Degeneration of hyperbolic surfaces and spectral gaps for large genus

Xuwen Zhu (Northeastern University)

“Geometrical Inverse Problems” Workshop

Joint with Yunhui Wu and Haohao Zhang

Outline

1. Spectrum on hyperbolic surfaces: review
2. Main result on spectral gaps
3. Degenerating hyperbolic surfaces and a Min-Max principle
4. Proof of the theorem

Hyperbolic surfaces and moduli spaces

- For a compact surface M with genus $g \geq 2$ with a given complex structure, there is a unique hyperbolic metric with finite area (Uniformization theorem, Gauss–Bonnet)
- The moduli space $\mathcal{M}_g$ is the set of all such complex structures (hence hyperbolic metrics) on a genus g surface up to diffeomorphism
- The surface can also have punctures (corresponding to marked points)
- If $2g + n - 2 > 0$, it corresponds to hyperbolic surfaces with cusps
- The moduli space $\mathcal{M}_{g,n}$ is a complex orbifold with dimension $3g - 3 + n$
- The hyperbolic metrics varies smoothly

Hyperbolic surfaces and moduli spaces

- For a compact surface M with genus $g \geq 2$ with a given complex structure, there is a unique hyperbolic metric with finite area (Uniformization theorem, Gauss–Bonnet)
- The moduli space $\mathcal{M}_g$ is the set of all such complex structures (hence hyperbolic metrics) on a genus g surface up to diffeomorphism
- The surface can also have punctures (corresponding to marked points)
- If $2g + n - 2 > 0$, it corresponds to hyperbolic surfaces with cusps
- The moduli space $\mathcal{M}_{g,n}$ is a complex orbifold with dimension $3g - 3 + n$
- The hyperbolic metrics varies smoothly

Compact and noncompact surfaces

- Cusped surfaces can be obtained as the degenerating limit of compact surfaces
- Take a nontrivial geodesic cycle in M, and let its length go to zero.

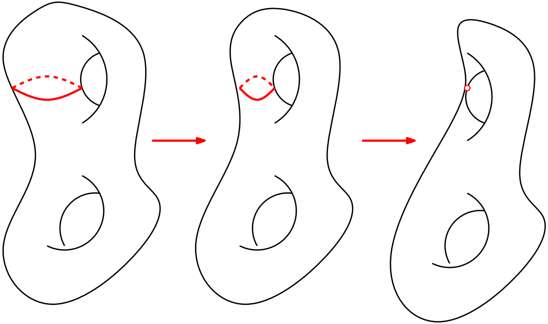


Figure: Degenerating surfaces with a geodesic cycle shrinking to a point

- Can also be seen from group actions on the universal cover

Eigenvalues of compact hyperbolic surfaces

- We study Δ, the Laplace–Beltrami operator w.r.t. hyperbolic metric
- When the surface is compact, there is a sequence of eigenvalues

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$$

- Varies smoothly on the moduli space [Buser, 1992]
- Distribution of eigenvalues are related to
  - genus
  - diameter
  - injectivity radius
  - ...

Spectrum of noncompact hyperbolic surfaces

- The spectrum for a hyperbolic surface with cusps are different
- Continuous spectrum $[\frac{1}{4}, \infty)$ + discrete eigenvalues $\{\lambda_i\}$
- Related to eigenvalues of compact surfaces under degeneration
- Related works: [Hejhal, 1990] [Ji, 1993] [Ji–Zworski, 1993] [Wolpert, 1987, 1992] ...

Small eigenvalues

- "Small" eigenvalues of hyperbolic surfaces: $\lambda \in (0, \frac{1}{4}]$
- Question:
  - Existence
  - Total number
  - Multiplicities...
- Literature: [McKean, 1972, 1974] [Randol, 1974] [Buser, 1982, 1984] [Brooks–Makover, 2001, 2004] [Otal–Rosas, 2009] [Mondal, 2015] [Ballmann–Mattheiesen–Mondal, 2016–]...
- Can have arbitrarily many small eigenvalues [Randol, 1974]
- Can have arbitrarily many inside $(0, \epsilon)$ [Buser, 1982]
- For genus g surface, $\lambda_{2g-2} > \frac{1}{4}$ [Otal–Rosas, 2009]
- $\lambda_1(X_{0,3}) > \frac{1}{4}$ [Otal–Rosas, 2009] [Ballmann–Mattheiesen–Mondal, 2016]
- There exists $X_{1,2}$ such that $\lambda_1(X_{1,2}) > \frac{1}{4}$ [Mondal, 2015]

