Phase-field approaches in elastic inverse problems

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Joint work with E. Beretta, C. Cavaterra, E. Rocca, M. Verani

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- Detection of defects (cavities, inclusions, cracks) inside an elastic body from boudary measurements.

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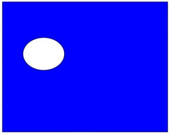

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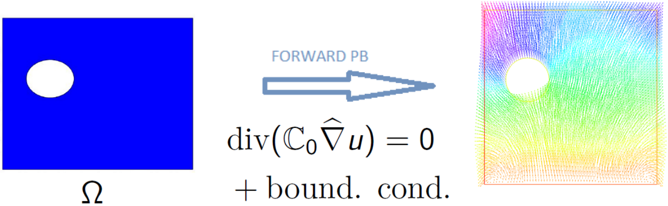

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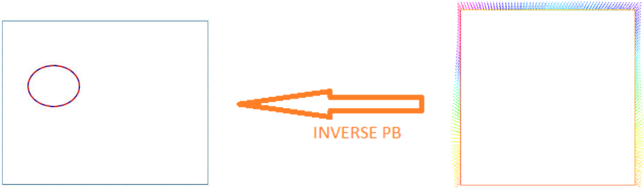

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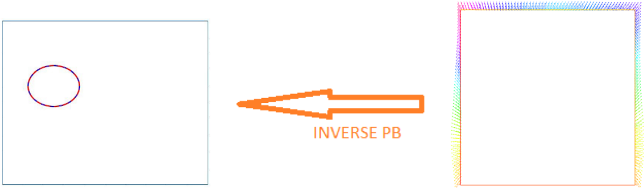
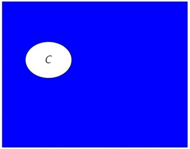


Possible applications: medical imaging, non-destructive testing of materials...

Detection of Cavities

- Ω is a bounded Lipschitz domain, $\partial\Omega := \Sigma_D \cup \Sigma_N$;

$$\begin{cases} \mathrm{div}(\mathbb{C}_0 \widehat{\nabla} u) = 0 & \text{in } \Omega \setminus \overline{C}, \\ (\mathbb{C}_0 \widehat{\nabla} u) n = 0 & \text{on } \partial C, \\ (\mathbb{C}_0 \widehat{\nabla} u) \nu = g & \text{on } \Sigma_N, \\ u = 0 & \text{on } \Sigma_D, \end{cases} \qquad (1)$$



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  - $\mathbb{C}_0$ is the fourth-order isotropic elasticity tensor, uniformly bounded, and strongly convex;
  - $C \Subset \Omega$ is a bounded Lipschitz domain (C= cavity);
  - $\widehat{\nabla} u = \frac{1}{2}(\nabla u + (\nabla u)^T)$;
  - $g \in L^2(\Sigma_N)$;

Forward Problem

Given $(C, \mathbb{C}_0, g) \rightsquigarrow$ find $u \in H^1_{\Sigma_D}(\Omega \setminus \overline{C})$.

Inverse Problem

Given $\mathbb{C}_0, g$, and u_m on $\Sigma_N \rightsquigarrow$ find C s.t. $u(C)\lfloor_{\Sigma_N} = u_m$ ($u(C)$ sol. to (1)).

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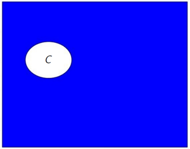

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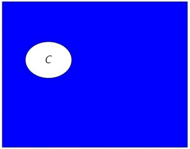

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Inverse pb: known results

- Uniqueness: a single pair of Cauchy data $\{g, u_m\}$ on Σ_N is sufficient to identify C, when
  - C is a Lipschitz domain;
  - $\mathbb{C}_0$ satisfies a $C^{0,1}$ regularity condition;

  (*Morassi-Rosset, Ang-Trong-Yamamoto, Lin-Wang-Nakamura,...*)

- Stability: very weak stability estimates (of log-log type) hold

$$d_H(C_1, C_2) \leq C(\log|\log(\|u_m^1 - u_m^2\|_{L^2(\Sigma_N)})|)^{-\eta},$$

$$\text{with} \quad C > 0 \quad \text{and} \quad 0 < \eta \leq 1$$

  when
  - C_1, C_2 are $C^{1,\alpha}$-domains;
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Remark: in analogy to the case of a scalar elliptic equation, the stability estimate is quite optimal.

