# Holonomy and Spectral Inverse Problems

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# Summary

# In this talk

- Let $(M, g)$ be a compact Riemannian manifold without boundary and $\mathcal{E} \to M$ a vector bundle equipped with a connection $\nabla^{\mathcal{E}}$. We address the following inverse problems:
  - Q1 To what extent does the holonomy of $\nabla^{\mathcal{E}}$ over closed geodesics determine the gauge-equivalence class $[\nabla^{\mathcal{E}}]$ of $\nabla^{\mathcal{E}}$?
  - Q2 Does the spectrum of the connection Laplacian $(\nabla^{\mathcal{E}})^*\nabla^{\mathcal{E}}$ determine the gauge class of $\nabla^{\mathcal{E}}$?

**We will show**

If $(M, g)$ has chaotic geodesic flow and $\nabla^{\mathcal{E}}$ is orthogonal, then:

- A1 Only the traces of holonomy suffice to determine the gauge-equivalence class $[\nabla^{\mathcal{E}}]$ locally and in many cases globally!
- A2 Similar results for Q2.

## Definition

A flow $\varphi_t : \mathcal{M} \to \mathcal{M}$ generated by a vector field $X$ is called Anosov if there is a continuous splitting $T\mathcal{M} = \mathbb{R}X \oplus E_u \oplus E_s$ into flow direction $\mathbb{R}X$, unstable/stable directions $E_{u/s}$ invariant under $d\varphi_t$, and there are constants $C, \nu > 0$ such that for all $x \in \mathcal{M}$, for some metric $| \bullet |$

$$|d\varphi_t(x)v| \leq \begin{cases} Ce^{-\nu t}|v|, & t \geq 0, v \in E_s(x), \\ Ce^{-\nu |t|}|v|, & t \leq 0, v \in E_u(x). \end{cases}$$

These flows model hyperbolic dynamics: sensitive (chaotic) upon a change in initial conditions. Restrictions on geometry/topology.

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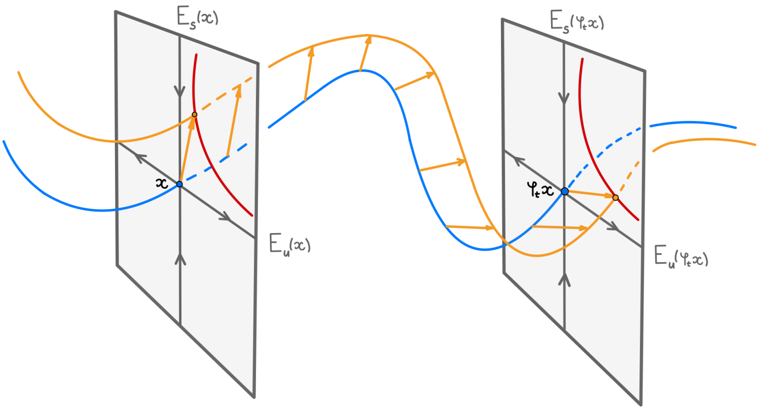
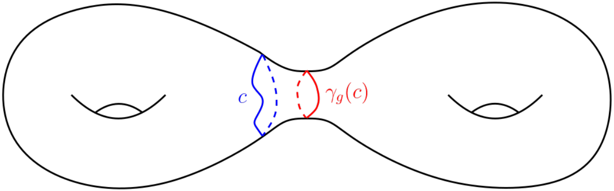
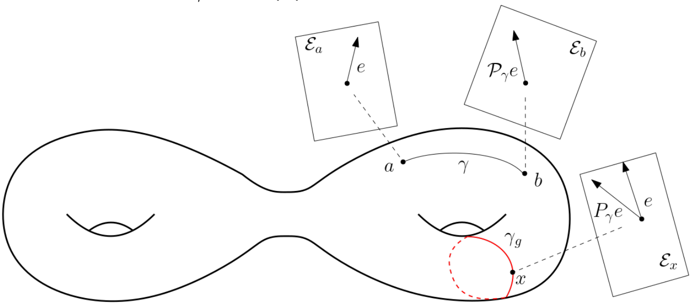


