Fractional graph Laplacian for image reconstruction

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Joint work with
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Outline

Outline

The model problem

We consider image reconstruction problems of the form

$$A\mathbf{x} + \boldsymbol{\eta} = \mathbf{b}^{\delta}, \tag{1}$$

where

- $A \in \mathbb{R}^{m \times n}$ is a severely ill-conditioned matrix (the singular values decay to zero rapidly and with no significant gap)
- $\boldsymbol{\eta} \in \mathbb{R}^{m}$ is the noise vector
- $\mathbf{b}^{\delta} \in \mathbb{R}^{m}$ is the measured data
- $\mathbf{x} \in \mathbb{R}^{n}$ is the unknown image to recover

Applications

- Image deblurring
- Computer tomography

The regularized $\ell^2 - \ell^q$ minimization problem

A regularized solution can be obtained by solving the $\ell^2 - \ell^q$ minimization problem

$$\min_{\mathbf{x}} \frac{1}{2} \left\| A\mathbf{x} - \mathbf{b}^{\delta} \right\|_2^2 + \frac{\mu}{q} \left\| L\mathbf{x} \right\|_q^q \tag{2}$$

where $0 < q \leq 2$, $L \in \mathbb{R}^{s \times n}$, $\mu > 0$, assuming that

$$\mathcal{N}(A) \cap \mathcal{N}(L) = \{0\}$$

Choice of L

- Usually is the discretization of a differential operator or a wavelet transform
- We consider the fractional graph Laplacian: $L \to L_{\omega}^{\alpha}, \alpha > 0$.

Outline

Weighted Graph

A weighted graph is represented by a pair $\omega = (\mathcal{V}, \mathcal{E})$, where

- $\mathcal{V}$ is the set of vertices
- $\mathcal{E} \subset \mathcal{V} \times \mathcal{V} \times \mathbb{R}$ associates the positive weight $\omega_{(i,j)}$ at the edge (i, j)

The adjacency matrix

Let $n = |\mathcal{V}|$, the number of nodes. The adjacency matrix $G \in \mathbb{R}^{n\times n}$ is

$$G_{i,j} = \begin{cases} \omega_{(i,j)} & \text{if } (i, j, \omega_{(i,j)}) \in \mathcal{E}, \\ 0 & \text{otherwise,} \end{cases}$$

A graph is undirected $\Leftrightarrow$ G is symmetric

Graph Laplacian

- Let G be symmetric
- Denote by D the (degrees) diagonal matrix

$$D_{j,j} = \sum_{i=1}^{n} G_{i,j}$$

and define the graph Laplacian L_ω as

$$L_\omega = \frac{D - G}{\|G\|_F}$$

where $\|\cdot\|_F$ is the Frobenius norm

Properties of L_ω

- L_ω is symmetric positive definite
- $\mathcal{N}(L_\omega) \supseteq \text{span}\{\mathbf{1}\}$

Construction of the Graph

Construction of a weighted and undirected graph for the image $X \in \mathbb{R}^{d \times d}$

- The *n*odes are the pixels of $X \Rightarrow n = d^2$
- To obtain a sparse matrix L_ω, connect two pixels $X_\mathbf{i}$ and $X_\mathbf{j}$ only if they are "spatially" close enough: $\|\mathbf{i} - \mathbf{j}\|_\infty \leq R$
- A common weight function is the Gaussian function: the weights depend on how similar the values (the intensity of two pixels) are

$$G_{\mathbf{i},\mathbf{j}} = \begin{cases} \mathrm{e}^{-(X_\mathbf{i} - X_\mathbf{j})^2/\sigma}, & \text{if } 0 < \|\mathbf{i} - \mathbf{j}\|_\infty \leq R,\ \mathbf{i} \neq \mathbf{j}, \\ 0, & \text{otherwise,} \end{cases}$$

- For instance, we use $R = 5$ and $\sigma = 10^{-3}$
- Applications
  - Denoising: [Pang and Cheung, IEEE Trans. Image Process. 2017]
  - Segmentations: [Calatroni et al., JMIV 2017]
  - Deblurring: [Bianchi et al., ETNA 2021]

Computation of the image

The construction of the graph needs a good approximation of the true image!

