Unique Continuation Properties and Uncertainty Principles for Discrete (Nonlocal) Elliptic Operators

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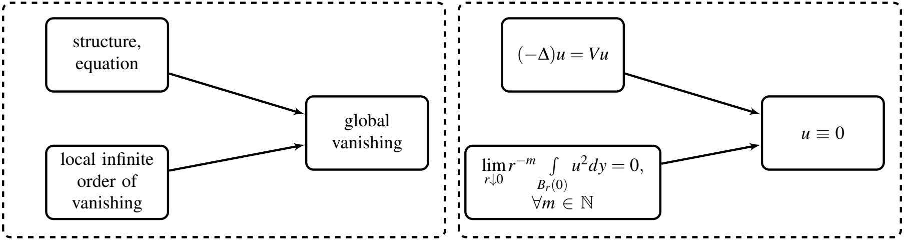
Unique Continuation, Uncertainty Principles and Rigidity

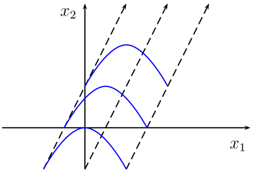


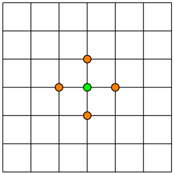
UCP ≈ Rigidity ≈ Uncertainty Principle.

Challenge: Discrete approximations of (nonlocal) elliptic equations $\rightsquigarrow$ high- but finite-dimensional problem $\rightsquigarrow$ loss of rigidity!

Question: How strong is this loss?

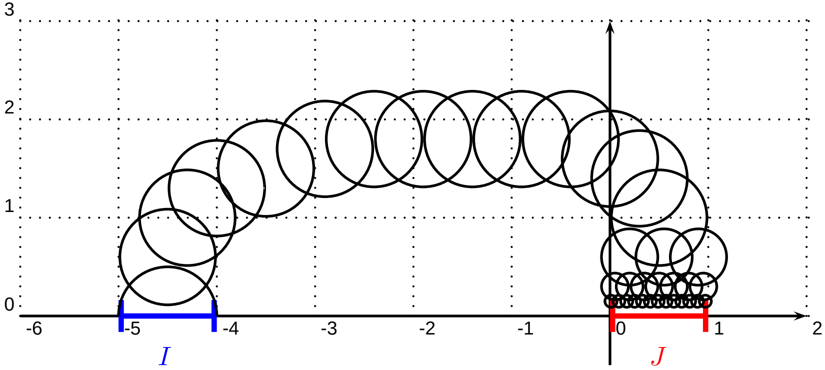
Unique Continuation





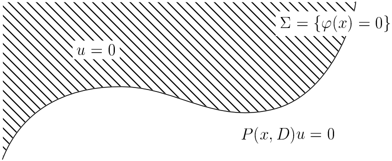
- Rigidity $\rightsquigarrow$ generalization of analyticity,
- control theory and inverse problems [Lions '88; Zuazua '04; Ammari-Uhlmann '04; Alessandrini-Vessella '05; Ghosh-Salo-Uhlmann '16; R.-Salo '20,...],
- uniqueness [Lenzmann-Silvestre '16],
- nodal domains of eigenfunctions [Donnelly-Fefferman '88; Logunov '18],
- strict domain monotonicity of eigenfunctions [de Figueiredo-Gossez '92],
- non-existence results for positive eigenvalues in Schrödinger operators [Jerison-Kenig '85; Koch-Tataru '06],
- localization properties [Bourgain-Kenig '05],
- compactness in nonlinear problems e.g. free boundary value problems [Koch-R.-Shi '16].

Unique Continuation and (Nonlocal) Inverse Problems



- Uniqueness properties and stability estimates, e.g. nonlocal Calderón problem, X-ray tomography [Ghosh-Salo-Uhlmann '20], [Salo-R. '20], [Ghosh-Salo-R.-Uhlmann '20], [García-Ferrero-R. '20], [R.'20], [Ilmavirta-Mönkkönen '19].

Unique Continuation for Schrödinger Operators

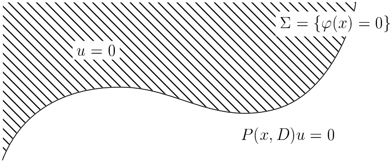


$$\boxed{\Delta u = Vu}$$

$$\left.\begin{array}{l} u(r) \sim r^k \\ \Delta u \sim r^{k-2} \end{array}\right\} \Rightarrow |V| = \frac{|\Delta u|}{|u|} \sim r^{-2}.$$

- Different qualitative forms of UCP: weak UCP, strong UCP, UCP from measurable sets, UCP from the boundary.
- [Carleman], [Jerison & Kenig '85], [Koch & Tataru '01]: Carleman estimates deducing different forms of the UCP.

