Ruelle zeta at zero for nearly hyperbolic 3-manifolds

Semyon Dyatlov (MIT)

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Overview

- Studying $m_{\mathrm{R}}(0)$: order of vanishing at 0 of the Ruelle zeta function for the geodesic flow on a negatively curved 3-manifold (Σ, g)
- $g = g_H$ hyperbolic $\implies$ $m_{\mathrm{R}}(0) = 4 - 2b_1(\Sigma)$ [Fried '86]
- $g =$ generic perturbation of g_H $\implies$ $m_{\mathrm{R}}(0) = 4 - b_1(\Sigma)$ [Cekić–Delarue–D–Paternain '20]
- This is in contrast with the case $\dim \Sigma = 2$ where $m_{\mathrm{R}}(0) = b_1(\Sigma) - 2$ for all negatively curved (Σ, g) [D–Zworski '17]
- Motivated by Fried's conjecture '87 relating the values at 0 of twisted dynamical zeta functions to analytic torsion

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Geodesic and contact flows

- (Σ, g) a compact connected oriented Riemannian n-dim manifold
- $M = S\Sigma$ the sphere bundle of (Σ, g), $\pi_\Sigma : M \to \Sigma$ projection map
- $\alpha_{(x,v)}(\xi) = \langle v, d\pi_\Sigma(x,v)\xi\rangle_g$ canonical 1-form on M
- α is a contact form: $d\,\mathrm{vol}_\alpha := \alpha \wedge (d\alpha)^{n-1}$ is nonvanishing
- Geodesic flow: $\varphi_t = e^{tX} : M \to M$ where $X \in C^\infty(M; TM)$ given by

$$\iota_X \alpha = 1, \qquad \iota_X d\alpha = 0$$

- g has negative sectional curvature $\implies$ φ_t is Anosov:

$$TM = E_0 \oplus E_u \oplus E_s, \qquad E_0 = \mathbb{R}X,$$

$$\exists C, \theta > 0 : \quad \|d\varphi_{-t}|_{E_u}\|, \|d\varphi_t|_{E_s}\| \le Ce^{-\theta t}, \quad t \ge 0$$

- Define $E_u^* := (E_0 \oplus E_u)^\perp$, $E_s^* := (E_0 \oplus E_s)^\perp$ subsets of T^*M

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Ruelle zeta function

Define the Ruelle zeta function

$$\zeta_{\mathrm{R}}(\lambda) = \prod_{\gamma} (1 - e^{-\lambda T_\gamma}), \quad \operatorname{Re}\lambda \gg 1$$

where the product is over all primitive closed geodesics γ of periods T_γ

- The function $\zeta_{\mathrm{R}}(\lambda)$ continues meromorphically to $\lambda \in \mathbb{C}$ [Giulietti–Liverani–Pollicott '13, D–Zworski '16]
  Conjectured by Smale '67; partial progress by Ruelle '76, Parry–Pollicott '90, Rugh '96, Fried '95
- Define the vanishing order $m_{\mathrm{R}}(0) \in \mathbb{Z}$:

$$\lambda^{-m_{\mathrm{R}}(0)} \zeta_{\mathrm{R}}(\lambda) \quad \text{holomorphic and nonvanishing at} \quad \lambda = 0$$

Question

Can we describe $m_{\mathrm{R}}(0)$ in terms of topological invariants of Σ, such as the Betti numbers $b_k(\Sigma) = \dim H^k(\Sigma; \mathbb{R})$?

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- More general zeta functions $\zeta_\rho(\lambda)$ twisted by a representation $\rho : \pi_1(\Sigma) \to \mathrm{U}(m)$; ζ_{R} corresponds to the trivial $\rho : \pi_1(\Sigma) \to \mathrm{U}(1)$
- ρ is called acyclic if $H^k_\rho(\Sigma; \mathbb{R}) = 0$ for all k
- Fried '86 studied the hyperbolic case (curvature $= -1$):

$$m_{\mathrm{R}}(0) = \begin{cases} b_1(\Sigma) - 2, & \dim \Sigma = 2 \\ 4 - 2b_1(\Sigma), & \dim \Sigma = 3 \end{cases}$$

  For ρ acyclic, he computed $m_\rho(0) = 0$ and $\zeta_\rho(0) = T_\rho^2$ where T_ρ is the analytic torsion. Fried's conjecture: same formula for $\zeta_\rho(0)$ holds for general locally homogeneous (Σ, g)
- Fried's conjecture proved for locally symmetric spaces by Shen '16, following Moscovici–Stanton '91, Bismut '11
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What happens for general (not locally symmetric) negatively curved Σ?

