An inverse boundary value problem for a nonlinear elastic wave equation

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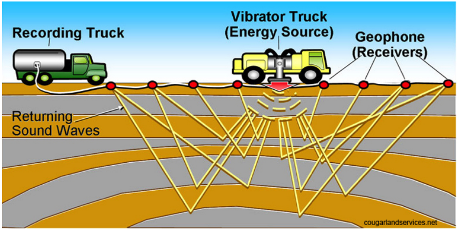
Seismic inversion

Recovery of subsurface geological structure from seismic records



seismic waves can be modeled by the elastic wave equation

Linear elastic wave equation

The linear elastic wave equation in isotropic medium

$$
\begin{aligned}
&\rho \frac{\partial^2 u}{\partial t^2} - \nabla \cdot S^L(x,u) = 0, \quad (t,x) \in (0,T) \times \Omega, \\
&u(t,x) = f(t,x), \quad (t,x) \in (0,T) \times \partial\Omega, \\
&u(0,x) = \frac{\partial}{\partial t} u(0,x) = 0, \quad x \in \Omega.
\end{aligned}
$$

Here $\Omega \subset \mathbb{R}^3$ is bounded.

- u: displacement, vector
- $\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$: strain
- $S^L(x,u) = \lambda(x)\mathrm{tr}\{\varepsilon(u)\}I + 2\mu(x)\varepsilon(u)$: stress
- ρ: density
- λ, μ: Lamé moduli
- λ, μ, ρ encode the mechanical properties of the elastic materials

Inverse problem for the linear equation

Define the Dirichlet-to-Neumann map

$$\Lambda^{lin} : f \mapsto S^L(x, u) \cdot \nu|_{(0,T)\times\partial\Omega},$$

where ν is the outer unit normal to the boundary. Assume T is large enough, and λ, μ, ρ are all smooth functions

Determine λ, μ, ρ from Λ^{lin}

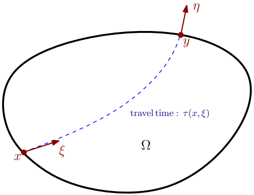
Boundary Control Method does not work!

Study the propagation of singularities of the solutions; related with certain geometrical inverse problems.

Reduction to geometrical inverse problems

There are two wavespeeds S-wave speed $c_S = \sqrt{\frac{\mu}{\rho}}$, P-wave speed $c_P = \sqrt{\frac{\lambda+2\mu}{\rho}}$.

Determination of c_S and c_P: lens data for c_S and c_P can be recovered from Λ^{lin}. (Rachele, 2000; Stefanov-Uhlmann-Vasy, 2017)



lens data: $\{\alpha(x,\xi) = (y,\eta)\} \cup \{\tau(x,\xi)\}$.

recover c from the lens data (lens rigidity problem):

- if $(\Omega, c^{-2}\mathrm{d}s^2)$ is simple (Muhometov-Romanov, 1978)
- if $(\Omega, c^{-2}\mathrm{d}s^2)$ admits a strictly convex function (the foliation condition) (Stefanov-Uhlmann-Vasy, 2016)

Determination of ρ

Using P-wave measurements: ($\lambda \neq 2\mu$) related with geodesic ray transform of 2-tensors (Rachele, 2000; Bhattacharyya, 2018).

(M, g) is a compact 3-dimensional Riemannian manifold with boundary ∂M. The geodesic ray transform of a symmetric 2-tensor f is

$$I_2 f(\gamma) = \int_\gamma f^{ij}(\gamma(t))\dot{\gamma}_i(t)\dot{\gamma}_j(t)\mathrm{d}t,$$

where γ runs over all geodesics with endpoints on ∂M.

s-injectivity of I_2 on 3-dimensional manifolds:

- generically true on simple manifolds (Stefanov-Uhlmann, 2005)
- under extra curvature conditions on simple manifolds (Sharafutdinov 94; Paternain-Salo-Uhlmann, 2015)
- true under the foliation condition (Stefanov-Uhlmann-Vasy, 2018)

Summary for the linear equation (dim=3)

Determination of $\frac{\lambda}{\rho}$ and $\frac{\mu}{\rho}$: related to the lens rigidity problem

- $(\Omega, c_{P/S}^{-2}\mathrm{d}s^2)$ is simple (Rachele, 2000);
- $(\Omega, c_{P/S}^{-2}\mathrm{d}s^2)$ admits a strictly convex function (Stefanov-Uhlmann-Vasy, 2017).

