The background of the slide is a photograph of a modern building. On the right side, there is a prominent tower with a glass-enclosed top section and a metallic, cylindrical structure on top. The building has multiple floors with large windows. In the foreground, there is a paved walkway leading towards the building, flanked by a low brick wall and some greenery. The sky is clear and blue.

Inverse fractional conductivity problem
University of Cambridge

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Outline

- 1 Inverse (fractional) conductivity problem
- 2 Nonlocal Calderón problems
- 3 Global uniqueness
- 4 Counterexamples for disjoint sets of measurements
- 5 Stability with full data
- 6 References

Collaborators and acknowledgements

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Inverse conductivity problem (Calderón, 1980)

- Is it possible to determine the electrical conductivity of a medium by making voltage and current measurements on its boundary?

$$\Delta u = 0 \text{ in } \Omega, \quad u|_{\partial\Omega} = f.$$

- Suppose one knows the DN map $\Lambda f = u|_{\partial\Omega}$, can we determine the electrical conductivity $\sigma : \Omega \rightarrow \mathbb{R}$ uniquely?
- Mathematical model for the **electrical impedance tomography (EIT)**.

Classical Calderón problem (n = 3)

Boundary determination (uniqueness for real-analytic) (Kohn-Vogelius, 1984).

Interior uniqueness when n = 3 (Sylvester-Uhlmann, 1987).

A reconstruction method (Nachman, 1988).

Logarithmic stability (Alessandrini, 1988) and optimality (Mandache, 2001).

Studied typically via the Liouville transformation :

$$r(x) \Delta u = \Delta u + q(x)u; \quad q = \Delta \phi(x):$$

The inverse problem is then solved using the complex geometric optics (CGO) solutions and their behaviour when $h \rightarrow 0$:

$$u(x) = e^{i \phi(x)}(1 + r(x)):$$

Some basic definitions

We say that an open set $\Omega \subset \mathbb{R}^n$ of the form $\Omega = \mathbb{R}^k \times \omega$, where $k > 0$ and $\omega \subset \mathbb{R}^k$ is a bounded open set, is a cylindrical domain.

We say that an open set $\Omega \subset \mathbb{R}^n$ is bounded in one direction if there exists a cylindrical domain $\Omega_1 \subset \mathbb{R}^n$ and a rigid Euclidean motion $A(x) = Lx + x_0$, where L is a linear isometry and $x_0 \in \mathbb{R}^n$, such that $\Omega \subset A(\Omega_1)$.

The fractional gradient is defined for all sufficiently regular functions by the formula

$$r^{-s}u(x; y) = \frac{C_{n;s}}{2} \frac{u(x) - u(y)}{|x - y|^{n=2+s+1}}(x - y)$$

and div_s denotes its adjoint operator. In particular, $\operatorname{div}_s(r^{-s}u) = (\quad)^{-s}u$ in the weak sense for all $u \in H^s(\mathbb{R}^n)$.

Fractional conductivity equation

Let $s \in (0; 1)$ and consider the Dirichlet problem for the fractional conductivity equation :

$$\begin{aligned} \operatorname{div}_s(r^{-s}u) &= 0 && \text{in } \Omega; \\ u &= f && \text{in } \Omega_e; \end{aligned} \quad (1)$$

where $\Omega_e := \mathbb{R}^n \setminus \bar{\Omega}$ is the exterior of the domain Ω ,
 $\gamma_{\pm} = \gamma_{\pm}^{1-2s}(x) \gamma_{\pm}^{1-2s}(y)$ depends on the global, elliptic,
 conductivity $\gamma \in L^1_+(\mathbb{R}^n)$.

We say $u \in H^s(\mathbb{R}^n)$ is a (weak) solution of (1) if the bilinear form

$$B(u; v) := \frac{C_{n;s}}{2} \int_{\mathbb{R}^{2n}} \frac{\gamma_{\pm}^{1-2s}(x) \gamma_{\pm}^{1-2s}(y)}{|x-y|^{n+2s}} (u(x) - u(y))(v(x) - v(y)) dx dy$$

vanishes for all $v \in C^1_c(\mathbb{R}^n)$ and $u - f \in H^s(\mathbb{R}^n) := \overline{C^1_c(\mathbb{R}^n)}^{H^s(\mathbb{R}^n)}$.