Small eigenvalues

- "Small" eigenvalues of hyperbolic surfaces: $\lambda \in (0, \frac{1}{4}]$
- Question:
  - Existence
  - Total number
  - Multiplicities...
- Literature: [McKean, 1972, 1974] [Randol, 1974] [Buser, 1982, 1984] [Brooks–Makover, 2001, 2004] [Otal–Rosas, 2009] [Mondal, 2015] [Ballmann–Mattheiesen–Mondal, 2016–]...
- Can have arbitrarily many small eigenvalues [Randol, 1974]
- Can have arbitrarily many inside $(0, \epsilon)$ [Buser, 1982]
- For genus g surface, $\lambda_{2g-2} > \frac{1}{4}$ [Otal–Rosas, 2009]
- $\lambda_1(X_{0,3}) > \frac{1}{4}$ [Otal–Rosas, 2009] [Ballmann–Mattheiesen–Mondal, 2016]
- There exists $X_{1,2}$ such that $\lambda_1(X_{1,2}) > \frac{1}{4}$ [Mondal, 2015]

Small eigenvalues

- "Small" eigenvalues of hyperbolic surfaces: $\lambda \in (0, \frac{1}{4}]$
- Question:
  - Existence
  - Total number
  - Multiplicities...
- Literature: [McKean, 1972, 1974] [Randol, 1974] [Buser, 1982, 1984] [Brooks–Makover, 2001, 2004] [Otal–Rosas, 2009] [Mondal, 2015] [Ballmann–Mattheiesen–Mondal, 2016–]...
- Can have arbitrarily many small eigenvalues [Randol, 1974]
- Can have arbitrarily many inside $(0, \epsilon)$ [Buser, 1982]
- For genus g surface, $\lambda_{2g-2} > \frac{1}{4}$ [Otal–Rosas, 2009]
- $\lambda_1(X_{0,3}) > \frac{1}{4}$ [Otal–Rosas, 2009] [Ballmann–Mattheiesen–Mondal, 2016]
- There exists $X_{1,2}$ such that $\lambda_1(X_{1,2}) > \frac{1}{4}$ [Mondal, 2015]

Why care about the spectrum

- Selberg's 3/16 conjecture regarding size of λ_1 [Selberg, 1965]
- Recent progress: [Gelbart–Jacquet, 1978] [Luo–Rudnick–Sarnak, 1995] [Kim–Shahidi, 2002] [Kim–Sarnak, 2003]
- Arithmetic vs non-arithmetic hyperbolic surfaces
  - Spectral gap
  - Embedded eigenvalues
- Relation to number theory, representation theory, geometry, etc.

Large genus limit

- Another type of question: what happens to eigenvalues when genus $g \to \infty$?
- $\lim_{g\to\infty} Prob(\lambda_1(X_g) \geq \frac{3}{16} - \epsilon) = 1$ [Wu–Xue, 2021] [Lipnowski–Wright, 2021]
- Related works: [Brooks–Markover, 2004] [Mirzakhani, 2013] [Hide, 2021] [Monk, 2021]
- Random covers of compact and noncompact hyperbolic surfaces [Magee, Naud, Puder, 2010–]
- Random covers of punctured hyperbolic surfaces has $\lambda_1 \geq \frac{1}{4} - \epsilon$ [Hide-Magee, 2021]
- There exists a sequence such that $\lambda_1(X_i) \to \frac{1}{4}$ with increasing genus [Hide–Magee, 2021]

Large genus limit

- Another type of question: what happens to eigenvalues when genus $g \to \infty$?
- $\lim_{g\to\infty} Prob(\lambda_1(X_g) \geq \frac{3}{16} - \epsilon) = 1$ [Wu–Xue, 2021] [Lipnowski–Wright, 2021]
- Related works: [Brooks–Markover, 2004] [Mirzakhani, 2013] [Hide, 2021] [Monk, 2021]
- Random covers of compact and noncompact hyperbolic surfaces [Magee, Naud, Puder, 2010–]
- Random covers of punctured hyperbolic surfaces has $\lambda_1 \geq \frac{1}{4} - \epsilon$ [Hide-Magee, 2021]
- There exists a sequence such that $\lambda_1(X_i) \to \frac{1}{4}$ with increasing genus [Hide–Magee, 2021]

Statement of the result

Our result concerns the gaps between eigenvalues when genus goes to infinity:

Theorem (Wu–Zhang–Z, 2022)

For any integer sequence $\{\eta(g)\}_{g=2}^{\infty}$ *such that* $\eta(g) \in [1, 2g-2]$,

$$\liminf_{g\to\infty} \sup_{X_g\in\mathcal{M}_g} (\lambda_\eta(X_g) - \lambda_{\eta-1}(X_g)) \geq \frac{1}{4}.$$

Related results

- There is another bound for $\eta = o(\ln(g))$:

$$\limsup_{g\to\infty} \sup_{X_g\in\mathcal{M}_g} (\lambda_\eta(X_g) - \lambda_{\eta-1}(X_g)) \leq \frac{1}{4}.$$