Unique Continuation for Schrödinger Operators



$$\boxed{h^{-2}\,\Delta_d\, u \quad = V u}$$

- Different qualitative forms of UCP: weak UCP, strong UCP, UCP from measurable sets, UCP from the boundary.
- [Carleman], [Jerison & Kenig '85], [Koch & Tataru '01]: Carleman estimates deducing different forms of the UCP.

Loss of Weak UCP for Discrete Harmonic Functions

$$
\begin{array}{ccccccc}
 & & & 2^{-m} & & & \\
 & & -2^{-m-1} & 0 & -2^{-m-1} & & \\
 & 2^{-m-1} & 0 & 0 & 0 & 2^{-m-1} & \\
-2^{-m} & 0 & 0 & 0 & 0 & 0 & -2^{-m} \\
 & 2^{-m-1} & 0 & 0 & 0 & 2^{-m-1} & \\
 & & -2^{-m-1} & 0 & -2^{-m-1} & & \\
 & & & 2^{-m} & & &
\end{array}
$$

[Fernández-Bertolin-Vega '17]

- **Structure preservation:** maximum principles; regularity estimates [Ciaurri, Gillespie, Roncal, Torrea, Varona '17];...
- **Differences in uncertainty principles:** Failure of qualitative (local) UCP [Fernández-Bertolin-Vega '17]; decay behaviour at infinity (Landis type results) [Fernández-Bertolin-Malinnikova '21]; stronger Liouville properties in 2D [Buhovsky, Logunov, Malinnikova, Sodin '17]; construction of complex geometric optics solutions [Ervedoza, De Gournay '11], ...

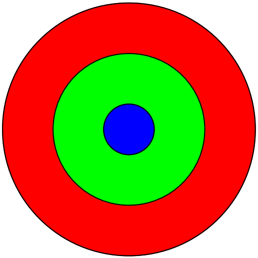
Question: Is the rigidity of elliptic equations completely lost?

The Quantitative Unique Continuation Property

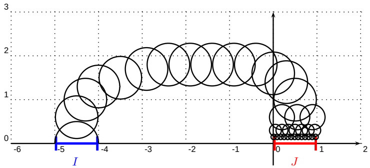
Theorem (Hadamard three balls theorem)

There exists $\alpha \in (0,1)$ such that for any harmonic function $u : B_5 \to \mathbb{R}$ it holds

$$\|u\|_{L^2(B_2)} \leq C\|u\|_{L^2(B_1)}^{\alpha}\|u\|_{L^2(B_4)}^{1-\alpha}.$$



- quantitative form of UCP,
- tool to control order of vanishing,
- tool to propagate information.



The Discrete Quantitative UCP

$$P_h f(n) := h^{-2}\,\Delta_d\, f(n) + h^{-1}\sum_{j=1}^{d} B_j(n) D^h_{+,j} f(n) + V(n) f(n)$$

Theorem (Fernández-Bertolin-Roncal-R.-Stan, Calc. Var. PDE '22)

There exists $\alpha \in (0,1)$, $c_0 > 0$, $h_0 \in (0,1)$ *and* $C > 1$ *such that for all* $h \in (0, h_0)$ *and* $u : (h\mathbb{Z})^d \to \mathbb{R}$ *with* $P_h u = 0$ *in* B_4 *we have*

$$\|u\|_{L^2(B_1)} \le C\big(\|u\|^{\alpha}_{L^2(B_{1/2})}\|u\|^{1-\alpha}_{L^2(B_2)} + e^{-c_0 h^{-1}}\|u\|_{L^2(B_2)}\big).$$

Earlier results:

- [Guadie-Malinnikova '13, '14]: analogous estimates for discrete harmonic functions; explicit representation formulas; optimality.
- [Mangoubi-Lippner '15, '17]: absolutely monotonic functions $\leftrightarrow$ UCP for discrete harmonic functions; optimality.