Inverse fractional conductivity problem

Let $\Omega \subset \mathbb{R}^n$ be an open set which is bounded in one direction and $0 < s < \min(1; n-2)$. Assume that $\sigma \in L^1(\Omega)$ satisfy $\sigma_0 > 0$.

For all $f \in X := H^s(\mathbb{R}^n) = \mathcal{H}^s(\Omega)$ there are unique weak solutions $u_f \in H^s(\mathbb{R}^n)$ of the fractional conductivity equation

$$\begin{aligned} \operatorname{div}_s(\sigma \nabla_s u) &= 0 && \text{in } \Omega; \\ u &= f && \text{in } \mathbb{R}^n \setminus \Omega. \end{aligned}$$

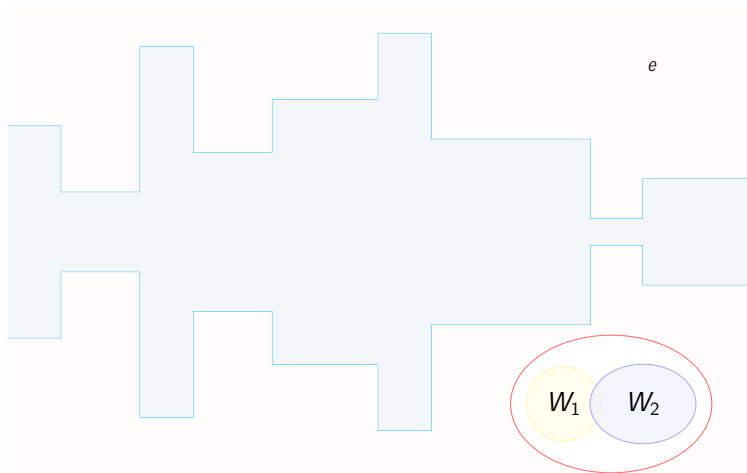
The exterior DN maps $\mathcal{B} : X \rightarrow X$ given by

$$\mathcal{B}(f; g) := \langle u_f; g \rangle;$$

where $u_f \in H^s(\mathbb{R}^n)$ is the unique solution to the fractional conductivity equation, is a well-defined bounded linear map.

The inverse fractional conductivity problem asks (Covi, 2020): Suppose that $\mathcal{B}_1 = \mathcal{B}_2$, does it imply that $\sigma_1 = \sigma_2$?

Geometric illustration of related domains



Terminology for abstract nonlocal Calderón's problems

Let $s \in \mathbb{R}$ and $B: H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \rightarrow \mathbb{R}$ be a bounded bilinear form:

- (i) We say that B has the **left UCP** on an open nonempty set $W \subset \mathbb{R}^n$ when the following holds: If $u \in H^s(\mathbb{R}^n)$, $u|_W = 0$ and $B(u, \cdot) = 0$ for all $\cdot \in C_c(W)$, then $u = 0$.
- (ii) We say that B has the **right UCP** on an open nonempty set $W \subset \mathbb{R}^n$ when the following holds: If $u \in H^s(\mathbb{R}^n)$, $u|_W = 0$ and $B(\cdot, u) = 0$ for all $\cdot \in C_c(W)$, then $u = 0$.
- (iii) We say that B is **local** when the following holds: If $u, v \in H^s(\mathbb{R}^n)$ and $\text{supp}(u) \cap \text{supp}(v) = \emptyset$, then $B(u, v) = 0$.

Abstract nonlocal Calderón problems

Lemma

Let $s \in \mathbb{R}$, and $\Omega \subset \mathbb{R}^n$ be open set such that $\Omega_e = \mathbb{R}^n \setminus \Omega$. Let $B: H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \rightarrow \mathbb{R}$ be a bounded bilinear form that is (strongly) coercive in $\tilde{H}^s(\Omega)$, that is, there exists some $c > 0$ such that $B(u, u) \geq c \|u\|_{H^s(\mathbb{R}^n)}^2$ for all $u \in \tilde{H}^s(\Omega)$. Then the following hold:

- 1 Existence of solutions: For any $f \in H^s(\mathbb{R}^n)$ and $F \in (\tilde{H}^s(\Omega))'$ there exists a unique $u \in H^s(\mathbb{R}^n)$ such that $u - f \in \tilde{H}^s(\Omega)$ and $B(u, \cdot) = F(\cdot)$ for all $\cdot \in \tilde{H}^s(\Omega)$. When $F = 0$, we denote this unique solution by u_f .
- 2 Let $X := H^s(\mathbb{R}^n)/\tilde{H}^s(\Omega)$ be the abstract trace space. Then the exterior DN map $\Lambda_B: X \rightarrow X'$ defined by $\Lambda_B[f][g] := B(u_f, g)$ for $[f], [g] \in X$ is a well-defined bounded linear map.

Runge approximation property

One may prove the following functional analytic theorem using the ideas of Ghosh–Salo–Uhlmann (2020), Cekić–Lin–Rüland (2020), Covi–Mönkkönen–R.–Uhlmann (2022):

Theorem (R.-Zimmermann, 2022)

Let $s \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$ be an open set such that $\Omega_e = \cdot$. Let $L, q: H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \rightarrow \mathbb{R}$ be bounded bilinear forms and assume that q is local and that $B_{L,q} := L + q$ is (strongly) coercive in $\tilde{H}^s(\Omega)$.

- i If L has the right UCP on a nonempty open set $W \subset \Omega_e$, then $R(W) := \{u_f - f; f \in C_c(W)\} \subset \tilde{H}^s(\Omega)$ is dense.
- ii If L has the left UCP on a nonempty open set $W \subset \Omega_e$, then $R(W) := \{u_g - g; g \in C_c(W)\} \subset \tilde{H}^s(\Omega)$ is dense.

Example (R.-Zimmermann, 2022)

Let us denote $B = B(0; \delta) \subset \mathbb{R}^n$ for any $\delta > 0$ and $n \geq 1$. For any $\delta > 0$, $s \in \mathbb{R}_+ \setminus \mathbb{Z}$ and $\Omega := \mathbb{R}^n \setminus \overline{B}$, the restriction to $\mathbb{R}^n \setminus \overline{B}$ of the unique solutions u_f to the equation $((-\Delta)^s + \delta)u = 0$ in $\mathbb{R}^n \setminus \overline{B}$ are dense in $\tilde{H}^s(\mathbb{R}^n \setminus \overline{B})$ with exterior conditions $f \in C_c(B)$.

$$((-\Delta)^s + \delta)u = 0$$

$$u = f$$

B

Generalized Ghosh–Salo–Uhlmann theorem

Theorem (R.-Zimmermann, 2022)

Let $s \in \mathbb{R}$, and $\Omega \subset \mathbb{R}^n$ be open such that $\Omega_e = \Omega \setminus \partial\Omega$. Let $L: H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \rightarrow \mathbb{R}$ be a bounded bilinear form with the following properties:

- 1 There exists a nonempty open set $W_1 \subset \Omega_e$ such that L has the right UCP on W_1 .
- 2 There exists a nonempty open set $W_2 \subset \Omega_e$ such that L has the left UCP on W_2 .
- 3 $W_1 \cap W_2 = \emptyset$.

Let $q_j: H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \rightarrow \mathbb{R}$, $j = 1, 2$, be local and bounded bilinear forms. Suppose that $B_{L,q_j} = L + q_j$ are (strongly) coercive in $\tilde{H}^s(\Omega)$. If the exterior data $\Lambda_{L,q_1}[f][g] = \Lambda_{L,q_2}[f][g]$ agree for all $f \in C_c(W_1)$ and $g \in C_c(W_2)$, then $q_1 = q_2$ in $\tilde{H}^s(\Omega) \times \tilde{H}^s(\Omega)$.