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We set our analysis in the following framework

- Unknown: $C \in \mathcal{C} := \{C \subset \overline{\Omega} :$ compact, simply connected, with ∂C Lipschitz, and $dist(C, \partial\Omega) \geq d_0 > 0\}$;

Main issues

– Nonlinearity

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Reconstruction algorithms: level set methods, topological derivative, shape derivative, monotonicity method, method of fundamental solutions,...
(*Ameur-Burger-Hackl, Ammari-Kang-Nakamura-Tanuma, Belhachmi-Meftahi, Ben Abda-Jaïem-Khalfallah-Zine, Ben Abda-Jaïem-Mejri, Bonnet-Constantinescu, Burger, Carpio-Rapún, Doubova, Fernández-Cara, Eberle-Harrach,Ikehata-Itou, Kaltenbacher, Kang-Kim-Lee, Karageorghis-Lesnic-Ma, Martínez–Castro-Faris-Gallego, Mejri, Sherina,...*)

Variational Approach

- Approach the inverse problem as a minimization problem

$$\min_{C\in\mathcal{C}} J(C) = \underbrace{\frac{1}{2}\int_{\Sigma_N} |u(C) - u_{meas}|^2 \, d\sigma(x)}_{\text{Misfit functional}}$$

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Scherzer et al., *Variational Methods in Imaging*, Applied Mathematical Sciences 167, Springer, 2009.

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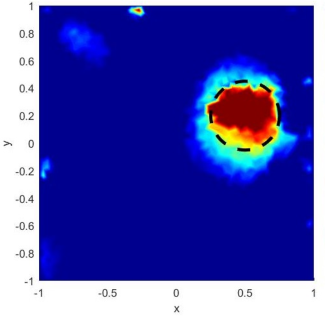

Poor reconstruction is clearly due to the ill-posedness of the inverse problem!

Scherzer et al., *Variational Methods in Imaging*, Applied Mathematical Sciences 167, Springer, 2009.

Variational Approach (cont.)

To mitigate the ill-posedness of the inverse problem a regularization term is needed.

- Add the perimeter of C as a regularization term in the functional (*Rondi, Deckelnick-Elliot-Styles, Beretta-Ratti-Verani, A.-Beretta-Cavaterra-Rocca-Verani*)

$$\min_{C\in\mathcal{C}} J_{reg}(C) = \underbrace{\frac{1}{2}\int_{\Sigma_N} |u(C)-u_{meas}|^2\, d\sigma(x)}_{\text{Misfit func.}} + \underbrace{\alpha \mathrm{Per}(C)}_{\text{Regularization func.}}$$

  - $u(C)$ is the solution to the boundary value problem (1);
  - $\alpha > 0$ is a regularization parameter;
  - $\mathrm{Per}(C)$ is the perimeter of C.

Analytical Results

$$\min_{C\in\mathcal{C}} J_{reg}(C) = \frac{1}{2}\int_{\Sigma_N} |u(C) - u_{meas}|^2 \, d\sigma(x) + \alpha \mathrm{Per}(C)$$

- Continuity properties of $u(C)$ with respect to perturbations of C;

Consequences:

- Existence of minima for $J_{reg}(C)$;
- Stability with respect to noisy data;
- Convergence of minimizers when $\alpha(\eta) = o(1)$ and $\frac{\eta^2}{\alpha(\eta)}$ is bounded, as $\eta \to 0$, to solution of the inverse problem;

How to proceed numerically?

...use suitable "relaxations" of the functional J_{reg} to overcome issues arising from non-convexity and non-differentiability of J_{reg}...