- Let $\mathcal{M} = SM = \{(x, v) \in TM \mid |v|_g = 1\}$ be the unit sphere bundle and define the geodesic flow $\varphi_t(x, v) = (\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t))$ on $SM$, where $\gamma_{x,v}(t)$ is the geodesic generated by the initial condition $(x, v)$.
- Examples of Anosov geodesic flows:
  - **Anosov ['67]**: if $(M, g)$ has negative sectional curvature;
  - $\exists$ examples with portions of positive curvature (**Eberlein ['73]**, **Donnay-Pugh ['03]**).
- If $(M, g)$ negatively curved, $\exists$ bijection between free homotopy classes $c \in \mathcal{C}$ and closed geodesics $\gamma_g(c)$ of length $L_g(c)$ in class $c$.



# Recall: connections on vector bundles

- Connection $\nabla^{\mathcal{E}}$ is a map $\nabla^{\mathcal{E}} : C^\infty(M, \mathcal{E}) \to C^\infty(M, T^*M \otimes \mathcal{E})$ such that in local coordinates $\nabla^{\mathcal{E}} = d + A$ for a matrix $A$ of 1-forms.
- If $\gamma : [a, b] \to M$ a curve, $e \in \mathcal{E}_a$, $s : [a, b] \to \mathcal{E}$ is the parallel transport of $e$ along $\gamma$ if $\nabla^{\mathcal{E}}_{\dot{\gamma}} s = 0$ (first order ODE) and $s(a) = e$, $\pi \circ s = \gamma$. Denote $\mathcal{P}_\gamma e := s(b) \in \mathcal{E}_b$.



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- $\nabla^{\mathcal{E}}$ is orthogonal if compatible with the inner product in the fibres of $\mathcal{E}$; it follows $\mathcal{P}_\gamma : \mathcal{E}_a \to \mathcal{E}_b$ is an orthogonal map.
- Vocabulary of the affine set $\mathcal{A}_{\mathcal{E}}$ of all orthogonal connections on $\mathcal{E}$:
  - Gauge group $\mathcal{G}(\mathcal{E})$:= the set of all orthogonal isomorphisms of $\mathcal{E}$;
  - $\mathcal{G}(\mathcal{E})$ acts on $\mathcal{A}_{\mathcal{E}}$ by pullback $p^*\nabla^{\mathcal{E}}(\bullet) := p^{-1}\nabla^{\mathcal{E}}(p\bullet)$;
  - Two connections $\nabla^{\mathcal{E}}_1$ and $\nabla^{\mathcal{E}}_2$ are gauge-equivalent if there is a $p \in \mathcal{G}(\mathcal{E})$ such that $p^*\nabla^{\mathcal{E}}_2 = \nabla^{\mathcal{E}}_1$;
  - The quotient $\mathbb{A}_{\mathcal{E}} := \mathcal{A}_{\mathcal{E}}/\mathcal{G}(\mathcal{E})$ is the moduli space of connections;
  - $\mathbb{A} := \sqcup_{\mathcal{E}} \mathbb{A}_{\mathcal{E}}$ is the moduli space of connections on all $\mathcal{E} \to M$.

# Primitive trace map

- $\mathcal{C}^\sharp := \{c_1^\sharp, c_2^\sharp, \dots\} \subset \mathcal{C}$ is the set of *primitive* free homotopy classes.
- $\mathrm{Hol}_{\nabla^\mathcal{E}}(c^\sharp) \in \mathrm{U}(x_{c^\sharp}) :=$ parallel transport along $\gamma_g(c^\sharp)$ at *some* $x_{c^\sharp} \in \gamma_g(c^\sharp)$. Note $\mathrm{Hol}_{\nabla^\mathcal{E}}(c^\sharp)$ depends up to conjugation on the choice of $x_{c^\sharp}$ and the gauge class $[\nabla^\mathcal{E}]$, but its trace *does not*.