Examples from the literature

- $(-\Delta)^s + w$ where $w \in L^\infty(\Omega)$ and Ω is bounded where $L(u, v) = ((-\Delta)^{s/2}u, (-\Delta)^{s/2}v)$ and $q(u, v) = \int_{\mathbb{R}^n} wuv dx$ (Ghosh–Salo–Uhlmann, 2016). An extension to certain Sobolev multiplier perturbations w (Rüland–Salo, 2017).
- $L^s + w$ where L^s is a fractional power of an elliptic 2nd order operator L and $w \in L^\infty(\Omega)$ and Ω is bounded (Ghosh–Lin–Xiao, 2017).
- $(-\Delta)^s + w + c \cdot \nabla$, c a vector field, has 0th and 1st order terms (Cekić–Lin–Rüland, 2018).
- Extension for general local linear lower order perturbations $(-\Delta)^s + P$, $s \in \mathbb{R}_+ \setminus \mathbb{Z}$, $m \in \mathbb{N}$ such that $2s > m$, by $P = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$ in $M_0(H^{s-|\alpha|} / H^{-s})$ (Covi–Mönkkönen–R.–Uhlmann, 2021).
- ...and much more

New examples (R.-Zimmermann, 2022)

- (Domains without Poincaré inequalities) For $(-\Delta)^s + q$ in Ω where $s \in \mathbb{R}_+ \setminus \mathbb{Z}$ and the potential q is uniformly positive and bounded, i.e. $q \in L_+(\mathbb{R}^n)$.
- (Higher order perturbations) For $(-\Delta)^t + (-\Delta)^{s/2} ((-\Delta)^{s/2} \cdot) + q$ in Ω where $t \in \mathbb{R}_+ \setminus \mathbb{Z}$, $s \in 2\mathbb{Z}$ and $t < s$, and $q \in L_+(\mathbb{R}^n)$.
- (A small fractional perturbation of the conductivity equation – with exterior data) $(-\Delta)^t + \operatorname{div}(\cdot)$ where $t \in (0, 1)$, $L_+(\mathbb{R}^n)$, Ω bounded in one direction. (One can plug in an elliptic $L(\Omega; \mathbb{R}^{n \times n})$ anisotropic conductivity as well.)
- Solutions to the related exterior value problems are dense in the corresponding spaces $\tilde{H}^s(\Omega)$, $\tilde{H}^s(\Omega)$ and $\tilde{H}^1(\Omega)$, respectively.
- ...many other results extend to domains bounded in one direction.

Solving the inverse fractional conductivity problem

Define $m := 1/2 - 1$ and call it the *background deviation* of .

Theorem (R.–Zimmermann, 2022)

Let $\Omega \subset \mathbb{R}^n$ be an open set which is bounded in one direction and $0 < s < \min(1, n/2)$. Assume that $\sigma_1, \sigma_2 \in L^\infty(\mathbb{R}^n)$ are uniformly elliptic with $m_1, m_2 \in H^{s, n/s}(\mathbb{R}^n)$. Suppose that $W \subset \Omega_e$ is a nonempty open set such that σ_1, σ_2 are continuous a.e. in W . Then $\Lambda_{\sigma_1}|_W = \Lambda_{\sigma_2}|_W$ for all $f \in C_c(W)$ if and only if $\sigma_1 = \sigma_2$ in \mathbb{R}^n .

- When $m \in H^{2s, n/2s}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$ earlier by Covi–R.–Zimmermann (2022).
- Brown conjectured (2003) that the classical Calderón problem is solvable for $W^{1,p}(\Omega)$ conductivities when $p > n$ and Haberman proved (2014) uniqueness when $W^{1,n}(\Omega)$, $n = 3, 4$.

Two fundamental properties of DN maps

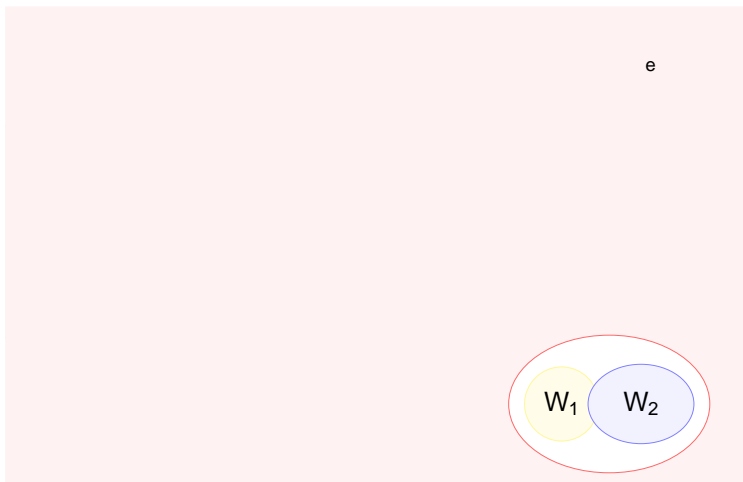
Theorem (Covi-R.-Zimmermann, R.-Zimmermann, 2022)