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Towards Numerical Algorithm

First step: Problem

$$\min_{C\in\mathcal{C}} J_{reg}(C) = \frac{1}{2}\int_{\Sigma_N} |u(C) - u_{meas}|^2 \, d\sigma(x) + \alpha \mathrm{Per}(C)$$

is equivalent to

$$\min_{\overline{v}\in X^{\mathcal{C}}_{0,1}(\Omega)} J_{reg}(\overline{v}) = \frac{1}{2}\int_{\Sigma_N} |u(\overline{v}) - u_{meas}|^2 \, d\sigma(x) + \alpha\, TV(\overline{v})$$

- $TV(\overline{v}) = \sup\left\{\int_\Omega \overline{v}\,\mathrm{div}(\varphi); \quad \varphi \in C^1_0(\Omega),\ \|\varphi\|_{L^\infty(\Omega)} \leq 1\right\}$;
- $X^{\mathcal{C}}_{0,1}(\Omega) := \{v \in BV(\Omega) \,:\, v = \chi_C \text{ a.e. in } \Omega,\ C \in \mathcal{C}\}$;
    - $BV(\Omega) = \{v \in L^1(\Omega) \,:\, TV(v) < \infty\}$.

Towards Numerical Algorithm (cont.)

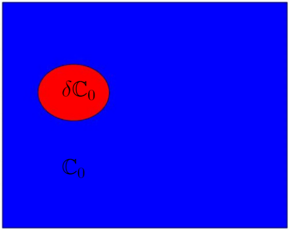
Second step (filling the cavity): let $\delta > 0$ be sufficiently small; then, consider

$$\min_{\overline{v}\in X^{\mathcal{C}}_{0,1}(\Omega)} \overline{J}_{reg}(\overline{v}) = \frac{1}{2}\int_{\Sigma_N} |u_\delta(\overline{v}) - u_{meas}|^2\, d\sigma(x) + \alpha\, TV(\overline{v})$$

where

$$\begin{cases} \mathrm{div}(\mathbb{C}_\delta(\overline{v})\widehat{\nabla} u_\delta(\overline{v})) = 0 & \text{in } \Omega, \\ (\mathbb{C}_\delta(\overline{v})\widehat{\nabla} u_\delta(\overline{v}))\nu = g & \text{on } \Sigma_N, \\ u_\delta(\overline{v}) = 0 & \text{on } \Sigma_D, \end{cases} \tag{2}$$

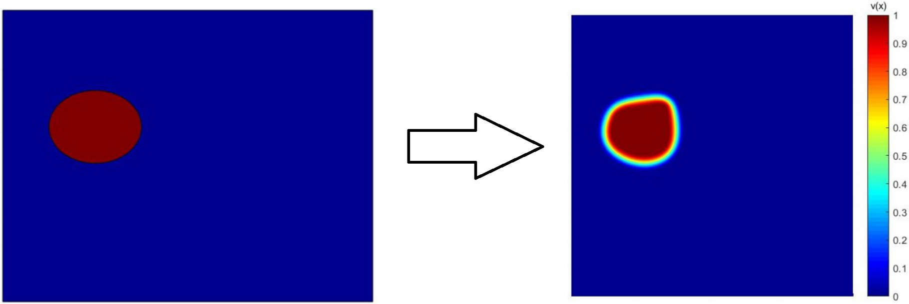
$$\mathbb{C}_\delta(\overline{v}) = \mathbb{C}_0 + (\mathbb{C}_1 - \mathbb{C}_0)\overline{v}, \quad \text{with} \quad \mathbb{C}_1 = \delta\mathbb{C}_0.$$



Allaire, Bourdain, Chambolle...