- D–Zworski '17: $m_{\mathrm{R}}(0) = b_1(\Sigma) - 2$ when $\dim \Sigma = 2$; applies to general contact Anosov flows in dimension 3
- Extended to surfaces with boundary by Hadfield '18, to the nonorientable case by Borns-Weil–Shen '20
- Cekić–Paternain '19: studied $m_{\mathrm{R}}(0)$ for general volume preserving Anosov flows on a 3-manifold M and showed it depends on the properties of the flow, not just on the topology of M
- Dang–Guillarmou–Rivière–Shen '20 proved Fried's conjecture on $\zeta_\rho(0)$ when Σ is any nearly hyperbolic 3-manifold
- Related works: Dang–Rivière '17, Chaubet–Dang '19, Küster–Weich '20

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Statement of the result

Theorem 1 [Cekić–Delarue–D–Paternain '20]

Let (Σ, g_H) be a compact connected oriented hyperbolic 3-manifold. Then:

1. If $g = g_H$ then $m_{\mathrm{R}}(0) = 4 - 2b_1(\Sigma)$
2. If g is a generic conformal perturbation of g_H then $m_{\mathrm{R}}(0) = 4 - b_1(\Sigma)$

Here generic conformal perturbation is understood as follows:

there exists an open dense $\mathscr{O} \subset C^\infty(\Sigma; \mathbb{R})$ such that
for any $a \in \mathscr{O}$ there exists $\varepsilon > 0$ such that for all $\tau \in (-\varepsilon, \varepsilon) \setminus \{0\}$
the metric $g = e^{\tau a} g_H$ has $m_{\mathrm{R}}(0) = 4 - b_1(\Sigma)$

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Spectral interpretation of zeta functions I

- General idea: "$\zeta(\lambda) = \det(\lambda - P)$" for some operator P
- This should be understood as $\partial_\lambda \log \zeta(\lambda) = \operatorname{tr}(\lambda - P)^{-1}$ with the right definition of trace
- Vanishing order of ζ at 0 $=$ dimension of the space of generalized eigenstates at 0 $\{u \mid \exists \ell : P^\ell u = 0\}$
- One can write the vanishing order $m_{\mathrm{R}}(0)$ of ζ_{R} using the dimensions of certain spaces of Pollicott–Ruelle generalized resonant forms $\mathrm{Res}_0^{k,\infty}$
- Our strategy is to describe $\mathrm{Res}_0^{k,\infty}$ in terms of the de Rham cohomology of Σ
- In this talk we will focus on the case $k = 1$

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Spectral interpretation of zeta functions II

- $M = S\Sigma$, $\dim \Sigma = 3$, $X \in C^\infty(M; TM)$ generates the geodesic flow
- Our operators: $P_{k,0} = \mathcal{L}_X$ acting on $\Omega_0^k := \{\omega \in \wedge^k T^*M \mid \iota_X \omega = 0\}$
- For certain anisotropic Sobolev spaces $\mathscr{H}, \mathscr{D}_P$ the operator $P_{k,0} - \lambda : \mathscr{D}_P(M; \Omega_0^k) \to \mathscr{H}(M; \Omega_0^k)$ is Fredholm of index 0

  Blank–Keller–Liverani '02, Liverani '04,'05, Baladi '05, Gouëzel–Liverani '06, Baladi–Tsujii '07, Butterley–Liverani '07
- We will use the microlocal/scattering theory approach: Faure–Roy–Sjöstrand '08, Faure–Sjöstrand '11, D–Zworski '16

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- The poles of $(P_{k,0} - \lambda)^{-1}$ are called Pollicott–Ruelle resonances
- Generalized resonant states at $\lambda = 0$:

$$\mathrm{Res}_0^{k,\infty} = \{u \in \mathscr{D}_P(M; \Omega_0^k) \mid \exists \ell : \mathcal{L}_X^\ell u = 0\}$$

