistency check

$$y^\delta := \frac{1}{T} z(T) = Au^\dagger + \frac{1}{T}\sqrt{\Sigma}\,(W(T) - W(0)),$$
$$\sqrt{\Sigma}\,(W(T) - W(0)) \sim \mathcal{N}_Y(0, T\Sigma).$$

Hence, the ending point of the observable process z carries the same information as the data observed in the original inverse problem with $\delta = \mathbf{1}/\sqrt{\mathbf{T}}$.

Kalman-Bucy filter: Non-stationary Asymptotical Regularization Method

The Kalman-Bucy filter:

$$\mathrm{d}m = CA^*\Sigma^{-1}(\mathrm{d}z - Am\mathrm{d}t), \quad m(0) = m_0;$$
$$\mathrm{d}C = -CA^*\Sigma^{-1}AC\mathrm{d}t, \quad C(0) = C_0.$$

Solve the Riccati equation to obtain

$$C(t) = (C_0^{-1} + tA^*\Sigma^{-1}A)^{-1}, \quad t > 0.$$

Thus the Kalman-Bucy filter is

$$\mathrm{d}m = (C_0^{-1} + tA^*\Sigma^{-1}A)^{-1}A^*\Sigma^{-1}(\mathrm{d}z - Am\mathrm{d}t), \quad m(0) = m_0.$$

3DVAR: Stationary Asymptotical Regularization Method

$$\mathrm{d}\zeta = \mathscr{C}A^*\Sigma^{-1}(\mathrm{d}z - A\zeta\mathrm{d}t), \quad \zeta(0) = m_0,$$
$$\mathrm{d}\mathscr{C} = 0, \quad \mathscr{C}(0) = C_0.$$

Non-stationary (and stationary) ARMs

Non-stationary ARM (Kalman-Bucy filter)

$$(u-m)(t) = e^{-\int_0^t (C_0^{-1}+sA^*\Sigma^{-1}A)^{-1}A^*\Sigma^{-1}A\mathrm{d}s}(u-m)(0)$$
$$-\int_0^t e^{-\int_s^t (C_0^{-1}+\tau A^*\Sigma^{-1}A)^{-1}A^*\Sigma^{-1}A\mathrm{d}\tau}(C_0^{-1}+sA^*\Sigma^{-1}A)^{-1}A^*\Sigma^{-1/2}\mathrm{d}W(s).$$

Stationary ARM (3DVAR)

$$(u-\zeta)(t) = e^{-C_0A^*\Sigma^{-1}At}(u-\zeta)(0) - \int_0^t e^{-\int_s^t C_0A^*\Sigma^{-1}A\mathrm{d}\tau}C_0A^*\Sigma^{-1/2}\mathrm{d}W(s).$$

Niu, Lu and Cheng [IPI 2019]

Main ingredients

Effective dimension

$$\mathscr{D}(\gamma) = \mathscr{D}_B(\gamma) := \operatorname{tr}\left((\gamma I + B^*B)^{-1} B^*B\right), \gamma > 0.$$

Stochastic integration (Infinite-dimensional Itô-isometry)

The stochastic integral $\Phi \to \int_0^t \Phi(s) \mathrm{d}\mathscr{W}(s)$ with respect to a Y-valued Q-Wiener process $\mathscr{W}(s)$ satisfies

$$\mathbb{E}\left\|\int_0^t \Phi(s) \mathrm{d}\mathscr{W}(s)\right\|_X^2 = \mathbb{E}\int_0^t \|\Phi(s)\|^2_{\mathscr{L}_2(Y_Q, X)} \mathrm{d}s < \infty$$

for $t \in [0, T]$.

L. Gawarecki and V. Mandrekar, *Stochastic Differential Equations in Infinite Dimensions with Applications to Stochastic Partial Differential Equations*. Springer-Verlag Berlin Heidelberg 2011.

Convergence rate of Non-stationary ARM

Let appropriate Assumptions hold, then the non-stationary ARM yields MSE estimates

- If the function $\lambda \mapsto \phi(\lambda)/\lambda^{p+1}$ is non-increasing, then

$$\mathbb{E}\|m(t)-u^\dagger\|^2 \le \phi^2\left(\left(\frac{\alpha}{t}\right)^{\frac{1}{p+1}}\right) + \alpha^{-\frac{1}{p+1}} t^{-\frac{p}{p+1}} \mathscr{D}\left(\frac{\alpha}{t}\right)$$

  for all $0 \le t \le T$.

- If there is a constant $c < \infty$ with $\phi(\lambda) \le c\lambda^{p+1}$ as $\lambda \to 0$, then

$$\mathbb{E}\|m(t)-u^\dagger\|^2 \le c\left(\frac{\alpha}{t}\right)^2 + \alpha^{-\frac{1}{p+1}} t^{-\frac{p}{p+1}} \mathscr{D}\left(\frac{\alpha}{t}\right)$$

  for all $0 \le t \le T$.

Convergence rate of Stationary ARM

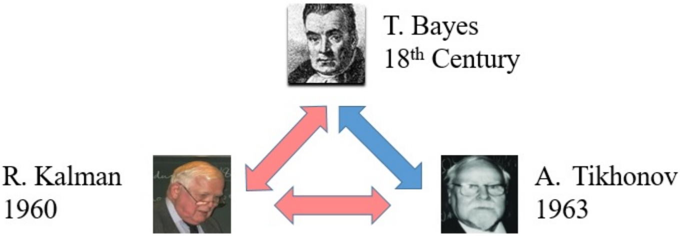
Let appropriate Assumptions hold, then the stationary ARM yields the MSE estimate

$$\mathbb{E}\|\zeta(t) - u^\dagger\|^2 \leq c\phi^2\left(\left(\frac{\alpha}{t}\right)^{\frac{1}{p+1}}\right) + \frac{1}{2}\alpha^{-1}\mathrm{tr}\,(\Omega)$$

for all $0 \leq t \leq T$ with a constant $c = \max\left\{(v_0/(p+1))^{v_0/(p+1)}, 1\right\}$.

Answers to what we want to know!

	Deterministic		Statistical
Steady equations	Tikhonov	$\Longleftrightarrow$	Bayesian approach
Discrete Dynamic	Tikhonov	$\Longleftrightarrow$	Kalman filter
	Iterative Tikhonov	$\Longleftrightarrow$	3DVAR
	Tikhonov	$\Longleftrightarrow$	4DVAR (offline)
Continuous Dynamic	**Tikhonov**	$\Longleftrightarrow$	**Kalman-Bucy filter**
	Showalter	$\Longleftrightarrow$	**3DVAR**



Outline

When inverse problems meet

- Randomness (Statistical inverse problems)
- Dynamic systems (Online inversion algorithms)

[Kügler 2008], [Boiger & Kaltenbacher 2016], [Parzer & Scherzer 2022] ...

Thank you for your attention and have a good time in Linz!

Iglesias, Lin, Lu and Stuart [CMS 2017]; Ding, Lu and Cheng [J. Complexity 2018];

Niu, Lu and Cheng [IPI 2019]; **Lu, Niu and Werner [JUQ 2021]**; Wang, Han and Lu [Q J R Meteorol Soc 2021].