Let $\Omega \subset \mathbb{R}^n$ be an open set which is bounded in one direction and $0 < s < \min(1, n/2)$. Assume that $\Lambda_1, \Lambda_2 \in L^s(\mathbb{R}^n)$ are uniformly elliptic with $m_1, m_2 \in H^{s, n/s}(\mathbb{R}^n)$. Assume that $W_1, W_2 \subset \Omega_e$ are nonempty open sets and that $\Lambda_1|_{W_1} = \Lambda_2|_{W_2}$ holds. If $W_1 \cap W_2 = \emptyset$, then $\Lambda_1 = \Lambda_2$ for all $f \in C_c(W_1)$ if and only if $\Lambda_1 = \Lambda_2$ in \mathbb{R}^n .

Theorem (Covi-R.-Zimmermann, R.-Zimmermann, 2022)

Let $\Omega \subset \mathbb{R}^n$ be an open set which is bounded in one direction and $0 < s < 1$. Assume that $\Lambda_1, \Lambda_2 \in L^s(\mathbb{R}^n)$ satisfy $\Lambda_1(x), \Lambda_2(x) > 0$. Suppose that $W \subset \Omega_e$ is a nonempty open set such that Λ_1, Λ_2 are continuous a.e. in W . If $\Lambda_1|_W = \Lambda_2|_W$ for all $f \in C_c(W)$, then $\Lambda_1 = \Lambda_2$ a.e. in W .

Recall the picture:



UCP of the DN maps 1/2

- (i) Low regularity fractional Liouville reduction when $2 \leq L_+^1(\mathbb{R}^n); m \in H^{s;n=s}(\mathbb{R}^n)$:

$$h(r^s u; r^{-s} i)_{L^2(\mathbb{R}^{2n})} = h(\cdot)^{s=2}(\cdot^{-1=2} u); (\cdot)^{s=2}(\cdot^{-1=2}) i)_{L^2(\mathbb{R}^n)} + h q(\cdot^{-1=2} u); (\cdot^{-1=2}) i; u; \in H^s(\mathbb{R}^n)$$

where

$$h q u; i = h(\cdot)^{s=2} m; (\cdot)^{s=2}(\cdot^{-1=2} u) i)_{L^2(\mathbb{R}^n)}$$

is a suitable Sobolev multiplier $\mathcal{M}(H^s; H^{-s})$.

- (ii) Reduction of DN maps: If $\gamma_1|_{W_1} \ll \gamma_2|_{W_2}$ and $\gamma_1 f|_{W_2} = \gamma_2 f|_{W_2}$ for all $f \in C_c^1(W_1)$, then $q_1 f|_{W_2} = q_2 f|_{W_2}$.
- (iii) Fractional Calderón problem for globally defined singular potentials (Ghosh{Salo{Uhlmann, Ralund{Salo): If $\gamma_1 f|_{W_2} = \gamma_2 f|_{W_2}$ for all $f \in C_c^1(W_1)$, then $q_1 = q_2$ in Ω .

UCP of the DN maps 2/2

- (i) Exterior determination for the fractional Schrödinger equation: $q_1 f|_{W_2} = q_2 f|_{W_2}$ for all $f \in C^1_c(W_1)$ and $W = W_1 \setminus W_2 \neq \emptyset$; , then $q_1 = q_2$ in W . This uses the earlier interior determination step, which already guarantees that $q_1 = q_2$ in W .