## Definition

Define the primitive trace map as:

$$\mathcal{T}^\sharp : \mathbb{A} \ni ([\mathcal{E}], [\nabla^\mathcal{E}]) \mapsto \left(\mathrm{Tr}\left(\mathrm{Hol}_{\nabla^\mathcal{E}}(c_1^\sharp)\right), \mathrm{Tr}\left(\mathrm{Hol}_{\nabla^\mathcal{E}}(c_2^\sharp)\right), \dots\right) \in \ell^\infty(\mathcal{C}^\sharp).$$

## Question (Holonomy Inverse Problem)

*When is the primitive trace map $\mathcal{T}^\sharp$ injective?*

# Polynomial Structures

- A map $p : \mathbb{S}^n \to \mathbb{S}^r$ is polynomial if it is the restriction of a polynomial map $\mathbb{R}^{n+1} \to \mathbb{R}^{r+1}$.
- Define $q(n)$ to be the least positive integer such that there exists a non-constant polynomial map $\mathbb{S}^n \to \mathbb{S}^{q(n)}$.
- **Examples:**
  - The inclusion map $\mathbb{S}^n \hookrightarrow \mathbb{S}^m$ is polynomial (of degree 1), so $q(n) \leq n$; so many polynomial maps from low to high dimensional spheres.
  - The Hopf fibrations $\mathbb{S}^3 \to \mathbb{S}^2$, $\mathbb{S}^7 \to \mathbb{S}^4$, and $\mathbb{S}^{15} \to \mathbb{S}^8$ are polynomial of degree 2; $z \mapsto z^k$ is polynomial $\mathbb{S}^1 \to \mathbb{S}^1$.
- Important result by **Wood ['68]**: "Assume $0 \leq r \leq n-1$ is such that there exists a power of 2 among $\{r+1, \dots, n\}$. Then, there is *no non-constant polynomial map* $\mathbb{S}^n \to \mathbb{S}^r$."
- Thus $\frac{n}{2} < q(n) \leq n$. The proof relies on theorems by **Cassels ['64]** and **Pfister ['65]** on sums of squares.

- Possible to completely classify *quadratic* polynomial maps between spheres, see **Yiu ['86, '94]**, which gives an upper bound on $q(n)$.
- Hopf construction: given a bilinear map $F : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^t$ such that $|F(x, y)|^2 = |x|^2|y|^2$, define

$$H : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^{t+1}, \quad H(x, y) := (|x|^2 - |y|^2, 2F(x, y)).$$

  which yields a quadratic map $\mathbb{S}^{r+s-1} \to \mathbb{S}^t$.
- Let $\rho(n)$ be the Radon-Hurwitz number given by

$$\rho\big((2b+1)2^{c+4d}\big) = 2^c + 8d, \quad 0 \leq c \leq 3;$$

  $\rho(n) - 1$ is the maximal number of independent vector fields on $\mathbb{S}^{n-1}$.

- Possible to construct a Hopf map $\mathbb{S}^{n+\rho(n+1)} \to \mathbb{S}^{n+1}$ by taking $F : \mathbb{R}^{n+1} \times \mathbb{R}^{\rho(n+1)} \to \mathbb{R}^{n+1}$

$$F(x, y) = y_0 x + y_1 J_1 x + \ldots + y_{\rho(n+1)-1} J_{\rho(n+1)-1} x,$$

  where $J_1, \ldots, J_{\rho(n+1)-1}$ are orthogonal almost-complex structures on $\mathbb{R}^{n+1}$ (coming from a Clifford algebra representation).
- Using the three Hopf fibrations, possible to show that:

$$q(2) = q(3) = 2, \quad q(4) = \ldots = q(7) = 4, \quad q(8) = \ldots = q(15) = 8.$$

- The first unknown value is $q(48)$ and we do not know if there is a map $\mathbb{S}^{48} \to \mathbb{S}^{47}$ (of degree at least 3 necessarily).

We are in shape to formulate our first main result:

**Theorem (C-Lefeuvre '21 & '22)**

*Assume $(M^{n+1}, g)$ has negative sectional curvature and $\mathcal{E} \to M$ a Euclidean vector bundle. Then, the primitive trace map $\mathcal{T}^\sharp$ is:*

(a) *If $n \geq 2$, locally injective near generic points in $\mathbb{A}$;*

(b) *globally injective under a low rank assumption $\operatorname{rank}(\mathcal{E}) \leq q(n)$.*

- *Generic* in (a) refers to an open and dense set in the quotient $C^N$ topology for $N$ large enough. More precisely, it is related to injectivity of the twisted X-ray transform studied in **C-L ['20, '21]**.
- Similar methods used in **C-L ['21]** to show ergodicity of the frame flow on $\mathcal{E}$ under a low rank assumption $\operatorname{rank} \mathcal{E} = \mathcal{O}(\sqrt{n})$.
- When $\dim M$ is odd, we also show that $\mathcal{T}^\sharp([\mathcal{E}], [\nabla^{\mathcal{E}}])$ determines $[\mathcal{E}]$.