- D–Zworski '16, using Hörmander's propagation of singularities, Melrose's radial estimates, and Atiyah–Bott–Guillemin trace formula:

$$m_{\mathrm{R}}(0) = \sum_{k=0}^{4} (-1)^k \dim \mathrm{Res}_0^{k,\infty}$$

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Resonance multiplicities

Theorem 1 follows from $m_{\mathrm{R}}(0) = \sum_{k=0}^{4}(-1)^k \dim \mathrm{Res}_0^{k,\infty}$ and

Theorem 2 [Cekić–Delarue–D–Paternain '20]

Let (Σ, g_H) be a compact connected oriented hyperbolic 3-manifold. Then the dimensions of $\mathrm{Res}_0^{k,\infty}$ are:

k	Hyperbolic	Perturbation
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2	$2b_1(\Sigma)+2$	$b_1(\Sigma)+2$
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$(d\alpha)^j \wedge : \mathrm{Res}_0^{2-j,\infty} \to \mathrm{Res}_0^{2+j,\infty}$ isomorphisms $\Rightarrow$ study $k = 0, 1, 2$

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Resonant and coresonant states

- Generalized resonant states:
  $\mathrm{Res}_0^{k,\infty} = \{u \in \mathscr{D}_P(M;\Omega_0^k) \mid \exists \ell : \mathcal{L}_X^\ell u = 0\}$
- $\mathcal{D}'_{E_u^*}(M;\Omega_0^k) = \{u \in \mathcal{D}'(M;\Omega_0^k) \mid \mathrm{WF}(u) \subset E_u^*\}$
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Closed resonant 1-forms

- $\mathrm{Res}_0^k = \{u \in \mathcal{D}'_{E_u^*}(M;\Omega_0^k) \mid \mathcal{L}_X u = 0\}, \quad \mathcal{L}_X = d\iota_X + \iota_X d$
- Closed forms: $\mathrm{Res}_0^k \cap \ker d = \{u \in \mathcal{D}'_{E_u^*}(M;\Omega^k) \mid \iota_X u = 0,\ du = 0\}$
- Cohomology map: $\pi_k : \mathrm{Res}_0^k \cap \ker d \to H^k(M;\mathbb{R}), \quad \pi_k(u) = [u]_{H^k}$
- π_k can be defined because $\mathcal{D}'_{E_u^*}$ is closed under $(d\delta + \delta d + 1)^{-1}$:
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Lemma: π_1 is an isomorphism

Injectivity: if $u \in \mathrm{Res}_0^1$ and $u = df$, $f \in \mathcal{D}'_{E_u^*}(M;\mathbb{R})$, then $Xf = \iota_X u = 0$, so $f \in \mathrm{Res}_0^0 = \mathbb{R}1$ and $u = df = 0$

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Resonant forms, hyperbolic case

- We know that $\mathcal{C} := \mathrm{Res}_0^1 \cap \ker d$ has dimension $b_1(M) = b_1(\Sigma)$
- We show every $u \in \mathrm{Res}_0^1$ is a section of $E_u^* = (E_0 \oplus E_u)^\perp \subset \Omega_0^1$
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- We show that $\mathrm{Res}_0^1 = \mathcal{C} \oplus \mathcal{I}(\mathcal{C})$ is $2b_1(\Sigma)$-dimensional and semisimplicity holds for $k = 1$, so $\dim \mathrm{Res}_0^{1,\infty} = 2b_1(\Sigma)$
- We also show that $\mathrm{Res}_0^2 = \mathrm{Res}_0^2 \cap \ker d = \mathbb{R}d\alpha \oplus \mathbb{R}\psi \oplus d\,\mathrm{Res}_0^1$ is $(b_1(\Sigma) + 2)$-dimensional where ψ is an explicit smooth 2-form
- We finally show $\dim \mathrm{Res}_0^{2,\infty} = 2b_1(\Sigma) + 2$: get $b_1(\Sigma)$ Jordan blocks

Resonant forms, hyperbolic case

- We know that $\mathcal{C} := \mathrm{Res}_0^1 \cap \ker d$ has dimension $b_1(M) = b_1(\Sigma)$
- We show every $u \in \mathrm{Res}_0^1$ is a section of $E_u^* = (E_0 \oplus E_u)^\perp \subset \Omega_0^1$
- The $\frac{\pi}{2}$-rotation $\mathcal{I} : E_u^* \to E_u^*$ commutes with $\mathcal{L}_X$ because the flow $\varphi_t = e^{tX}$ is conformal on E_u^*: $|d\varphi_t(\rho)^{-T}\xi| = e^t|\xi|$, $\xi \in E_u^*(\rho)$
- Thus $\mathcal{I}$ acts on $\mathrm{Res}_0^1 = \{u \in \mathcal{D}'_{E_u^*}(M; \Omega_0^1) \mid \mathcal{L}_X u = 0\}$
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Resonant forms for perturbations

