Boundary Recovery of Anisotropic Electromagnetic Parameters for the Time-Harmonic Maxwell's Equations

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Introduction: Time-harmonic Maxwell's equations

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Our goal is to determine the electrical parameters.

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Infer the electrical parameters in a region M from measurements of the tangential components of E and H on the boundary of the region ∂M.

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- The tangential component of one field is controlled at the boundary.
- The tangential component of the other field is measured at the boundary. If $\iota : \partial M \mapsto M$ is the inclusion map:
  - $\Lambda_\varepsilon : \iota^* H \to \iota^* E$ is the **impedance map**;
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- The electrical parameters are anisotropic (see next slide).
- We will assume that ω is fixed at a positive value such that the boundary value problems are well-posed.

Introduction: The anisotropic parameters

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- If ε and μ are scalars, the equations are **isotropic**.
- If ε and μ are $(1,1)$ tensor fields, the equations are **anisotropic**.

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Assumptions

We assume that ε and μ are real-valued, smooth and anisotropic. We also assume that they are both **symmetric** in the sense that

$$\langle a, \varepsilon b\rangle_g = \langle \varepsilon a, b\rangle_g, \quad \langle a, \mu b\rangle_g = \langle \mu a, b\rangle_g$$

for any covectors (or vectors) a and b.

Introduction: Independence from g

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g independent equations

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$$*_{\hat{\mu}} \mathrm{d}E = i\omega H, \quad \delta_{\hat{\mu}} H = 0.$$

Pause for notation!

We finally introduce coordinates and indexed expressions!

- We use standard notation for the components

$$g = g_{ij}dx^i dx^j, \quad g^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}.$$

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- We use real inner products and norms despite the fact that vectors can be complex valued. For example:

$$\langle a, b \rangle_g = g_{ij} a^i b^j, \quad |a|_g^2 = \langle a, a \rangle_g = g_{ij} a^i a^j.$$

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  Note: since a can be complex, $|a|_g^2$ can be zero for non-zero a.
- We use indices with a tilde, e.g. $\tilde{i}$, to indicate the index can only take values $\tilde{i} = 1$ or 2.

Pause! Summary of tensors

(M, g) a Riemannian manifold with boundary.

- Electrical parameter ε

Tensor	Order	Coordinate expression
ε	$(1,1)$	ε^i_j
$\varepsilon^\sharp$	$(2,0)$	$\varepsilon^{ij} = \varepsilon^i_k g^{kj}$
$(\varepsilon^{-1})_\flat$	$(0,2)$	$(\varepsilon^{-1})_{ij} = (\varepsilon^{-1})^k_j g_{ki}$
$\hat{\varepsilon}$	$(2,0)$	$\varepsilon^{ij}/\det(\varepsilon^k_l)$
$\hat{\varepsilon}^{-1}$	$(0,2)$	$(\varepsilon^{-1})_{ij}\det(\varepsilon^k_l)$

Same for μ.

Result summary

Well-posedness

Maxwell's system (solution for E and H with specified $\iota^* E$) is well-posed for anisotropic parameters except for on a discrete set of ω.

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Non-uniqueness

- Transformation of the parameters $\hat{\varepsilon}$ and $\hat{\mu}$ by a diffeomorphism that fixes the boundary ∂M does not change Λ_ε.
- The determinant of g in boundary normal coordinates for either $(\varepsilon^{-1})_\flat$ or $(\mu^{-1})_\flat$ cannot be determined from Λ_ε in a neighbourhood of ∂M.

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Tangential components

The tangential components of $\hat{\varepsilon}$ and $\hat{\mu}$ are uniquely determined by the principal symbols of Λ_ε and Λ_μ.

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Non-tangential components

Under some hypotheses (details later), if the boundary maps are the same for two sets of electrical parameters then the hat metrics are the same at the boundary in boundary normal coordinates for $\hat{\varepsilon}$.

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Full jet

If in boundary normal coordinates for ε the determinant of g is known and two sets of electrical parameters are the same at the boundary, then all derivatives of the parameters also agree at the boundary.

Brief literature review: Maxwell's equations

- Corresponding time dependent problems in the isotropic case, as well as for other systems of equations, have been considered and solved using the boundary control method [Belishev, Kurylev, Lassas, Oksanen, Paternain ... others].

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- Numerical work for the anisotropic case has been done [Costabel 1991 ... others].

Brief literature review: Methods

Our method analyses the forward operator as a pseudodifferential operator including determination of the principal symbol. This method can be traced back:

- Boundary recovery for anisotropic Calderón problem [Lee, Uhlmann 1989].
- Elastic system [Nakamura, Uhlmann 1997].
- Isotropic Maxwell's system [McDowall 1997], with complex parameter [McDowall 2000] and for the full boundary jet [Joshi, McDowall 2000].
- Harmonic differential forms [Lionheart, Joshi 2005].