- **Counterexample:** On $(M^{4m}, g)$ set $\Lambda^{\pm} = \{\star \in \Lambda^{2m} TM \mid \star\alpha = \pm\alpha\}$ and equip with the Levi-Civita connection $\nabla^{\pm}$. Then we show $\mathcal{T}^{\sharp}(\Lambda^{+}, \nabla^{+}) = \mathcal{T}^{\sharp}(\Lambda^{-}, \nabla^{-})$, but $[\nabla^{+}] \neq [\nabla^{-}]$ (and $[\Lambda^{+}] \neq [\Lambda^{-}]$ when $m = 1$).
- **Previous results:**
  - **Paternain ['09, '10, '12, '13]** classified transparent connections (parallel transport over all closed geodesics is the identity) on surfaces and showed their abundance on bundles with rank $\mathcal{E} = 2$; see also **Guillarmou-P-Salo-Uhlmann ['16]**;
  - studied with the convex foliation condition by **P-S-U-Zhou ['18]** and on simple surfaces **P-S-U ['12]**;
  - Analogous marked length spectrum problem: study injectivity of $\mathcal{L}^{\sharp} : \mathbb{M}_{<0} \ni g \mapsto (L_g(c_1^{\sharp}), L_g(c_2^{\sharp}), \dots) \in \ell^{\infty}(\mathcal{C}^{\sharp})$.

- Length spectrum: the set of lengths of closed geodesics counted with multiplicities. It is simple if all closed geodesics have distinct lengths (generic condition).
- Connection Laplacian is the operator $\Delta_{\mathcal{E}} := (\nabla^{\mathcal{E}})^* \nabla^{\mathcal{E}}$. It is 2nd order elliptic, self-adjoint, non-negative, acting on $C^\infty(M, \mathcal{E})$, with discrete spectrum $\mathrm{spec}(\Delta_{\mathcal{E}}) = \{0 \leq \lambda_0(\nabla^{\mathcal{E}}) \leq \lambda_1(\nabla^{\mathcal{E}}) \leq \dots\}$ counted with multiplicities.
- $\mathrm{spec}(\Delta_{\mathcal{E}})$ depends only on $[\nabla^{\mathcal{E}}]$ and defines the spectrum map:

$$\mathcal{S} : \mathbb{A}_{\mathcal{E}} \ni [\nabla^{\mathcal{E}}] \mapsto \mathrm{spec}(\Delta_{\mathcal{E}}) \in \mathbb{R}_{\geq 0}^{\mathbb{N}}.$$

- Trace formula of **Duistermaat-Guillemin** applied to $\Delta_{\mathcal{E}}$ reads (assuming simple length spectrum; $P_\gamma$ is the Poincaré map):

$$\lim_{t \to L_g(c)} (t - L_g(c)) \sum_{j \geq 0} e^{-it\sqrt{\lambda_j}} = \frac{L_g(c)\,\mathrm{Tr}\,(\mathrm{Hol}_{\nabla^{\mathcal{E}}}(c))}{2\pi |\det(\mathrm{id} - P_{\gamma_g(c)})|^{1/2}}. \qquad (1.1)$$

- Consequence of (1.1) and the previous theorem is:

**Theorem (C-Lefeuvre '21 & '22)**

*Assume* $(M^{n+1}, g)$ *has negative sectional curvature with simple length spectrum. Then, the spectrum map* $\mathcal{S}$ *is:*

(a) *If* $\dim M \geq 3$, *locally injective near generic points in* $\mathbb{A}$*;*

(b) *Globally injective on* $\mathbb{A}$ *under the low rank assumption* $\operatorname{rank}(\mathcal{E}) \leq q(n)$.