Determination of ρ separately: related to some tensor tomography problem

- $\lambda \neq 2\mu$, $(\Omega, c_P^{-2}\mathrm{d}s^2)$ is simple, and has some explicit upper bound on the sectional curvature (Rachele, 2003);
- $\lambda \neq 2\mu$, $(\Omega, c_P^{-2}\mathrm{d}s^2)$ admits a strictly convex function (Bhattacharyya, 2018).

Nonlinear elastic wave equations

The nonlinear elastic wave equations

$$\rho\frac{\partial^2 u}{\partial t^2} - \nabla\cdot S(x,u) = 0, \quad (t,x)\in(0,T)\times\Omega,$$
$$u(t,x) = f(t,x), \quad (t,x)\in(0,T)\times\partial\Omega,$$
$$u(0,x) = \frac{\partial}{\partial t}u(0,x) = 0, \quad x\in\Omega.$$

The stress tensor S has the form (Gol'dberg 1961)

$$\begin{aligned} S_{ij} =& S^L_{ij} + \frac{\lambda+\mathscr{B}}{2}\frac{\partial u_m}{\partial x_n}\frac{\partial u_m}{\partial x_n}\delta_{ij} + \mathscr{C}\frac{\partial u_m}{\partial x_m}\frac{\partial u_n}{\partial x_n}\delta_{ij} + \frac{\mathscr{B}}{2}\frac{\partial u_m}{\partial x_n}\frac{\partial u_n}{\partial x_m}\delta_{ij} \\ &+ \mathscr{B}\frac{\partial u_m}{\partial x_m}\frac{\partial u_j}{\partial x_i} + \frac{\mathscr{A}}{4}\frac{\partial u_j}{\partial x_m}\frac{\partial u_m}{\partial x_i} + (\lambda+\mathscr{B})\frac{\partial u_m}{\partial x_m}\frac{\partial u_i}{\partial x_j} \\ &+ \left(\mu+\frac{\mathscr{A}}{4}\right)\left(\frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j} + \frac{\partial u_i}{\partial x_m}\frac{\partial u_j}{\partial x_m} + \frac{\partial u_i}{\partial x_m}\frac{\partial u_m}{\partial x_j}\right) + \mathcal{O}(u^3). \end{aligned}$$

Determine $\lambda, \mu, \rho, \mathscr{A}, \mathscr{B}, \mathscr{C}$ from the (nonlinear) Dirichlet-to-Neumann map

$$\Lambda : f \to S(x,u)\cdot\nu|_{(0,T)\times\partial\Omega}.$$

Recent development in nonlinear equations

- Recovery of a Lorentzian metric g in the semilinear wave equation $\Box_g u + au^2 = 0$ from the *source-to-solution map* (Kurylev-Lassas-Uhlmann, 2018; inverse problem for the corresponding linear equation is still open! the nonlinearity helps!)

- Other nonlinear hyperbolic equations
  - Einstein's equation (Kurylev, Lassas, Oksanen, Uhlmann, Wang)
  - Yang-Mills equations (Chen, Lassas, Oksanen, Paternain)
  - etc.

The main result

Theorem (Uhlmann-Z, 2021, 2022)

Assume $T > 2\,diam_S(\Omega)$, $\partial\Omega$ is strictly convex with respect to $c_S^{-2}ds^2$ and $c_P^{-2}ds^2$, and either of the following conditions holds

1. $(\Omega, c_{P/S}^{-2}ds^2)$ *is simple;*
2. $(\Omega, c_{P/S}^{-2}ds^2)$ *admits a strictly convex function.*

Then the Dirichlet-to-Neumann map determines $\lambda, \mu, \rho, \mathscr{A}, \mathscr{B}, \mathscr{C}$ in $\overline{\Omega}$ uniquely.