Many others have contributed and have related work!

Well-posedness

Natural system

Maxwell's equations can be written in a matrix form as

$$\begin{pmatrix} 0 & \delta_{\hat{\mu}} \\ i\omega & *_{\hat{\varepsilon}}\mathrm{d} \\ -*_{\hat{\mu}}\mathrm{d} & i\omega \\ \delta_{\hat{\varepsilon}} & 0 \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

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Augmented elliptic system

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- We can apply standard methods for elliptic systems to prove operator with appropriate boundary conditions is self-adjoint.
- This allows us to prove the well-posedness result.
- The method requires finding the singular values and vectors of the principal symbol, which is also related to our later analysis.
- Somersalo uses a similar augmented system.

Non-uniqueness

Diffeomorphism invariance

This type of invariance is common in geometric inverse problems following from:

- Maxwell's equations can be written invariantly with respect to $\hat{\varepsilon}$ and $\hat{\mu}$;
- Diffeomorphism which fix the boundary do not affect $\iota^* E$ or $\iota^* H$.

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Determinant of g in boundary normal coordinates

Follows from diffeomorphism invariance since:

- Maxwell's equations can be made independent from g;
- We can choose diffeomorphisms which fix the boundary and change the determinant of g arbitrarily in a neighbourhood of the boundary.

Boundary normal coordinates and tangential components

Boundary normal coordinates

- Boundary normal coordinates for a metric are found by taking a coordinate chart on ∂M and then distance to the boundary as the third coordinate.
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Tangential components

- In boundary normal coordinates for g, the covector metric has a matrix of the form
$$\begin{pmatrix} \tilde{g}^{\tilde{i}\tilde{j}} & 0 \\ 0 & 1 \end{pmatrix}.$$
- $\tilde{g}$ is the tangential component of g, a covector metric on ∂M. It is invariantly defined on ∂M.

Principal symbols: Outline of method

Working in a set of boundary coordinates (i.e. $\partial M = \{x^3 = 0\}$):

1. Note that for solutions of Maxwell's system D_{x^3} is a pseudodifferential operator acting on H and E restricted to constant x^3. This follows [Nakamura, Uhlmann 1997].

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6. Use inductive method to recover full-jet. This uses method from [Joshi, McDowall 2000].

Principal symbol of D_{X^3}

- From Maxwell's equations, H satisfies

$$L_H H = \Big(- *_{\hat{\varepsilon}} *_{\hat{\mu}} \mathrm{d}\delta_{\hat{\mu}} + *_{\hat{\varepsilon}} \mathrm{d} *_{\hat{\varepsilon}} \mathrm{d} \Big) H - \omega^2 *_{\hat{\varepsilon}} *_{\hat{\mu}} H = 0.$$

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- If the principal symbol of L_H is $\sigma_p(L_H) = M_H$ which we consider as a function of ξ_3, then the principal symbol of D_{x^3} acting on H is

$$B^{(1)} = \int_{\Gamma^+} \xi_3 M_H(\xi_3)^{-1} \, d\xi_3 \left(\int_{\Gamma^+} M_H(\xi_3)^{-1} \, d\xi_3 \right)^{-1}$$

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  where Γ^+ is a contour in $\mathbb{C}$ enclosing all solutions of $\det(M_H)(\xi_3) = 0$ with positive real part.
- This formula can also be used to show that $B^{(1)}$ is a smooth function of position x and $\tilde{\xi} = (\xi_1, \xi_2)$.

Principal symbol of D_{χ^3}

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- Indeed, consider the covectors
$$\xi, \quad \zeta = \hat{\varepsilon}^{-1}\hat{\mu}\xi, \quad \chi = *_{\hat{\varepsilon}}(\xi \wedge \zeta),$$
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- We have
$$B^{(1)}\chi_{\hat{\varepsilon}} = \xi_{\hat{\varepsilon}3}\chi_{\hat{\varepsilon}}, \quad B^{(1)}\xi_{\hat{\mu}} = \xi_{\hat{\mu}3}\xi_{\hat{\mu}}.$$