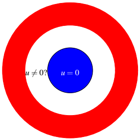
Key Ideas – A Carleman Estimate

Theorem (Fernández-Bertolin-Roncal-R.-Stan, Calc. Var. PDE '22)

There exists $h_0, \delta_0 \in (0,1)$ *with* $h_0 < \delta_0$, $C > 1$ *and* $\tau_0 > 1$ *such that for all* $h \in (0, h_0)$ *and* $\tau \in (\tau_0, \delta_0 h^{-1})$ *we have*

$$\tau^3 \|e^{\tau\phi} u\|_{L^2} + \tau \|e^{\tau\phi} h^{-1} D_s u\|_{L^2}^2 + \tau^{-1} \|e^{\tau\phi} h^{-2} D_s^2 u\|_{L^2}^2 \leq C \|e^{\tau\phi} h^{-2} \Delta_d u\|_{L^2}^2.$$

- Subelliptic estimate.
- Would be possible to extend to variable coefficient metrics.
- Challenge: discrete length scale; close to continuum for $\tau \in (\tau_0, \delta_0 h^{-1})$.



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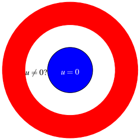
- Spectrum of $h^{-2}\Delta_d$ and Δ only close for limited range.
- Study conjugate operator $e^{\tau\phi} h^{-2} \Delta_d e^{-\tau\phi}$; close to continuum operator $e^{\tau\phi} \Delta e^{-\tau\phi}$ if $\tau \in (\tau_0, \delta_0 h^{-1})$.
- perturbative strategy: localization and freezing arguments.
- Expect: for $\tau \geq \delta_0 h^{-1}$ genuinely discrete effects.



Global UCP for the Fractional Laplacian

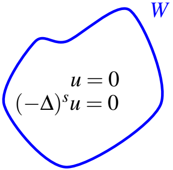
$$(-\Delta)^s f(x) := \mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}u)(x) = C P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy.$$

Theorem (Global UCP, Ghosh-Salo-Uhlmann '21)

Let $u \in H^r(\mathbb{R}^n)$ with $r \in \mathbb{R}$. Assume that for some open set $\Omega \subset \mathbb{R}^n$

$$u = 0 \text{ and } (-\Delta)^s u = 0.$$

Then $u \equiv 0$.



- Further (weak, strong, measurable) UCP: [Fall-Felli '14, Seo '14, R. '15, Yu '17, García-Ferrero-R. '19, Ghosh-Salo-R.-Uhlmann '21].

- genuinely nonlocal property.
- relies on boundary UCP and Carleman estimates [R. 15].

Global UCP for the Fractional Laplacian

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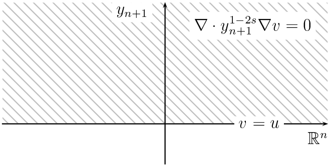

- Further (weak, strong, measurable) UCP: [Fall-Felli '14, Seo '14, R. '15, Yu '17, García-Ferrero-R. '19, Ghosh-Salo-R.-Uhlmann '21].

- dual: Runge approximation properties [Dipierro-Savin-Valdinoci '17, '19], [Ghosh-Salo-Uhlmann '20], [R.-Salo '20], [Ghosh-Salo-R.-Uhlmann '20] .

Global UCP for the Fractional Laplacian

$$(-\Delta)^s f(x) := \mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}u)(x) = C P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy.$$

Rigidity $\leftrightarrow$ Antilocality

$u \equiv 0, (-\Delta)^s u \equiv 0$ in W
$\Rightarrow u \equiv 0$ in $\mathbb{R}^n$

duality $\longleftrightarrow$ well-posedness

Flexibility $\leftrightarrow$ Approximation

$\overline{\mathcal{R}} := \overline{\{u = P_s f; f \in C_c^\infty(W)\}}$
$= L^2(\Omega)$
P_s Poisson op.

- Persistence of global UCP for discrete counterpart?

The Fractional Discrete Laplacian I

Fourier symbol representation

$$\mathcal{F}_{h\mathbb{Z}}((-\Delta_d)^s u)(h\xi) := h^{-2s}\left(\sum_{j=1}^{d} 4\sin^2(h\xi_j/2)\right)^s \mathcal{F}_{h\mathbb{Z}} u(h\xi).$$

Semi-discrete Caffarelli-Silvestre Extension

$$\begin{aligned}(\partial_t t^{1-2s}\partial_t + t^{1-2s}\,\Delta_d)\tilde{u} = 0 \text{ in } (h\mathbb{Z})^d \times \mathbb{R}_+,\\ \tilde{u} = u \text{ on } (h\mathbb{Z})^d \times \{0\},\end{aligned}$$

and $(-\Delta_d)^s u_j := C_s \lim_{t\to 0} t^{1-2s}\partial_t \tilde{u}_j(t)$.