Approximation of Characteristic Functions

Third step (consider a phase-field variable)



$\overline{v} = \chi_C$ $\qquad$ $v \in \mathcal{K}(\Omega)$

- $\mathcal{K}(\Omega) = \{v \in H^1(\Omega) : \ 0 \leq v(x) \leq 1 \ a.e. \ in \ \Omega, \ v(x) = 0 \ a.e. \ in \ \Omega_1\},$
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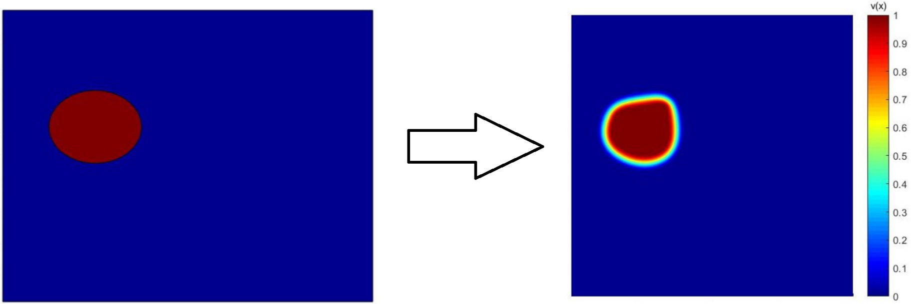

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Perimeter functional: Let $P : L^1(\Omega) \to [0, +\infty]$ s.t.

$$P(\widetilde{v}) = \begin{cases} TV(\widetilde{v}) & \text{if } \widetilde{v} \in X_{0,1}(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

- $X_{0,1}(\Omega) := \{\widetilde{v} \in BV(\Omega) : \widetilde{v} = \chi_C \text{ a.e. in } \Omega\}$.

Modica-Mortola functional: For any $\varepsilon > 0$, let $M_\varepsilon : L^1(\Omega) \to [0, +\infty]$ s.t.

$$M_\varepsilon(v) = \begin{cases} \frac{4}{\pi} \int_\Omega \left(\varepsilon |\nabla v|^2 + \frac{1}{\varepsilon} v(1-v) \right) & \text{if } v \in \mathcal{K}(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

Modica-Mortola (1977)

M_ε Γ-converges to P as $\varepsilon \to 0$.

Issue: by Modica-Mortola, as $\varepsilon \to 0$, the limit $\widetilde{v}$ is the characteristic function of a finite perimeter set only.

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Phase-field Approach

For $\varepsilon, \delta > 0$, find

$$\min_{v \in \mathcal{K}(\Omega)} J_{\delta,\varepsilon}(v) := \frac{1}{2}\int_{\Sigma_N} |u_\delta(v) - u_{meas}|^2 + \frac{4\alpha}{\pi}\int_\Omega \left(\varepsilon|\nabla v|^2 + \frac{1}{\varepsilon}v(1-v)\right)$$

- $\mathcal{K}(\Omega) = \{v \in H^1(\Omega) : \ 0 \leq v(x) \leq 1 \ a.e. \ in \ \Omega, \ v(x) = 0 \ a.e. \ \ in \ \ \Omega_1\}$;
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- $u_\delta(v)$ solution to

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where

$$\mathbb{C}_\delta(v) = \mathbb{C}_0 + v(\delta - 1)\mathbb{C}_0.$$

Analytical Results

Analytical results (A.-Beretta-Cavaterra-Rocca-Verani (2022))

- Existence of solutions $v = v_{\delta,\varepsilon} \in \mathcal{K}(\Omega)$, for any δ, $\varepsilon > 0$.
- Necessary optimality condition: any minimizer $v_{\delta,\varepsilon} \in \mathcal{K}(\Omega)$ satisfies

$$J'_{\delta,\varepsilon}(v_\varepsilon)[\omega - v_\varepsilon] \geq 0, \qquad \forall \omega \in \mathcal{K}(\Omega),$$

where,

$$J'_{\delta,\varepsilon}(v)[\vartheta] = \int_\Omega \vartheta(\mathbb{C}_0 - \mathbb{C}_1)\widehat{\nabla} u_\delta(v) : \widehat{\nabla} p_\delta(v) + \frac{8\alpha\varepsilon}{\pi}\int_\Omega \widehat{\nabla} v : \widehat{\nabla}\vartheta + \frac{4\alpha}{\varepsilon\pi}\int_\Omega (1-2v)\vartheta.$$

and $p_\delta \in H^1_{\Sigma_D}(\Omega)$ is the solution to the *adjoint problem*

$$\int_\Omega \mathbb{C}_\delta(v)\widehat{\nabla} p_\delta(v) : \widehat{\nabla}\psi = \int_{\Sigma_N} (u_\delta(v) - u_{meas})\psi, \qquad \forall \psi \in H^1_{\Sigma_D}(\Omega).$$