- This is computed solving the $\ell_2 - \ell_2$ problem

$$\mathbf{x}_\mu = \arg\min_{\mathbf{x}} \left\| A\mathbf{x} - \mathbf{b}^\delta \right\|_2^2 + \mu \left\| L\mathbf{x} \right\|_2^2, \quad (3)$$

  where

- L is a discretization of the 2D gradient:

$$L = \begin{bmatrix} L_1 \otimes I \\ I \otimes L_1 \end{bmatrix} \in \mathbb{R}^{2n \times n}, \quad (4)$$

  where $\otimes$ is the Kronecker product, $I \in \mathbb{R}^{d \times d}$ is the identity matrix, and L_1 is the discretization of the first derivative.

- μ is determined by GCV
- The $\ell_2 - \ell_2$ problem in (3) is solved by the Generalized Krylov (GKS) method proposed in [Lampe et al. LAA 2012]. We fix the size of the Krylov subspace equal to 50 (accurate solution is not necessary).

Outline

Algorithm $\ell^2 - \ell^q$ with Graph Laplacian

Algorithm to solve $\ell^2 - \ell^q$ with L_ω:

1. Compute a cheap initial approximation $\mathbf{x}_\mu$ by GKS, where μ is estimated by GCV
2. Construct the Graph Laplacian matrix L_ω associated to $\mathbf{x}_\mu$
3. Solve the $\ell^2 - \ell^q$ problem (2) with L_ω by the MM-GKS (majorization-minimization GKS) method proposed in [Huang et al., BIT 2017]

Smoothed functional

For $q \leq 1$ the functional in (2) is non-smooth $\rightarrow$ smooth the functional.

Fix $\varepsilon > 0$ (i.e., $\varepsilon = 0.1$) and $\Phi_{q,\varepsilon}(t) = \left(\sqrt{t^2 + \varepsilon^2}\right)^q$ s.t.

$$\|\mathbf{x}\|_q^q \approx \sum_{i=1}^{n} \Phi_{q,\varepsilon}(x_i).$$

Define the functional

$$\mathcal{J}_\varepsilon(\mathbf{x}) := \frac{1}{2}\left\|A\mathbf{x} - \mathbf{b}^\delta\right\|_2^2 + \frac{\mu}{q}\sum_{i=1}^{n} \Phi_{q,\varepsilon}((L\mathbf{x})_i)$$

and replace the problem (2) with its smoothed version

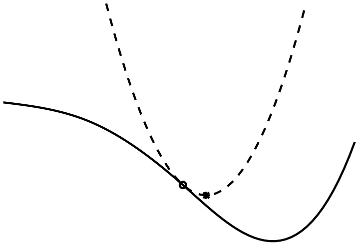
$$\min_{\mathbf{x}} \mathcal{J}_\varepsilon(\mathbf{x}) \tag{5}$$

solved by the majorization-minimization method.

Majorization-minimization method

For computing a sequence $\mathbf{x}^{(k)}$ that converges to a stationary point of $\mathcal{J}_\varepsilon$:

1. At each step the functional $\mathcal{J}_\varepsilon$ is majorized by a quadratic function $\mathcal{Q}(\mathbf{x}, \mathbf{x}^{(k)})$ that is tangent to $\mathcal{J}_\varepsilon$ at $\mathbf{x}^{(k)}$
2. $\mathbf{x}^{(k+1)}$ is the unique minimizer of $\mathcal{Q}(\mathbf{x}, \mathbf{x}^{(k)})$