- Consider now the perturbed metric $g_\tau = e^{\tau a} g_H$, $a \in C^\infty(\Sigma; \mathbb{R})$
- Define $\pi_\Sigma : M = S\Sigma \to \Sigma$; $\mathcal{J} : M \to M$, $\mathcal{J}(x, v) = (x, -v)$
- We still have $\dim(\mathrm{Res}_0^1 \cap \ker d) = b_1(\Sigma)$, need to show that all non-closed elements of Res_0^1 are moved by the perturbation
- A first variation calculation shows that we need nondegeneracy of

$$du \in d(\mathrm{Res}_0^1), \quad du_* \in d(\mathrm{Res}_{0*}^1) \quad \mapsto \quad \int_M (\pi_\Sigma^* a)\alpha \wedge du \wedge du_*$$

- Take for simplicity $b_1(\Sigma) = 1$, then enough to show

$$u \in \mathrm{Res}_0^1, \quad du \neq 0 \quad \Longrightarrow \quad \int_M (\pi_\Sigma^* a)\alpha \wedge du \wedge \mathcal{J}^*(du) \neq 0$$

- That's true for generic a as long as $\pi_{\Sigma*}(\alpha \wedge du \wedge \mathcal{J}^*(du)) \neq 0$ where $\pi_{\Sigma*} : \mathcal{D}'(M; \Omega^k) \to \mathcal{D}'(\Sigma; \Omega^{k-2})$ is the pushforward on forms

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Nontriviality of first variation

- Working only with the hyperbolic metric now
- Given $u \in \mathrm{Res}_0^1$, $du \neq 0$, need $\pi_{\Sigma*}(\alpha \wedge du \wedge \mathcal{J}^*(du)) \neq 0$
- Write $\pi_{\Sigma*}(\alpha \wedge du \wedge \mathcal{J}^*(du)) = F\, d\,\mathrm{vol}_g$ for some $F \in \mathcal{D}'(\Sigma; \mathbb{R})$
- Difficult to show that $F \neq 0$ because cannot evaluate F at points

Main identity

We have $Q_4 F = -\frac{1}{6}\Delta_g |\sigma|_g^2$ where

- $\sigma = \pi_{\Sigma*}(\alpha \wedge du)$ is a nonzero harmonic 1-form on Σ
- $Q_4 f(x) = \int_{\mathbb{H}^3} \cosh^{-4} d_{\mathbb{H}^3}(x,y) f(y)\, d\,\mathrm{vol}_g(y)$ descends to $Q_4 : \mathcal{D}'(\Sigma) \to C^\infty(\Sigma)$ where $\Sigma = \Gamma \backslash \mathbb{H}^3$

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Two conjectures

Conjecture 1

Let (Σ, g) be a generic negatively curved compact connected oriented 3-manifold. Then:

- semisimplicity holds and $d(\mathrm{Res}_0^k) = 0$ for all $k = 0, \dots, 4$
- $\dim \mathrm{Res}_0^0 = 1$, $\dim \mathrm{Res}_0^1 = b_1(\Sigma)$, $\dim \mathrm{Res}_0^2 = b_1(\Sigma) + 2$
- $m_{\mathrm{R}}(0) = 4 - b_1(\Sigma)$

The set of g satisfying Conjecture 1 is open:

$\dim \mathrm{Res}_0^{1,\infty} \leq b_1(\Sigma)$, $\dim \mathrm{Res}_0^{2,\infty} \leq b_1(\Sigma) + 2 \quad \Longrightarrow \quad$ Conjecture 1 holds

Conjecture 2

Let $\rho : \pi_1(\Sigma) \to \mathrm{U}(m)$ be acyclic: $H^\bullet_\rho(\Sigma; \mathbb{R}) = 0$. Then $\mathrm{Res}_0^k = 0$ for all k

Conjecture 2 + DGRS '20 $\quad \Longrightarrow \quad \zeta_\rho(0)$ is locally constant under perturbations of ρ, g, which could lead to a solution of Fried's conjecture

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Thank you for your attention!