previous result: assume $\rho \equiv 1$, uniqueness of λ, μ, $\mathscr{A}$, $\mathscr{B}$ under the simplicity condition (de Hoop-Uhlmann-Wang, 2020: nonlinear interaction of distorted plane waves)

First version of our result

Assume λ, μ, ρ are already known (note that one can recover Λ^{lin} from Λ)

Theorem (Uhlmann-Z, 2021)

Assume $T > 2\,diam_S(\Omega)$, $\partial\Omega$ is strictly convex with respect to $c_S^{-2}ds^2$ and $c_P^{-2}ds^2$, and either of the following conditions holds

1. $(\Omega, c_{P/S}^{-2}ds^2)$ *is simple;*
2. $(\Omega, c_{P/S}^{-2}ds^2)$ *admits a strictly convex function.*

Assume that λ, μ, ρ are already known. Then the Dirichlet-to-Neumann map determines $\mathscr{A}, \mathscr{B}, \mathscr{C}$ in $\overline{\Omega}$ uniquely.

Second order linearization and an integral identity

We have (using integration by parts)

$$\int_0^T \int_{\partial\Omega} \left(\frac{\partial^2}{\partial\epsilon_1 \partial\epsilon_2} \Lambda(\epsilon_1 u^{(1)} + \epsilon_2 u^{(2)})|_{\epsilon_1=\epsilon_2=0} \right) u^{(0)}\, dSdt$$
$$= \int_0^T \int_{\Omega} \mathcal{G}(\nabla u^{(1)}, \nabla u^{(2)}, \nabla u^{(0)})\, dxdt,$$

where $u^{(1)}, u^{(2)}, u^{(0)}$ are solutions to the linear elastic wave equations.

- $\mathcal{G}$ contains information of $\mathscr{A}, \mathscr{B}, \mathscr{C}$
- general strategy: construct special solutions $u^{(1)}, u^{(2)}, u^{(0)}$ and try to extract information about $\mathscr{A}, \mathscr{B}, \mathscr{C}$
- a lot of freedoms in choosing $u^{(1)}, u^{(2)}, u^{(0)}$: *P-P-P, P-P-S, P-S-S, P-S-P, S-S-P, S-S-S*

Explicit form of $\mathscr{G}$

$$
\begin{aligned}
&\mathcal{G}(\nabla u^{(1)}, \nabla u^{(2)}, \nabla u^{(0)}) = (\lambda + \mathscr{B})(\nabla u^{(1)} : \nabla u^{(2)})(\nabla \cdot u^{(0)}) + 2\mathscr{C}(\nabla \cdot u^{(1)})(\nabla \cdot u^{(2)})(\nabla \cdot u^{(0)}) \\
&+ \mathscr{B}\left((\nabla \cdot u^{(1)})(\nabla u^{(2)} : \nabla^T u^{(0)}) + (\nabla \cdot u^{(2)})(\nabla u^{(1)} : \nabla^T u^{(0)}) + (\nabla u^{(1)} : \nabla^T u^{(2)})(\nabla \cdot u^{(0)}\right) \\
&+ \mathscr{B}(\nabla u^{(1)} : \nabla^T u^{(2)})(\nabla \cdot u^{(0)}) + \frac{\mathscr{A}}{4}\left(\frac{\partial u_j^{(1)}}{\partial x_m}\frac{\partial u_m^{(2)}}{\partial x_i} + \frac{\partial u_j^{(2)}}{\partial x_m}\frac{\partial u_m^{(1)}}{\partial x_i}\right)\frac{\partial u_i^{(0)}}{\partial x_j} \\
&+ (\lambda + \mathscr{B})\left((\nabla \cdot u^{(1)})(\nabla u^{(2)} : \nabla u^{(0)}) + (\nabla \cdot u^{(2)})(\nabla u^{(1)} : \nabla u^{(0)})\right) \\
&+ \left(\mu + \frac{\mathscr{A}}{4}\right)\left(\frac{\partial u_m^{(1)}}{\partial x_i}\frac{\partial u_m^{(2)}}{\partial x_j} + \frac{\partial u_m^{(2)}}{\partial x_i}\frac{\partial u_m^{(1)}}{\partial x_j} + \frac{\partial u_i^{(1)}}{\partial x_m}\frac{\partial u_j^{(2)}}{\partial x_m} + \frac{\partial u_i^{(2)}}{\partial x_m}\frac{\partial u_j^{(1)}}{\partial x_m} + \frac{\partial u_i^{(1)}}{\partial x_m}\frac{\partial u_m^{(2)}}{\partial x_j} + \frac{\partial u_i^{(2)}}{\partial x_m}\frac{\partial u_m^{(1)}}{\partial x_j}\right)\frac{\partial u_i^{(0)}}{\partial x_j}.
\end{aligned}
$$