The Fractional Discrete Laplacian II

Heat semi-group and kernel representation

$$(-\Delta_d)^s u_j = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t\Delta_d} u_j - u_j) \frac{dt}{t^{1+s}}$$
$$= \sum_{m \in \mathbb{Z}^d, m \neq j} (u_j - u_m) K_s^h(j-m), \text{ with}$$

$$K_s^h(m) = \frac{1}{h^{2s}} \frac{1}{|\Gamma(-s)|} \int_0^\infty e^{-2dt} \prod_{j=1}^d I_{m_j}(2t) \frac{dt}{t^{1+s}}, \ m \in \mathbb{Z}^d \setminus \{0\},$$

$$K_s^h(0) = 0.$$

- Expression of K_s^h explicit in 1D.

Failure of the (Naive) Global UCP; Exterior UCP

Theorem (Failure of Naive Global UCP, FB-R-R. '22)

Let $X \subset (h\mathbb{Z})^d$ be a finite set of cardinality $n \in \mathbb{N}$. Then there exists a non-trivial function $u : (h\mathbb{Z})^d \to \mathbb{R}$ such that $u_j = 0 = (-\Delta_d)^s u_j$ for $j \in X$.

- For $j \in X$:

$$(-\Delta_d)^s u_j = \sum_{m \in \mathbb{Z}^d, m \neq j} u_m K_s^h(j-m) \stackrel{!}{=} 0.$$

- Set $u_j = 0$ for all $j \in \mathbb{Z}^d \setminus X \cup M$, where $X \cap M = \emptyset$ and $\#M > n+1$.
- Homogeneous system of n equations with $n+1$ unknowns $\rightsquigarrow$ solvability and existence.

Failure of the (Naive) Global UCP; Exterior UCP

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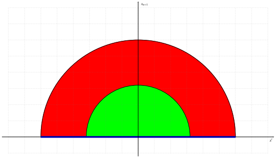
Theorem (Persistence of Exterior UCP, FB-R-R. '22)

Assume that $u \in H^r((h\mathbb{Z})^d)$ for some $r \in \mathbb{R}$ and that $\mathrm{supp}(u)$, $\mathrm{supp}((-\Delta_d)^s u) \subset B_R$ for some $R > 0$. Then $u \equiv 0$.

- Idea: Presence of branch-cut; similar arguments in [Isakov '90], [R.-Salo '20].
- Analogous argument in upper half-plane yields weak UCP from upper half-plane.
- How badly does the global UCP fail in the discrete case?

Quantitative UCP from the Boundary I

$$(-\Delta)^{\frac{1}{2}}u(x) = \lim_{t\to 0} \partial_t \tilde{u}(x,t).$$



Theorem (Jerison-Lebeau '99)

Let $V \in L^{\infty}(\mathbb{R}^{d+1}_+)$ and let $\tilde{u}$ be a solution to

$$(\Delta + \partial_t^2)\tilde{u} = V\tilde{u} \text{ in } \mathbb{R}^{d+1}_+, \; \tilde{u} = u \text{ on } \mathbb{R}^d \times \{0\}.$$

Then there exists $C > 0$, $\alpha \in (0,1)$ such that

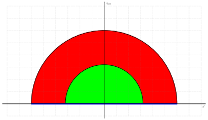
$$\|\tilde{u}\|_{L^2(B_1^+)} \leq C \|\tilde{u}\|^{1-\alpha}_{L^2(B_4^+)} \left(\|u\|_{H^1(B_4')} + \|\partial_t \tilde{u}\|_{L^2(B_4')} \right)^{\alpha}.$$

- Quantitative UCP for Dirichlet and Neumann data for Laplacian.

- More generally available for weighted operators [R.-Salo '20].