- (ii) We may then use the assumption that $t_j|_W = t_2|_W$ and the knowledge (in the sense of distributions/as Sobolev multipliers)

$$\frac{(\cdot) \circledast \left(\begin{smallmatrix} 1=2 \\ 1 \end{smallmatrix} \right)}{1} = q_1 = q_2 = \frac{(\cdot) \circledast \left(\begin{smallmatrix} 1=2 \\ 2 \end{smallmatrix} \right)}{2} \quad \text{in } W$$

and a UCP of the fractional Laplacians : If $u \in H^{r,p}(\mathbb{R}^n)$ for $r \in \mathbb{R}; p \in [1; \infty)$ and $(\cdot) \circledast u = u = 0$ in a nonempty open $V \subset \mathbb{R}^n$, $t \in \mathbb{R}_+ \setminus \mathbb{N}$, then $u = 0$ in \mathbb{R}^n (Kar{Zimmermann, 2022 + based on several other works). Here $n-s > 2$.

- (iii) Altogether, $q_1 = q_2$.

Exterior determination 1/2

- i) Define the Dirichlet energy first as

$$E(u) := B(u; u) = \int_{\mathbb{R}^{2n}} r^s u \cdot r^s u \, dx dy.$$

Notice that $E(u_f) = \int_{\mathbb{R}^n} f \cdot f \, dx$ where u_f is the unique solution of the fractional conductivity equation with the exterior condition f .

- ii) Elliptic estimate: Let $W \subset \mathbb{R}^n$, $\text{dist}(W; \infty) > 0$, $|W| < \infty$. If $f \in C_c^1(W)$ and $u_f \in H^s(\mathbb{R}^n)$ is the unique solution of

$$\begin{aligned} ((-\Delta)^s + q)u &= 0 && \text{in } \mathbb{R}^n; \\ u &= f && \text{in } W; \end{aligned}$$

then

$$\|u_f\|_{H^s(\mathbb{R}^n)} = \|u_f - f\|_{H^s(\mathbb{R}^n)} + C\|f\|_{L^2(W)}$$

for some $C(n; s; |W|; \text{dist}(W; \infty)) > 0$.

Exterior determination 2/2

- (i) This uses the quadratic definition of the fractional Laplacian

$$(-\Delta)^s f, = \frac{C_{n,s}}{2} \int_{\mathbb{R}^{2n}} \frac{(f(x) - f(y))(|x - y|^{-n+2s})}{|x - y|^{n+2s}} dx dy.$$

Similar argument can be made for the conductivity equation.

- (ii) **Construction of special solutions:** $\{C_N\}$ such that $C_N \in L^2(W)$ and $\int_{\mathbb{R}^n} C_N = 0$ as $N \rightarrow \infty$ and $\int_{\mathbb{R}^n} C_N H^s(\mathbb{R}^n) = 1$ for all $N \in \mathbb{N}$. Let $u_N \in H^s(\mathbb{R}^n)$ be the unique solutions to the conductivity equation with $u_N|_{\Omega_e} = C_N$. The elliptic energy estimate and the given properties of the exterior conditions give that $E(u_N)$ and $E(C_N)$ are equal as $N \rightarrow \infty$. These exterior conditions are similar to the boundary conditions considered by Kohn and Vogelius (1984).
- (iii) **Energy concentration property:** Given any $x_0 \in W$, one may show that there exists such sequences $\{C_N\}$ so that $E(C_N) \rightarrow \delta(x_0)$ as $N \rightarrow \infty$.

Counterexamples

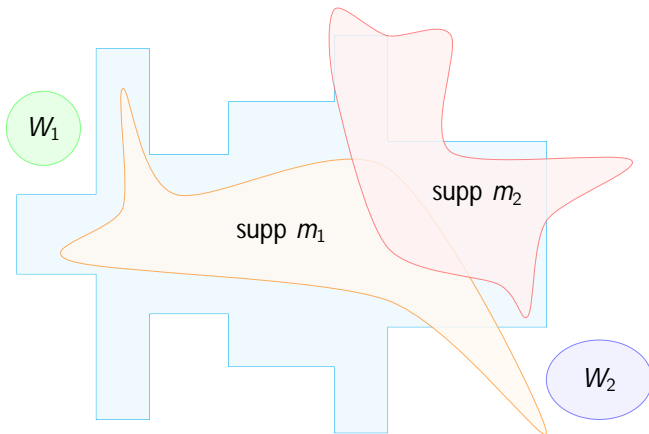
Our uniqueness result for the partial data problem is complemented with the following general counterexamples:

Theorem (R.-Zimmermann, 2022)

Let $\Omega \subset \mathbb{R}^n$ be an open set which is bounded in one direction, $0 < s < \min(1, n/2)$. For **any** nonempty open **disjoint sets** $W_1, W_2 \subset \Omega_e$ with $\text{dist}(W_1, W_2, \Omega) > 0$ there exist two different conductivities $\gamma_1, \gamma_2 \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ such that $\gamma_1(x), \gamma_2(x) > 0$, $m_1, m_2 \in H^{s, n/s}(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$, and $\Lambda_{\gamma_1} f|_{W_2} = \Lambda_{\gamma_2} f|_{W_2}$ for all $f \in C_c(W_1)$.

The problem remains open for any nonempty open disjoint sets $W_1, W_2 \subset \Omega_e$ with $\text{dist}(W_1, W_2, \Omega) = 0$.

Graphical illustration



Sketch of the proof 1/2

Using the fractional Liouville reduction one can **characterize** the **invariance of data**, for **any** disjoint data the following holds:

Lemma (R.-Zimmermann, 2022)

Let $\Omega \subset \mathbb{R}^n$ be an open set which is bounded in one direction and $0 < s < \min(1, n/2)$. Assume that $\Lambda_1, \Lambda_2 \in L^\infty(\mathbb{R}^n)$ with background deviations m_1, m_2 satisfy $\Lambda_1(x), \Lambda_2(x) - m_0 > 0$ and $m_1, m_2 \in H^{s, n/s}(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$. Finally, assume that $W_1, W_2 \subset \Omega_e$ are nonempty disjoint open sets and that $\Lambda_1/w_1 = \Lambda_2/w_1 = w_2$ holds. Then there holds $\Lambda_1 f/w_2 = \Lambda_2 f/w_2$ for all $f \in C_c(W_1)$ **if and only if** $m_0 := m_1 - m_2 \in H^s(\mathbb{R}^n)$ is the unique solution of

$$\begin{aligned} (-\Delta)^s m + q_2 m &= 0 && \text{in } \Omega, \\ m &= m_0 && \text{in } \Omega_e. \end{aligned}$$

Sketch of the proof 2/2

- Take $\epsilon > 0$. Now, by the invariance of data and searching for $m_1 = (m_1 + 1)^2$, the problem reduces to finding a **s-harmonic function** in Ω , i.e. $m_1 \in H^{s, n/s}(\mathbb{R}^n) \cap H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ which solves

$$(-\Delta)^s m_1 = 0 \quad \text{in } \Omega, \quad m_1 = m_0 \quad \text{in } \Omega_e, \quad (2)$$

with the "positivity" condition $m_1 \geq \frac{1}{2} - 1$ and any suitable exterior condition m_0 vanishing in $\overline{W_1} \setminus \overline{W_2}$.

- One may first look for a $H^s(\mathbb{R}^n)$ function which is s-harmonic in a slightly larger domain Ω and vanishes near $\overline{W_1} \setminus \overline{W_2}$. Using a **mollification argument** one finds a smooth s-harmonic function solving (2) with the right regularity properties, as $n/s > 2$.
- Finally, using the linearity of the equation and a **scaling argument**, the positivity condition can be made to hold.