- Two results combined, for $\eta = o(\ln(g))$

$$\lim_{g\to\infty} \sup_{X_g\in\mathcal{M}_g} (\lambda_\eta(X_g) - \lambda_{\eta-1}(X_g)) = \frac{1}{4}.$$

- For $\eta = 1$, this is the result in [Hide–Magee, 2021]

$$\lim_{g\to\infty} \sup_{X_g\in\mathcal{M}_g} \lambda_1(X_g) = \frac{1}{4}.$$

Related results

- There is another bound for $\eta = o(\ln(g))$:

$$\limsup_{g\to\infty} \sup_{X_g\in\mathcal{M}_g} (\lambda_\eta(X_g) - \lambda_{\eta-1}(X_g)) \leq \frac{1}{4}.$$

- Two results combined, for $\eta = o(\ln(g))$

$$\lim_{g\to\infty} \sup_{X_g\in\mathcal{M}_g} (\lambda_\eta(X_g) - \lambda_{\eta-1}(X_g)) = \frac{1}{4}.$$

- For $\eta = 1$, this is the result in [Hide–Magee, 2021]

$$\lim_{g\to\infty} \sup_{X_g\in\mathcal{M}_g} \lambda_1(X_g) = \frac{1}{4}.$$

Related results

- Upper bound of eigenvalues for hyperbolic surfaces [Cheng, 1975]

$$\lambda_i(X_g) \leq \frac{1}{4} + i^2 \cdot \frac{16\pi^2}{\mathrm{Diam}^2(X_g)}.$$

  and $\mathrm{Diam}(X_g) \geq C \ln(g)$.

- There exists degeneration of compact hyperbolic surfaces into a connected cusp surface such that

$$\limsup_{t\to\infty} \lambda_1(X_t) \geq 3/16$$

  [Buser–Burger–Dodziuk, 1988]

Components of the proof

- Degeneration of hyperbolic surfaces
  - Analysis: Min-Max principle for eigenvlaues
- A theorem of Schoen–Wolpert–Yau
- Finding the correct components of punctured surfaces with large first eigenvalue
  - Geometry: decomposition into punctured surfaces
  - Eigenvalue estimate for each component

Degenerating hyperbolic surfaces

Locally the geometry near the shrinking cycle is described by the normal crossing model:

$$(z, w) \in \mathbb{C}^2, \quad zw = t$$

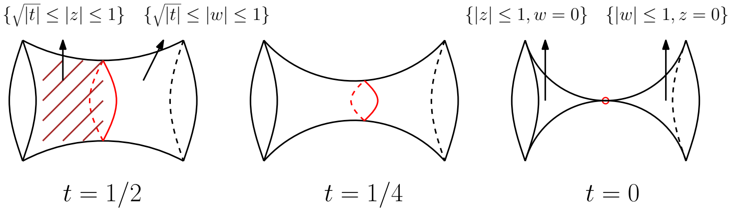


Figure: Local geometry of $zw = t$, with coordinate patch z and w

Results on degenerating hyperbolic metrics

Theorem(Melrose–Z, 2018, 2019)

- The degenerating hyperbolic metrics are polyhomogeneous on a space with new variables.
- The metric is uniformly bounded on any compact set outside the degenerating area.

- This implies control of eigenfunctions on any compact set outside the degenerating area.

A Min-Max principle

Proposition

- *Take* $\lim_{t\to 0} X_g(t) = X_g(0) \in \partial\mathcal{M}_g$
- $X_g(0)$ *has* k *connected components, i.e.,* $X_g(0) = Y_1 \sqcup Y_2 \cdots \sqcup Y_k$ *where* $k \geq 2$.
- *Let* $\lambda_1(Y_i)$ *be the first non-zero eigenvalue of* Y_i
- *denote* $\bar{\lambda}_1(*) = \min\left\{\lambda_1(*), \frac{1}{4}\right\}$ *for* $* = Y_1, \cdots, Y_k$.

Then

$$\liminf_{t\to 0} \lambda_k(X_g(t)) \geq \min_{1\leq i\leq k} \{\bar{\lambda}_1(Y_i)\}.$$

- There is a version for degenerating sequence with non-separating limit in [Buser–Burger–Dodziuk, 1988]
- Why need $\bar{\lambda}_1$ and $1/4$: example of $X_{0,3}$

Proof of the Min-max principle

- Eigenfunctions are uniformly bounded (Sobolev–Gårding inequality)
- There is a subsequence $\{\phi_k\}$ that converges to $\Delta\phi_0 = \lambda(0)\phi_0$
- Uniform bounds $\Rightarrow$ ϕ_0 bounded in H^1
- $(\lambda(0), \phi_0)$ must satisfy one of the two conditions:
  1. ϕ_0 is an eigenfunction of $\Delta_{X_g(0)}$ and also restricts to at least one of the components Y_k as an eigenfunction; or
  2. $\phi_0 = 0$ everywhere on $X_g(0)$ and $\lambda(0) = \frac{1}{4}$.