Quantitative UCP from the Boundary II

$$(-\Delta)^{\frac{1}{2}}u(x) = \lim_{t\to 0} \partial_t \tilde{u}(x,t).$$



Theorem (Fernández-Bertolin-Roncal-R. '22)

Let $V \in L^\infty(\mathbb{R}^{d+1}_+)$ *and let* $\tilde{u}$ *be a solution to*

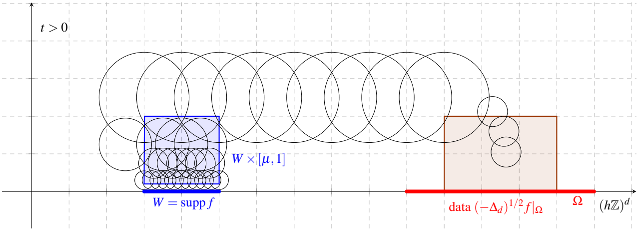
$$(\Delta_d + \partial_t^2)\tilde{u} = V\tilde{u} \text{ in } (h\mathbb{Z})^d \times \mathbb{R}_+, \ \tilde{u} = u \text{ on } (h\mathbb{Z})^d \times \{0\}.$$

Then there exists $C > 0$, $\alpha \in (0,1)$ *such that*

$$\|\tilde{u}\|_{L^2(B_1^+)} \leq C\|\tilde{u}\|^{1-\alpha}_{L^2(B_4^+)} \left(\|u\|_{H^1(B_4')} + \|\partial_t\tilde{u}\|_{L^2(B_4')}\right)^\alpha + e^{-Ch^{-1}}\|\tilde{u}\|_{L^2(B_4^+)}.$$

- Only exponentially small correction term needed!

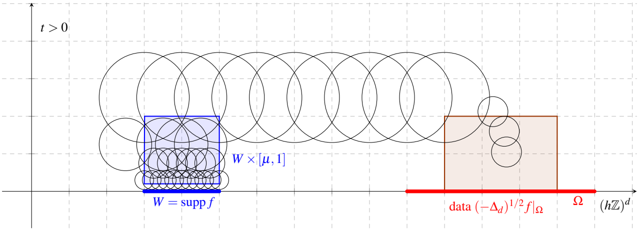
An Application to a Linear Inverse Problem



Inverse Problem: Given $(-\Delta_d)^{\frac{1}{2}} f|_\Omega$ recover $f \in C_c^\infty(W \cap (h\mathbb{Z})^d)$.

- Related to problems from X-ray tomography.
- For fixed $h > 0$ Lipschitz stability; continuum problem exponentially ill-posed.
- **Objective:** Uniform stability estimates in lattice spacing $h > 0$!

An Application to a Linear Inverse Problem



Theorem (Fernández-Bertolin-Roncal-R. '22)

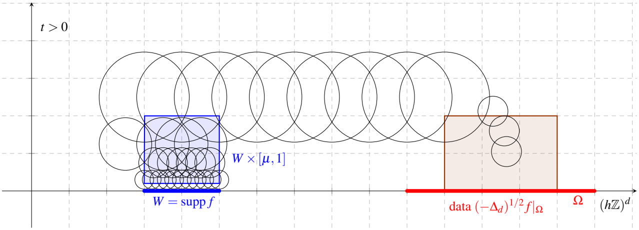
Let $\epsilon := \frac{\|(-\Delta_d)^{\frac{1}{2}} f\|_{L^2(\Omega)}}{\|f\|_{H^1(W)}}$. *There exists* $\nu \in (0,1)$ *such that if*

$$0 < h_0 < |\log(\epsilon)|^{-1+\nu} |\log(-C \log(\epsilon))|^{-1},$$

then the following estimate holds:

$$\|f\|_{L^2(W)} \leq C |\log(\epsilon)|^{-\nu} \|f\|_{H^1(W)} + C \exp(-C h^{-1} |\log(\epsilon)|^{-1+\nu}) \|f\|_{H^1(W)}.$$

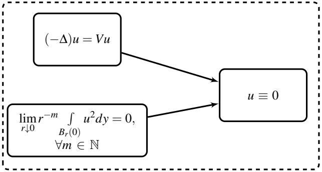
An Application to a Linear Inverse Problem



- In the limit $h_0 \to 0$ matches analogous results from [García-Ferrero-R. '20], [R.-Salo '20].
- Boundary-bulk interpolation inequality.
- Chain of balls arguments via doubling + optimization step.
- Limiting factor: lattice size.

Summary

- Discretization counteracts UCP rigidity.
- Naive UCP results fail in general.
- Possible to **robustly** recover UCP up to exponentially small errors.



Further questions:

- Dependence on lattice structures?
- Useful for nonlinear inverse problems?
- Duality results?
- Variants of UCP at more global scales?