Retrieving coupled Yang-Mills-Higgs fields

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Workshop 4 "Geometrical Inverse Problems"
Special Semester on "Tomography Across the Scales"
RICAM, The 8th November 2022

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  - Mass gained via coupling with Higgs bosons (Yukawa coupling)

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- To describe massive bosons, a mass term $\frac{m^2}{2}\langle A, A \rangle_{\mathrm{Ad}}$ is required in the Lagrangian. However, $A \notin \Omega^1(M, \mathrm{Ad})$.

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where the bilinear form $J_\rho : \mathcal{W} \times \mathcal{W} \to \mathfrak{g}$ is defined by

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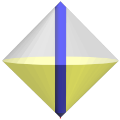
Retrieving coupled Yang-Mills-Higgs fields



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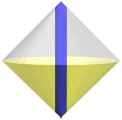

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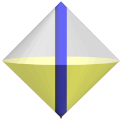

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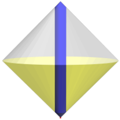

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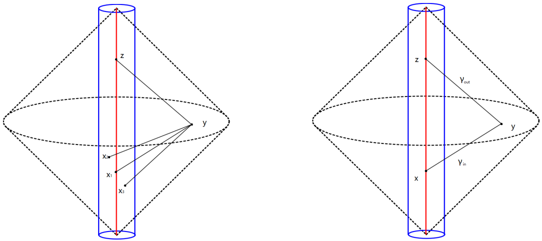

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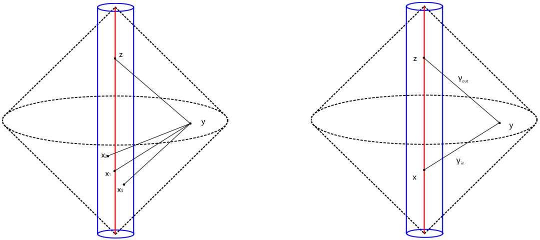

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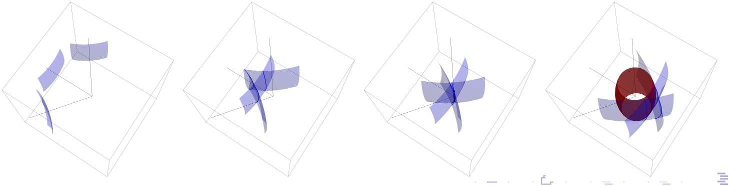

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  - The non-degeneracy of J_ρ (i.e. ρ_* is fully charged) yields Φ.

Danke schön!