Principal symbols of boundary maps

Using $B^{(1)}$ and the corresponding expression applied to E, we can find from Maxwell's equations the principal symbols of Λ_ε and Λ_μ:

$$\sigma_p(\Lambda_\varepsilon)(F) = -\frac{*_{\iota^*\hat{\varepsilon}}(\tilde{\xi} \wedge F)}{\omega\langle \nu_{\hat{\varepsilon}}, \xi_{\hat{\varepsilon}}\rangle_{\hat{\varepsilon}}}\tilde{\xi},$$

$$\sigma_p(\Lambda_\mu)(G) = \frac{*_{\iota^*\hat{\mu}}(\tilde{\xi} \wedge G)}{\omega\langle \nu_{\hat{\mu}}, \xi_{\hat{\mu}}\rangle_{\hat{\mu}}}\tilde{\xi}.$$

- Here $\nu_{\hat{\varepsilon}}$ is the inner unit conormal to the boundary in the $\hat{\varepsilon}$ metric.
- It is possible to determine the tangential components of $\hat{\varepsilon}$ and $\hat{\mu}$ from these which proves the tangential result.

Non-tangential recovery

For the next steps, we carefully analyse components in each set of boundary normal coordinates and the relationship between them.

Covectors	BNCs for $\hat{\mu}$	BNCs for $\hat{\varepsilon}$
$\xi_{\hat{\mu}}$	$\xi_{\hat{\mu},\tilde{\mu}}$	$\xi_{\hat{\mu},\tilde{\varepsilon}}$
$\chi_{\hat{\mu}}$	$\chi_{\hat{\mu},\tilde{\mu}}$	$\chi_{\hat{\mu},\tilde{\varepsilon}}$
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Based on the equivalence of the third component of *E* or *H* fields in appropriate boundary normal coordinates we get the lemma.

Lemma

If $(\hat{\varepsilon}, \hat{\mu})$ and $(\hat{\varepsilon}', \hat{\mu}')$ are electrical parameters with the same boundary mappings, then at the boundary

$$\left(\xi_{\hat{\mu}',\tilde{\varepsilon}3} + \xi_{\hat{\varepsilon},\tilde{\varepsilon}3}\right)\chi_{\hat{\mu},\tilde{\varepsilon}3} = \left(\xi_{\hat{\mu},\tilde{\varepsilon}3} + \xi_{\hat{\varepsilon},\tilde{\varepsilon}3}\right)\chi_{\hat{\mu}',\tilde{\varepsilon}3},$$
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Based the lemma, we have a result about boundary recovery of non-tangential parts.

Theorem

If $\Lambda_{\hat{\varepsilon}} = \Lambda_{\hat{\varepsilon}'}$, then in boundary normal coordinates for $\hat{\varepsilon}/\hat{\varepsilon}'$ at ∂M at least one of the following cases holds:

1. The metrics $\tilde{\mu}$ and $\tilde{\varepsilon}$ are multiples and $\hat{\mu}'^{3\tilde{j}} = c\hat{\mu}^{3\tilde{j}}$ for some $c \in \mathbb{R}$.
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Comments on theorem:

- The proof is based on explicit expressions of the components from the previous lemma.
- We do not believe that a stronger result is possible from further analysis of the lemma or indeed the principal part of the E or H fields.
- Looking at lower order symbols could be beneficial.

Determination of boundary jet

Theorem

Take two sets of electromagnetic parameters (ε, μ) and (ε', μ'). Suppose that in boundary normal coordinates for $\varepsilon^\sharp/\varepsilon'^\sharp$, $|g| = |g'|$. If the boundary mappings are the same and the parameters agree at the boundary, then, also in boundary normal coordinates for $\varepsilon^\sharp/\varepsilon'^\sharp$,

$$\partial^\kappa_{x_3} \varepsilon^\sharp = \partial^\kappa_{x_3} \varepsilon'^\sharp, \quad \partial^\kappa_{x_3} \mu^\sharp = \partial^\kappa_{x_3} \mu'^\sharp \quad \text{at } x_3 = 0,$$

for any $\kappa \geq 1$.

Comments:

- Proof is based on the method from [Joshi, McDowall 2000].
- A special class of pseudodifferential operators vanishing to different orders at the boundary is introduced.
- Based on this, the proof proceeds inductively in κ.

Conclusion

- We have explicit formulas for the principal symbols of Λ_ε and Λ_μ and found that they imply tangential recovery of the parameters $\hat{\varepsilon}$ and $\hat{\mu}$.

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- We also have explicit formulas for the principal symbols of the maps from boundary data to the normal component of E and H fields in appropriate boundary normal coordinates. This is used for the non-tangential results.
- We do not believe a stronger result can be found based only on principal symbols but analysis of lower order symbols could help.
- We successfully applied the inductive method introduced in [Joshi, McDowall 2000] to this case.
- **To do:**
  1. Fill in gap in non-tangential recovery.
  2. Look at recovery in the interior ... .

The end

Thank you for your attention!