The quadratic tangent majorant

- Define the quadratic tangent majorant of $\mathcal{J}_\varepsilon$ in $\mathbf{x}^{(k)}$ [Huang et al., BIT 2017]

$$\mathcal{Q}^F(\mathbf{x}, \mathbf{x}^{(k)}) = \frac{1}{2}\left\|A\mathbf{x} - \mathbf{b}^\delta\right\|_2^2 + \frac{\mu\varepsilon^{q-2}}{2}\left(\|L\mathbf{x}\|_2^2 \; -2\left\langle \boldsymbol{\omega}_F^{(k)}, L\mathbf{x}\right\rangle\right) + c,$$

where $\boldsymbol{\omega}_F^{(k)} = \mathbf{u}^{(k)}\left(1 - \left(\frac{(\mathbf{u}^{(k)})^2+\varepsilon^2}{\varepsilon^2}\right)^{q/2-1}\right)$ for $\mathbf{u}^{(k)} = L\mathbf{x}^{(k)}$

- Defining $\nu = \mu\varepsilon^{q-2}$, the next iterate $\mathbf{x}^{(k+1)}$ can be obtained by

$$\mathbf{x}^{(k+1)} = \arg\min_{\mathbf{x}} \left\| \begin{bmatrix} A \\ \nu^{1/2}L \end{bmatrix}\mathbf{x} - \begin{bmatrix} \mathbf{b}^\delta \\ \nu^{1/2}\boldsymbol{\omega}_F^{(k)} \end{bmatrix} \right\|_2^2 \qquad (6)$$

- An approximate solution of (6) can be computed by GKS
- ν is automatically estimated at each iteration by discrepancy principle:

$$\nu \rightarrow \nu_k$$

Minimization step

- Let $V_k \in \mathbb{R}^{n\times k}$ with orthonormal columns, we look for $\mathbf{y}^{(k+1)} \in \mathbb{R}^k$ s. t.

$$\mathbf{x}^{(k+1)} = V_k \mathbf{y}^{(k+1)}$$

- Using the economic QR factorizations

$$AV_k = Q_A R_A, \quad \text{and} \quad \left(W_{\text{reg}}^{(k)}\right)^{\frac{1}{2}} LV_k = Q_L R_L,$$

estimate μ_k by the discrepancy principle and solve

$$\mathbf{y}^{(k+1)} = \arg\min_{\mathbf{y}\in\mathbb{R}^k} \frac{1}{2} \left\| R_A \mathbf{y} - Q_A^T \mathbf{b}^\delta \right\|_2^2 + \frac{\mu_k}{2} \| R_L \mathbf{y} \|_2^2 .$$

- Enlarge the subspace by

$$V_{k+1} = \left[V_k, \frac{\mathbf{r}^{(k+1)}}{\|\mathbf{r}^{(k+1)}\|_2} \right]$$

where

$$\mathbf{r}^{(k+1)} = A^T \left(A\mathbf{x}^{(k+1)} - \mathbf{b}^\delta \right) + \lambda L^T W_{\text{reg}}^{(k)} L\mathbf{x}^{(k+1)}$$

Restarting

This restarting strategy is proposed in [Buccini and Reichel, APNUM in press]

- Most of the coefficients of $\mathbf{y}^{(k+1)}$ almost vanish and so restart V_k every r iterations: if $k \equiv 0 \mod r$ set
$$V_k = \mathbf{x}^{(k)}/\|\mathbf{x}^{(k)}\|$$
- It holds
$$\mathcal{J}_\varepsilon(\mathbf{x}^{(k+1)}) \leq \mathcal{J}_\varepsilon(\mathbf{x}^{(k)})$$
and exists a converging subsequence $\mathbf{x}^{(k_j)}$
- In practice, the entire sequence converges saving a lot of memory
- A standard choice is $r = 30$

Outline

Fractional Graph Laplacian

New step

The computed solution can be used to construct a new Graph Laplacian with enhanced diffusion $\Longrightarrow$ $L_\omega^\alpha, \ \alpha > 0$

- L_ω is sparse and positive definite
- Functions on L_ω can be well approximated in a small Krylov subspace by Lanczos method [Susnjara et al., arXiv 2015]
- The fractional graph Laplacian introduces non-local dynamics [Benzi et al., JCN 2020]
- Nonlinear fractional diffusion models for image reconstruction, see e.g. [Guo et al., IPI 2019], [Bahador et al., CMA 2022]