- S-waves are divergence-free
- can only recover $\mathscr{C}$ using P-P-P

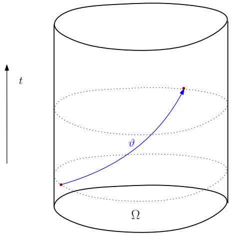
Gaussian beam solutions

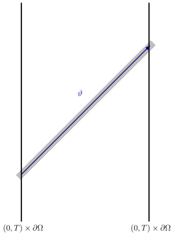
Solutions of the form

$$u(t,x) = \underbrace{e^{\mathrm{i}\varrho\varphi(t,x)}\mathfrak{a}_\varrho(t,x)}_{\textit{principle term}} + \underbrace{R_\varrho(t,x)}_{\textit{remainder}},$$

with a large parameter ϱ.

- the principal term is supported near a null geodesic ϑ in $((0,T)\times\Omega, -\mathrm{d}t^2 + c_{P/S}^2\mathrm{d}s^2)$
- φ: phase function, complex-valued
- $\mathrm{Im}(D^2\varphi)(X,X) > 0$ if X is normal to ϑ
- $R_\varrho \to 0$ as $\varrho \to +\infty$

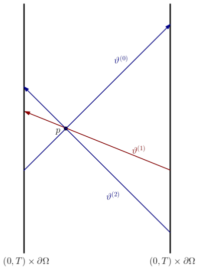




Recovery of $\mathscr{A}, \mathscr{B}$

- $u_\varrho^{(1),P} = e^{\mathrm{i}\varrho\varphi^{(1),P}}\mathfrak{a}_\varrho^{(1)} + R_\varrho^{(1)}$ representing P-waves;
- $u_\varrho^{(2),S} = e^{\mathrm{i}\varrho\varphi^{(2),S}}\mathfrak{a}_\varrho^{(2)} + R_\varrho^{(2)}$ representing S-waves;
- $u_\varrho^{(0),S} = e^{\mathrm{i}\varrho\varphi^{(0),S}}\mathfrak{a}_\varrho^{(0)} + R_\varrho^{(0)}$ representing S-waves.

The three waves intersect at a single point p



Pointwise recovery

Extract the oscillatory integral

$$\int_0^T \int_\Omega e^{\mathrm{i}\varrho(\varphi^{(1),P}+\varphi^{(2),S}+\varphi^{(0),S})}\mathcal{A}_\varrho(t,x)\mathrm{d}x\mathrm{d}t + o(1),$$

where $\mathcal{A}_\varrho$ is supported in a neighborhood of p. Need

$$\nabla(\varphi^{(1),P} + \varphi^{(2),S} + \varphi^{(0),S})(p) = 0$$

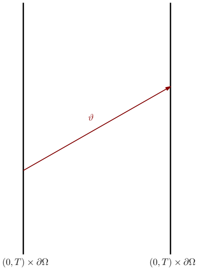
to apply the method of stationary phase to recover $\mathcal{A}_\varrho(p)$. Can be done by choosing $\vartheta^{(1)}, \vartheta^{(2)}, \vartheta^{(0)}$ properly.
(Impossible to have $\nabla(\varphi^{(1),P} + \varphi^{(2),P} + \varphi^{(0),P})(p) = 0$!)

Recover the parameters $\mathscr{A}$ and $\mathscr{B}$ at the point p (actually at x_p, $p = (t_p, x_p)$).

