SSVM 2023

The 9th International Conference on
Scale Space and Variational Methods in Computer Vision

Santa Margherita di Pula (CA), Sardinia, Italy

Dates: May 21-25 2023

Organising committee

- Luca Calatroni (CNRS, FR)
- Marco Donatelli (Univ. Insubria, IT)
- Serena Morigi (Univ. Bologna, IT)
- Marco Prato (UniMoRe, IT)
- Giuseppe Rodriguez (Univ. Cagliari, IT)
- Matteo Santacesaria (Univ. Genova, IT)

Important dates

Paper submission:	January 16, 2023
Notification of acceptance:	February 27, 2023

Hotel Flamingo

Proceedings

Papers accepted for the conference will appear in the conference proceedings that will be published in Springer's Lecture Notes in Computer Science series. The proceedings will be available at the conference. Prospective authors are invited to submit a full-length twelve-page paper electronically via the SSVM'23 Paper Submission Web Page. All papers will undergo a double-blind peer-review procedure. At the conference the papers will be presented as posters or talks.

http://events.unibo.it/ssvm2023/

Compressed sensing for the sparse Radon transform

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(UniGe)

Matteo Santacesaria
(UniGe)

S. Ivan Trapasso
(PoliTo)

Outline

The sparse Radon transform

Compressed sensing

Compressed sensing for the sparse Radon transform

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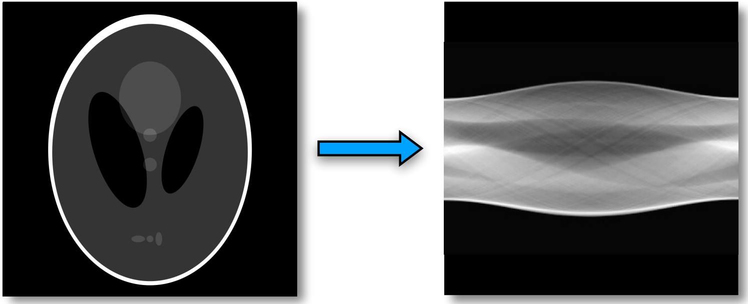
Compressed sensing for the sparse Radon transform

The Radon transform

$$\mathcal{R}u(\theta, s) = \int_{\theta^\perp} u(y + s\theta)dy$$



The Radon transform[1]

- Domain: $\mathcal{B}_1 = B(0,1) \subseteq \mathbb{R}^2$

[1]Natterer, The Mathematics of Computerized Tomography, 2001
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$$\mathcal{R} \colon L^2(\mathcal{B}_1) \to L^2(\mathbb{S}^1 \times [-1,1]), \qquad \mathcal{R}u(\theta, s) = \mathcal{R}_\theta u(s)$$

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- Ill-posedness/inversion:

$$\|\mathcal{R}u\|_{L^2} \asymp \|u\|_{H^{-\frac{1}{2}}}$$

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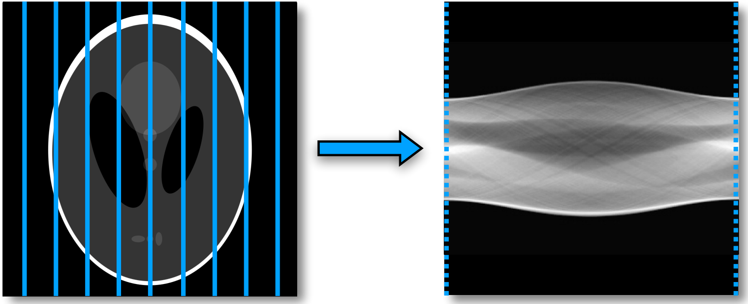