Computation of L_ω^α

- The step k of the MM-GKS method computes the product $L_\omega^\alpha V_k$

Precomputing phase

- For a small $m \in \mathbb{N}$, e.g., $m = 10$, compute the Lanczos factorization

$$L_\omega Q_m = Q_m T_m + \beta_m \mathbf{q}_{m+1} \mathbf{e}_m^T$$

- Compute the spectral decomposition of the projected matrix $T_m = U_m D_m U_m^T$ such that

$$L_\omega \approx Q_m U_m D_m U_m^T Q_m^T$$

- The product

$$L_\omega^\alpha V_k = [L_\omega^\alpha \mathbf{v}_1, \ldots, L_\omega^\alpha \mathbf{v}_k]$$

is approximated adding the new vector

$$L_\omega^\alpha \mathbf{v}_k \approx Q_m U_m D_m^\alpha U_m^T Q_m^T \mathbf{v}_k$$

Outline

Setting

Algorithm parameters

- Graph Laplacian: $R = 5$ and $\sigma = 10^{-3}$ (weights)
- MM-GKS: $\varepsilon = 0.1$ (smoothing) and $r = 30$ (restarting)

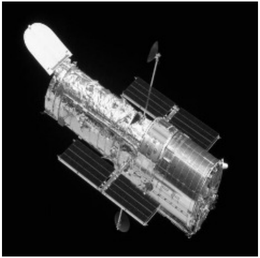
Model and methods

The same $\ell^2 - \ell^q$ problem in (2) with:

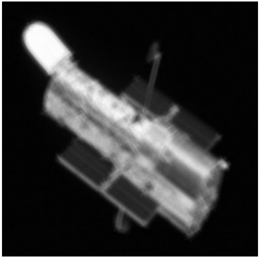
- q=0.1
- TV: MM-GKS with L defined as in (4) as proposed in [Buccini and Reichel, APNUM, in press]

Example Deblurring

- Hubble Space Telescope blurred and noisy (238×238)
- Gaussian noise such that $\|\boldsymbol{\eta}\|_2 = 0.01 \|\mathbf{b}\|_2$
- zero-Dirichlet boundary conditions



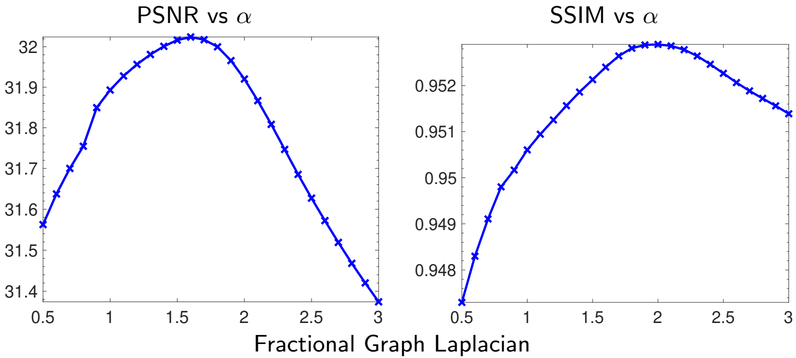
True image



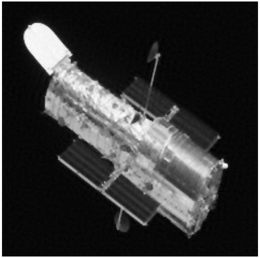
Observed image

Restoration Errors

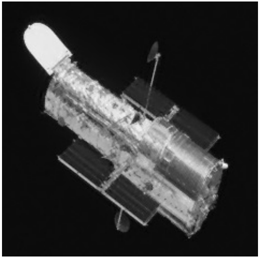
Method	RRE	PSNR	SSIM
$\ell_2 - \ell_2$	0.1318	27.49	0.8695
TV	0.0933	30.49	0.9114
Graph Laplacian	0.0857	31.23	0.9445
Fractional Graph Laplacian ($\alpha = 1.6$)	0.0782	32.02	0.9524