Recovery of $\mathscr{C}$

- $u_\varrho^{(1),P} = e^{\mathrm{i}\varrho\varphi^{(1),P}}\mathfrak{a}_\varrho^{(1)} + R_\varrho^{(1)}$ representing P-waves;
- $u_\varrho^{(2),P} = e^{\mathrm{i}\varrho\varphi^{(2),P}}\mathfrak{a}_\varrho^{(2)} + R_\varrho^{(2)}$ representing P-waves;
- $u_\varrho^{(0),P} = e^{\mathrm{i}\varrho\varphi^{(0),P}}\mathfrak{a}_\varrho^{(0)} + R_\varrho^{(0)}$ representing P-waves.

The three waves are concentrated near the same null geodesic ϑ.

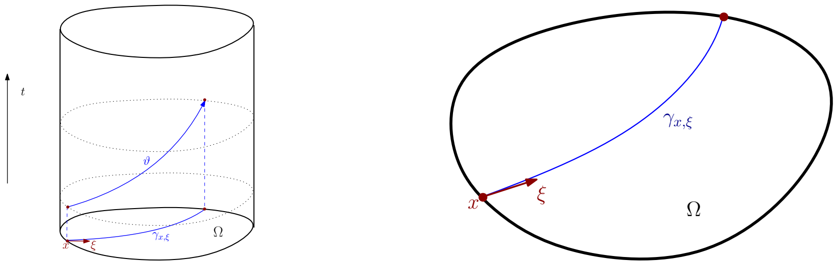


Weighted geodesic ray transform

Obtain the Jacobi weighted ray transform

$$\int_{\gamma_{x,\xi}} \mathscr{C} c_P^{-9/2} \rho^{-3/2} (\det Y(t))^{-1/2} \mathrm{d}t$$

where $\gamma_{x,\xi}$ is the projection of ϑ onto $(\Omega, c_P^{-2}\mathrm{d}s^2)$. $Y(t)$: some complex tensor field along $\gamma_{x,\xi}$ satisfying the Jacobi equation: many weights



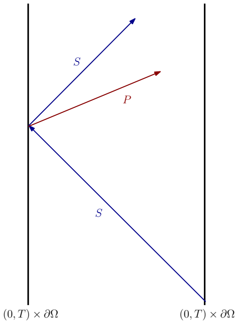
recover $\mathscr{C}$ from the above ray transform (Feizmohammadi-Oksanen, 2020)

Second version of our result

- assume that $\frac{\lambda}{\rho}$ and $\frac{\mu}{\rho}$ are already recovered from Λ^{lin}.
- need extra technical assumptions to determine ρ in the linear model
- to use the nonlinearity to determine ρ

Difficulties

- geometries are known – the trajectories of the waves are known
- linear model is not fully known – hard to control the reflection of the waves
  - mode conversion at the boundary
  - evanescent waves
- carefully choose the trajectories to avoid multiple intersections



Determination of the parameters

Amplitudes of P and S waves (leading order term):

$$|\mathbf{a}_P| = \det(Y_P)^{-1/2} c_P^{-3/2} \rho^{-1/2}, \quad |\mathbf{a}_S| = \det(Y_S)^{-1/2} c_S^{-3/2} \rho^{-1/2}$$

- use S-S-P waves to recover $\rho^{-3/2}(\lambda + \mathscr{B})$ and $\rho^{-3/2}(4\mu + \mathscr{A})$
- use P-S-P waves to recover $\rho^{-3/2}(3\mu + \lambda + \mathscr{A} + 2\mathscr{B})$
- determine $\rho^{-3/2}(\lambda + \mu)$ from above
- ρ is determined since $\frac{\lambda+\mu}{\rho}$ is known
- $\mathscr{A}$ and $\mathscr{B}$ can be determined also
- determine $\mathscr{C}$ finally

Summary

Recovery of the six parameters $\lambda, \mu, \rho, \mathscr{A}, \mathscr{B}, \mathscr{C}$:

- recover Λ^{lin} from Λ by first order linearization
- recovery of $\frac{\lambda}{\rho}, \frac{\mu}{\rho}$ from Λ^{lin}: reduced to lens rigidity problem
- recovery of $\rho, \mathscr{A}, \mathscr{B}$ from second order linearization of Λ: pointwise recovery
- recovery of $\mathscr{C}$: invert a weighted ray transform