Sets in the proof

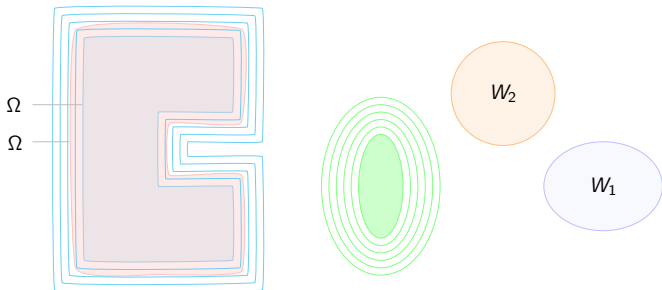


Figure: We construct in the first step a nonzero s -harmonic background deviation $\tilde{m}_1 \in H^s(\mathbb{R}^n)$ in the set Ω' , which has a smooth boundary and lies in the deformed annulus $\Omega_{3\epsilon} \setminus \overline{\Omega}_{2\epsilon}$, and then obtain by mollification a nonzero smooth s -harmonic function $m_1 := \tilde{m}_1 \cdot \chi_\epsilon$ in the set Ω . The set $\Omega \setminus \overline{W_1} \setminus \overline{W_2}$ is used to construct a cutoff function $\chi \in C_c^\infty(\mathbb{R}^n)$ with $\chi|_\Omega = 1$, which \tilde{m}_1 has as an exterior value and its support contained in $\Omega_{5\epsilon} \setminus \overline{\Omega}_{\epsilon}$. Next scale so that $\|m_1\|_{L^\infty(\mathbb{R}^n)} = 1/2$ and set $\epsilon_0 = 1/4$.

Stability estimate in the exterior

Write $A := A_{H^s(\Omega_e)}(H^s(\Omega_e))$. The exterior determination argument is constructive and leads to the following stability estimate:

Theorem (Covi-R.-Zimmermann, R.-Zimmermann, 2022)

Let $\Omega \subset \mathbb{R}^n$ be a domain bounded in one direction and $0 < s < 1$. Assume that $\lambda_1, \lambda_2 \in L^\infty(\mathbb{R}^n)$ satisfy $\lambda_1(x), \lambda_2(x) > 0$, and are continuous a.e. in Ω_e . There exists a constant $C > 0$ depending only on s such that

$$\| \lambda_1 - \lambda_2 \|_{L^\infty(\Omega_e)} \leq C \| \Lambda_{\lambda_1} - \Lambda_{\lambda_2} \|.$$

The argument is "local" in the exterior. Therefore, similar holds with the partial data in $W \subset \Omega_e$.

Stability estimate in the interior

Let $0 < s < \min(1; n/2)$, $\epsilon > 0$ and assume that $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain. Suppose that the conductivities $\sigma_1, \sigma_2 \in L^1(\Omega)$ with background deviations m_1, m_2 fulfill the following conditions:

- (i) $0 < \sigma_1(x) < \sigma_2(x) < \epsilon^{-1}$ for some $0 < \epsilon < 1$,
- (ii) $m_1, m_2 \in H^s(\Omega)$ and there exist $C_1, C_2 > 0$ such that

$$\|m_i\|_{H^{4s+2}(\Omega)} \leq C_i \|\sigma_i - \sigma\|_{L^1(\Omega)}$$

for $i = 1, 2$.

If $\sigma_1 \geq \sigma_2$ (max(1=2; 2s=n); 1) and there holds $\|\sigma_1 - \sigma_2\|_{L^1(\Omega)} \leq \epsilon$ for some $0 < \epsilon < \frac{1}{2}\epsilon_0$, then we have

$$\|\sigma_1 - \sigma_2\|_{L^q(\Omega)} \leq C \|\sigma_1 - \sigma_2\|_{L^1(\Omega)}$$

for all $1 < q \leq \frac{2n}{n-2s}$, where $\omega(x)$ is a logarithmic modulus of continuity satisfying

$$\omega(x) \leq C_j \log x_j; \quad \text{for } 0 < x_j < 1;$$

for some constants $C_j > 0$ depending only on $s; n; \sigma_1, \sigma_2; \epsilon_0$ and ϵ .

About the proof

- 1 The proof is based on one of the possible uniqueness proofs with full data.
- 2 The proof uses the stability estimate for the corresponding Schrödinger problem by Roland (Salo (2020)).
- 3 The proof uses the earlier exterior stability estimate, which also is related to having L^1 (L^1) a priori bound in the exterior.
- 4 Other key properties to show (resembling Alessandrini's work) are " $k_{q_1} \leq C k_{q_2} \leq C k_{q_1}$ " up to a constant depending of the a priori bounds (the real estimate looks a bit different), and the identity

$$\operatorname{div}_s(\mathbf{r}^s) = \frac{1}{2} (q_2 - q_1) \quad \text{in } \mathbb{R}^n;$$

$$\text{where } \mathbf{r}^s := \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 2 \end{pmatrix} = \frac{1}{2} \cdot$$

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