A Parabolic Obstacle Problem

Natural strategy: to find a phase-field critical point $v \in \mathcal{K}(\Omega)$ satisfying $J'_{\delta,\varepsilon}(v)[\omega - v] \geq 0, \quad \forall \omega \in \mathcal{K}(\Omega)$ ($\rightsquigarrow$ i.e. to find at least a local minimum of $J_{\delta,\varepsilon}$) we use the following Parabolic Obstacle Problem:

- find $v(\cdot, t) \in \mathcal{K}(\Omega)$, $t \geq 0$ s.t. $v(\cdot, 0) = v_0$ and

$$\int_\Omega \partial_t v(\omega - v) + J'_{\delta,\varepsilon}(v)[\omega - v] \geq 0, \quad \forall \omega \in \mathcal{K}, t \in (0 + \infty). \qquad (3)$$

In fact,

- choosing $\omega = v(\cdot, t - \Delta t)$ in (3);
- dividing by Δt;
- sending $\Delta t \to 0$

$$\|v_t\|^2 + J'_{\delta,\varepsilon}(v) v_t \leq 0, \quad \text{that is} \quad \frac{d}{dt} J_{\delta,\varepsilon}(v(\cdot, t)) \leq 0$$

If $\lim_{t \to +\infty} v(\cdot, t) := v_\infty$ exists, we expect that v_∞ is a solution of $J'_{\delta,\varepsilon}(v)[\omega - v] \geq 0$.

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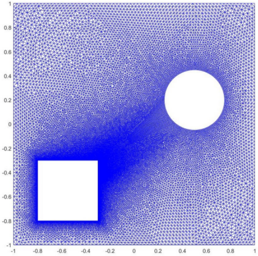
Algorithm & Numerical Results

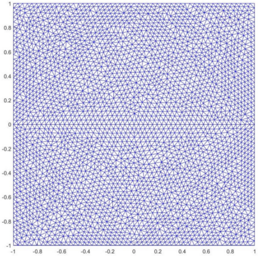
Algorithm 1 Discrete Parabolic Obstacle Problem

Set $n = 0$ and $v_h^0 = v_0$, the initial guess for the inclusion
while $\|v_h^n - v_h^{n-1}\| > \mathrm{tol}$ **do**
  find $u_h(v_h^n)$ solution of the forward problem with $v = v_h^n$
  find $p_h(v_h^n)$ solution of the adjoint problem with $v = v_h^n$
  find v^{n+1} solving the parab. obstacle prob.;
  update $n = n + 1$;
end while

Meshes and Refinement

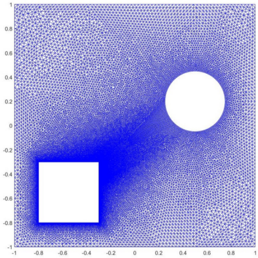


(a) Mesh $\mathcal{T}_h^{ref}$: forward problem.



(b) Mesh $\mathcal{T}_h$: inverse problem.

Meshes and Refinement



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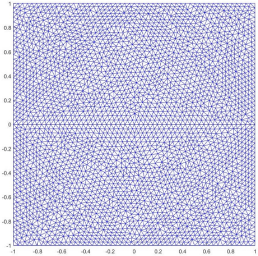
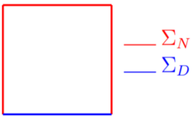
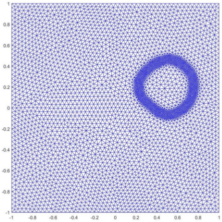


(b) Mesh $\mathcal{T}_h$: inverse problem.