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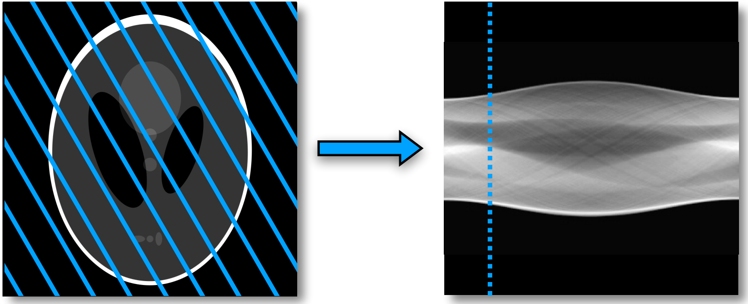

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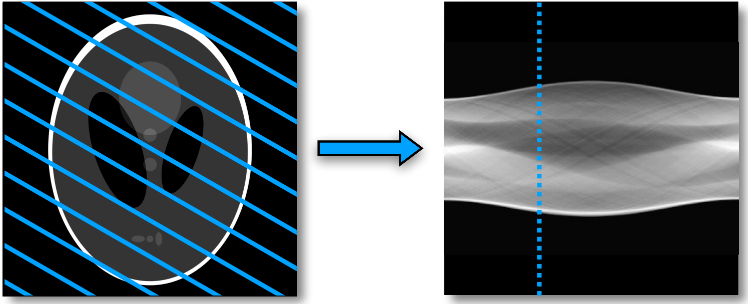

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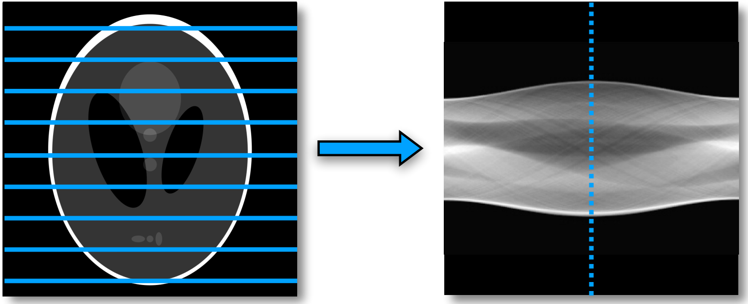
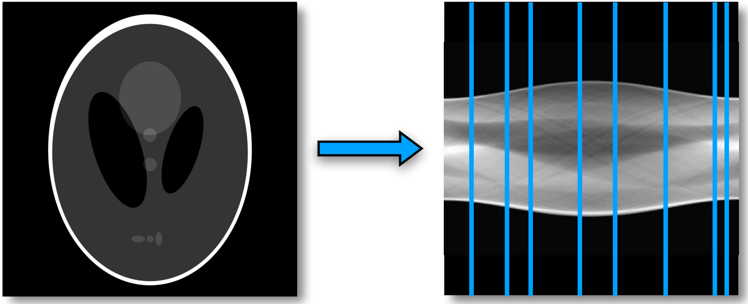


The sparse Radon transform

$$\left(\mathcal{R}u^\dagger(\theta_1,\cdot),\ldots,\mathcal{R}u^\dagger(\theta_m,\cdot)\right), \quad \theta_1,\ldots,\theta_m \overset{\text{i.i.d.}}{\sim} \nu \text{ uniform on } \mathbb{S}^1$$



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The sparse Radon inverse problem

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- Subsampled measurements $\implies$ need a-priori information on $u^\dagger$
- Natural assumption: $u^\dagger$ is sparse

(Some) related literature

$$\big(\mathcal{R}u^\dagger(\theta_1,\cdot),\ldots,\mathcal{R}u^\dagger(\theta_m,\cdot)\big) \in L^2(-1,1)^m \qquad \longrightarrow \qquad u^\dagger \in L^2(\mathcal{B}_1)$$

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Empirical works:

- Siltanen et al, *Statistical inversion for medical x-ray tomography with few radiographs*, 2003
- Hämäläinen et al, *Sparse Tomography*, 2013
- Jørgensen and Sidky, *How little data is enough? Phase-diagram analysis of sparsity-regularized X-ray computed tomography*, 2015
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Theoretical works:

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A theory of sparse Radon transform?