A theorem on eigenvalues under degeneration

Theorem (Schoen–Wolpert–Yau '80)

For any compact hyperbolic surface X_g of genus g and integer $i \in (0, 2g-2)$, the i-th eigenvalue satisfies

$$\alpha_i(g) \cdot \ell_i \leq \lambda_i \leq \beta_i(g) \cdot \ell_i$$

and

$$\alpha(g) \leq \lambda_{2g-2}$$

where ℓ_i is the minimal possible sum of the lengths of simple closed geodesics in X_g which cut X_g into $i+1$ connected components.

Intuition: if one cuts a compact hyperbolic surface into n pieces, then $\lambda_1, \ldots, \lambda_{n-1}$ will go to 0 while λ_n will stay away from 0.

Degeneration into "good" pieces

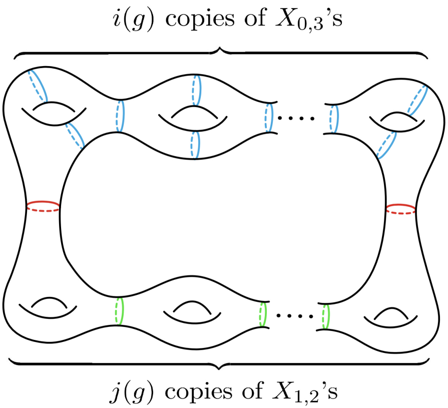


Figure: An example of the degeneration of a genus g surface into $i(g)$ copies of $X_{0,3}$'s and $j(g)$ copies of $X_{1,2}$'s by pinching all the simple geodesics marked in the picture

Degeneration into “good” pieces

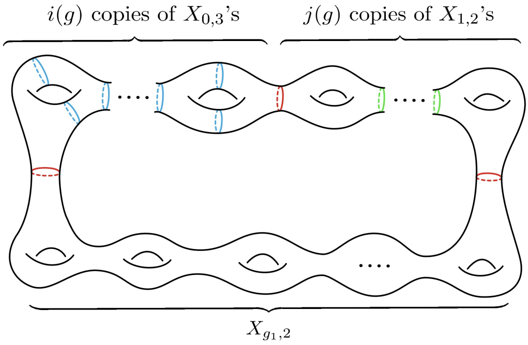


Figure: An example of decomposing a surface of genus g into $i(g)$ copies of $X_{0,3}$'s, $j(g)$ copies of $X_{1,2}$'s and a copy of $X_{g_1,2}$

Sketch of proof for the main theorem

First we show lower bound

Proposition

For all $i \geq 1$,

$$\inf_{X_g \in \mathcal{M}_g} \left(\lambda_i(X_g) - \lambda_{i-1}(X_g)\right) = 0.$$

Split into 3 cases:

Example: $1 \leq i \leq 2g - 3$. Choose a closed hyperbolic surface close to $\underbrace{X_{0,3} \sqcup \cdots \sqcup X_{0,3}}_{2g-2 \text{ copies}}$, then $\lambda_i(\mathcal{X}_g)$ is close to 0.

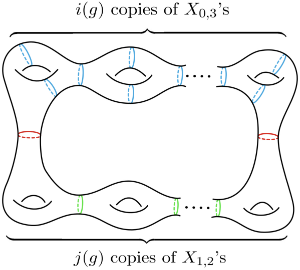
Sketch of proof for the main theorem

Then we show the upper bound (split into 4 cases):

Example: $\eta(g) \in [g+1, 2g-3]$. Choose $X_g(t) : (0,1) \to \mathcal{M}_g$ be a family of closed hyperbolic surfaces such that

$$\lim_{t\to 0} X_g(t) = \underbrace{X_{0,3} \sqcup \cdots \sqcup X_{0,3}}_{i(g)\text{ copies}} \sqcup \underbrace{Z_{1,2} \sqcup \cdots \sqcup Z_{1,2}}_{j(g)\text{ copies}} \in \partial\mathcal{M}_g$$

where $i(g)$ and $j(g)$ are two non-negative integers satisfying $i(g) + j(g) = \eta(g)$.



More questions

- Removal of assumption on the range of η?
  Conjecture: For any sequence $\{\eta(g)\}$,

$$\liminf_{g\to\infty} \sup_{X_g\in\mathcal{M}_g} (\lambda_\eta(X_g) - \lambda_{\eta-1}(X_g)) \geq \frac{1}{4}.$$

- Can one say more precise information about small eigenvalues?

Thank you for your attention!