(a) Boundary condition in numerical experiments: Neumann boundary conditions are assigned on the red part. Homogeneous Dirichlet conditions are assigned on the blue part.



(b) Refinement of the mesh around the reconstructed domain.

- Some numerical results (initial guess $v_0 \equiv 0$)

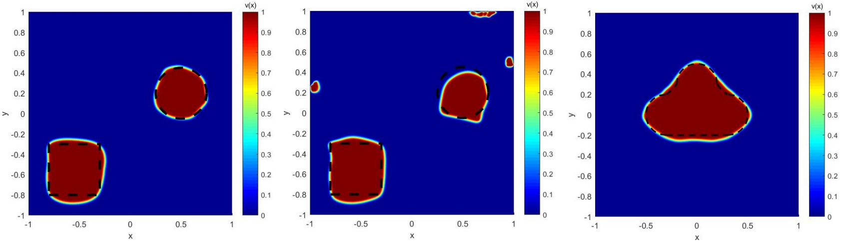


Figure: Example 1: noise 2%. Example 2: noise 5%. Example 3: no noise.

Before concluding...an alternative

The use of the misfit functional is not the only possible choice.
An energy-gap functional can be used.
Consider the two boundary value problems

$$\begin{cases} \mathrm{div}(\mathbb{C}_0 \widehat{\nabla} u_N) = 0 & \text{in } \Omega \setminus C \\ (\mathbb{C}_0 \widehat{\nabla} u_N) n = 0 & \text{on } \partial C \\ (\mathbb{C}_0 \widehat{\nabla} u_N) \nu = g & \text{on } \Sigma_N \\ u_N = 0 & \text{on } \Sigma_D, \end{cases} \quad \text{and} \quad \begin{cases} \mathrm{div}(\mathbb{C}_0 \widehat{\nabla} u_D) = 0 & \text{in } \Omega \setminus C \\ (\mathbb{C}_0 \widehat{\nabla} u_D) n = 0 & \text{on } \partial C \\ u_D = u_{meas} & \text{on } \Sigma_N \\ u_D = 0 & \text{on } \Sigma_D. \end{cases}$$

Kohn-Vogelius type functional

$$\min_{C \in \mathcal{C}} J_{KV}(C) := \underbrace{\frac{1}{2} \int_{\Omega \setminus C} \mathbb{C}_0 \widehat{\nabla}(u_N(C) - u_D(C)) : \widehat{\nabla}(u_N(C) - u_D(C))\, dx}_{\text{Kohn-Vogelius func.}} + \alpha \mathrm{Per}(C)$$

...one can repeat an analogous analysis as done in the previous slides (*A. (2022)*)

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Numerical results - Kohn-Vogelius func.

- Some numerical results (initial guess $v_0 \equiv 0$)

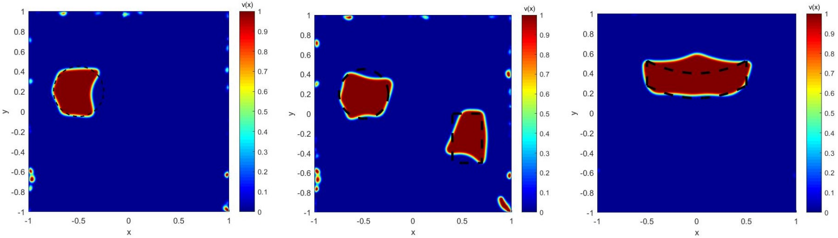

Figure: Example 1: noise 5%. Example 2: noise 5%. Example 3: noise 2%.

Conclusions

- We have introduced a phase-field approach in elastic inverse problems;
- The method is more versatile than others since no a priori information is needed (initial guess could also be $v_0 = 0$);

Open problems:

- Prove Γ-convergence of $J_{\delta,\varepsilon}$ to J as $\delta, \varepsilon \to 0$;
- Extend analytical and numerical results to other differential operators (e.g. evolution PDE systems, non-linear forward problems...);
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Thank you for your attention