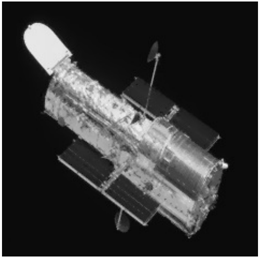
Restored images



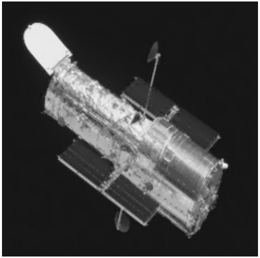
TV



$\alpha = 1$

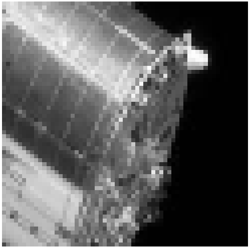


$\alpha = 1.5$

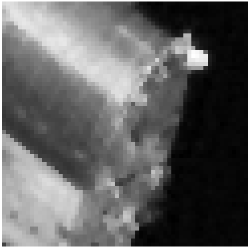


$\alpha = 2$

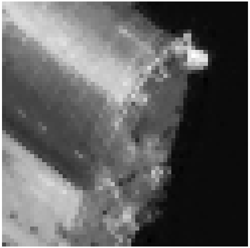
Zoom



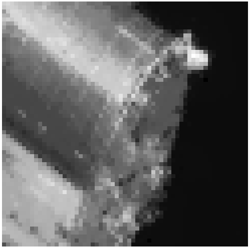
True



TV



$\alpha = 1$

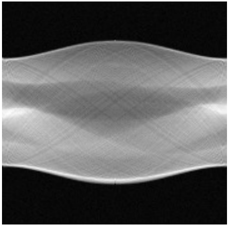


$\alpha = 1.8$

Example Tomography

- Shepp-Logan Phantom with 179 parallel beams at 180 equispaced angles between 0 and π
- Gaussian noise such that $\|\boldsymbol{\eta}\|_2 = 0.02\,\|\mathbf{b}\|_2$
- Example created using the IRtools toolbox [Gazzola, Hansen, Nagy, 2017]

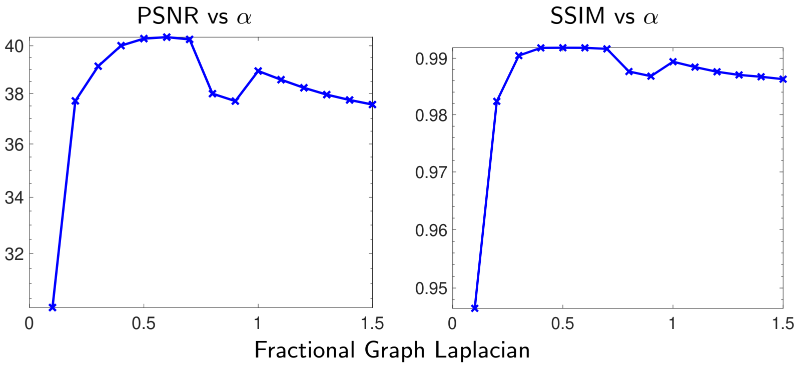
True image (128×128)



Sinogram (179×180)

Restoration Errors

Method	RRE	PSNR	SSIM
$\ell_2 - \ell_2$	0.1482	28.80	0.6333
TV	0.0544	37.51	0.9600
Graph Laplacian	0.0582	36.91	0.9844
Fractional Graph Laplacian ($\alpha = 0.6$)	0.0391	40.38	0.9918



Restored images

$\ell_2 - \ell_2$

TV

$\alpha = 1$

$\alpha = 0.5$

Outline

Conclusions and future work

The proposed algorithm

- use the fractional graph Laplacian as regularization operator in the $\ell^2 - \ell^q$ problem for imaging problems
- can be applied to large scale problems thanks to the projection into Krylov subspaces of fairly small dimension
- some parameters are fixed and other automatically estimated $\rightarrow$ no tuning is required for a fixed α

Research directions

- Automatic estimation and/or variable α
- Add the nonnegative constraint using [Buccini, Pasha, Reichel, 2021]
- Non-Gaussian noise $\rightarrow \ell^p - \ell^q$ problem

References

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Fraction Graph Laplacian for image reconstruction
in progress