Main question:

number of measurements (sample complexity) $\longleftrightarrow$ sparsity of $u^\dagger$

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Compressed sensing!

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  Discussions started at "100 years of the Radon transform", RICAM

WARNING

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Main result at the end!

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SPOILER

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$m \gtrsim$ **sparsity**

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- Unknown signal: $u^\dagger \in \mathbb{R}^M$
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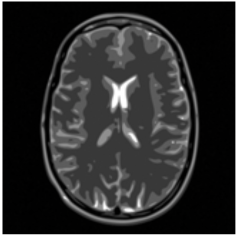

$u^\dagger$

Measured frequencies

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- In practice, **compressibility**:

$$u = v + \text{small}, \qquad v \in \Sigma_s.$$

Real-world signals are compressible

Figure: Left: original image - Right: image obtained (roughly) by keeping only the 1% largest coefficients with respect to a discrete wavelet basis (JPEG-2000 compression standard)

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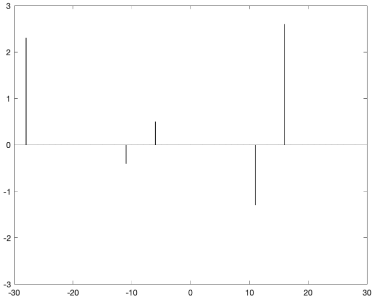

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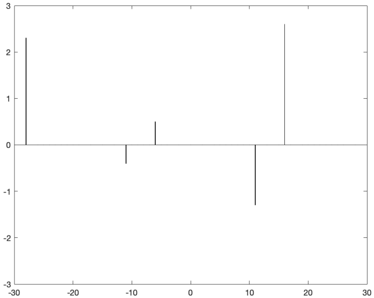

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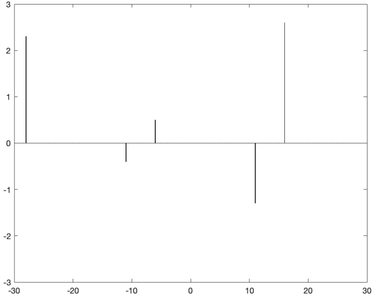

Coherence:

$$B := \max_{n,l} |\langle \Phi_n, \psi_l \rangle|, \qquad \text{ideally: } B = \frac{1}{\sqrt{M}}$$

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Recovery estimate[3]

- $u^\dagger \in \mathbb{R}^M$: unknown signal
- $u^\dagger$ is s-sparse w.r.t. $\{\Phi_n\}_n$
- $(Au)_l = \langle u, \psi_l \rangle$, $l = 1, \dots, m$: subsampled isometry

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- measurements: $y = Au^\dagger$

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Outline

The sparse Radon transform

Compressed sensing

Compressed sensing for the sparse Radon transform

Back to the sparse Radon transform

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1. Forward map $\mathcal{R}$ affects sparsity[5]

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- Solution: many dictionaries and operators of interest are 'compatible'

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2. Ill-posed problem: g-RIP

- Classical CS: Restricted Isometry Property (RIP)

$$(1-\delta)\|u\|^2 \leqslant \|Au\|_2^2 \leqslant (1+\delta)\|u\|^2, \quad u \in \Sigma_s$$

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A few comments

- This theorem is a particular case of an abstract result dealing with:
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  - Hilbert space-valued measurements
  - ill-posed inverse problems
- Explicit estimates with
  - noisy data
  - compressible (and not sparse) $u^\dagger$
  - regularization with sampling: $m = m(\text{noise})$

Conclusions

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- Rigorous theory of compressed sensing for subsampled isometries (e.g. MRI)
- Empirical evidence for compressed sensing Radon transform

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Future

- Fan-beam geometry
- Wavelets $\rightarrow$ shearlets, curvelets, etc.
- Generalisation to other ill-posed problems
- Nonlinear problems
- Compressed sensing with generative models