- **Kuwabara ['90]**: counterexamples to injectivity of $\mathcal{S}$ for line bundles on covers of surfaces (simple length spectrum condition violated).
- Famous question of **Kac ['66]**: "*Can one hear the shape of a drum?*" (counterexamples exist on hyperbolic surfaces). Shape $\leftrightarrow$ magnetic field.
- Classical result of **Guillemin-Kazhdan ['80]**: $q \in C^\infty(M)$ determined from $\mathrm{spec}(-\Delta_g + q)$ (see also **Croke-Sharafutdinov ['98], P-S-U ['14]**).

# Summary

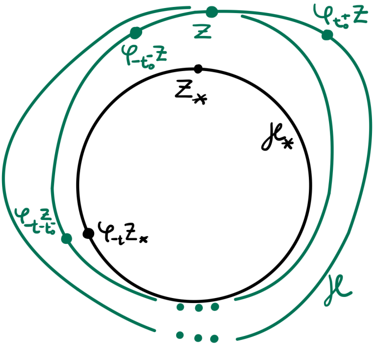
1. Non-Abelian Livšic theory (dynamical systems): show that if $\mathcal{T}^\sharp(\mathcal{E}_1, \nabla^{\mathcal{E}_1}) = \mathcal{T}^\sharp(\mathcal{E}_2, \nabla^{\mathcal{E}_2})$, then $\pi^*\nabla^{\mathcal{E}_1}$ and $\pi^*\nabla^{\mathcal{E}_2}$ are *dynamically equivalent*, that is, there exists $p : \mathcal{E}_2 \to \mathcal{E}_1$ such that

$$(\pi^*\nabla)_X^{\mathrm{Hom}(\mathcal{E}_2,\mathcal{E}_1)} p = 0,$$

   where $X$ is the geodesic vector field and $\pi : SM \to M$ the projection. (That is, for all geodesics $\gamma$, $\nabla_{\dot\gamma}^{\mathcal{E}_1} p(\gamma, \dot\gamma) = p(\gamma, \dot\gamma)\nabla_{\dot\gamma}^{\mathcal{E}_2}$.)
2. For the local result: show that in a neighbourhood of a generic connection, by a convexity argument (on the level of elliptic operators) the unique Pollicott-Ruelle resonance close to zero controls the distance in the moduli space.
3. Fourier analysis: by **G-P-S-U ['16]**, $p \in C^\infty(SM; \mathrm{Hom}(\mathcal{E}_2, \mathcal{E}_1))$ has finite Fourier content, i.e. when restricted to an arbitrary sphere $S_xM \subset SM$, $p : S_xM \to SO(r)$ is a polynomial map.
4. Algebraic geometry: assuming $r \leq q(n)$, $p$ is constant in each fibre and so $p$ is a gauge-equivalence.

- Let $z_\star = (x_\star, v_\star) \in SM$ be a fixed closed geodesic and $\mathcal{H}$ the set of all homoclinic orbits to $z_\star$. Define Parry's free monoid **G** and representation $\rho : \mathbf{G} \to \mathrm{SO}(\mathcal{E}_{x_\star})$:

$$\mathbf{G} := \{\gamma_1^{m_1}...\gamma_k^{m_k} \mid k \in \mathbb{N}, m_1, ..., m_k \in \mathbb{N}_0, \gamma_1, ..., \gamma_k \in \mathcal{H}\}.$$



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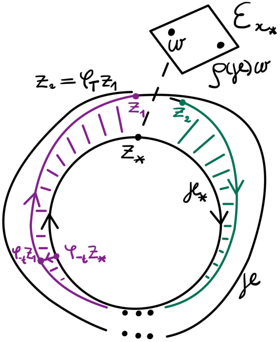

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- If $\mathcal{T}^\sharp(\mathcal{E}_1, \nabla^1) = \mathcal{T}^\sharp(\mathcal{E}_2, \nabla^2)$, then their Parry's representations are conjugate by $p_\star : (\mathcal{E}_2)_{x_\star} \to (\mathcal{E}_1)_{x_\star}$. Possible to push $p_\star$ along $\mathcal{H}$ to a smooth $p$ dynamically conjugating the connections!

Thank you for your attention!