Radon transforms supported in hypersurfaces

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Plan of talk

Example of a distribution whose Radon transform is supported on the set of tangents to a circle.

Theorem:
Such examples do not exist for other domains than ellipsoids.

Some words on the proof.

Related *local* questions.

The Radon transform

Define $$Rf(L) = \int_L f\,ds, \quad f \in C_c(\mathbb{R}^2), \quad L \text{ line in } \mathbb{R}^2.$$

More generally: L hyperplane in $\mathbb{R}^n$, ds area measure on L.

The hyperplanes are parametrized by $(\omega, p) \in S^{n-1} \times \mathbb{R}$ so that

$$L = L(\omega, p) \quad \text{is the hyperplane} \quad \{x \in \mathbb{R}^n;\ x \cdot \omega = p\}.$$

Then we write

$$Rf(\omega, p) = Rf(L(\omega, p)) = \int_{x \cdot \omega = p} f\,ds, \quad (\omega, p) \in S^{n-1} \times \mathbb{R}.$$

Note that $Rf(\omega, p) = Rf(-\omega, -p)$.

The Radon transform of a *distribution* f in $\mathbb{R}^n$ is defined by

$$\langle Rf, \varphi \rangle = \langle f, R^* \varphi \rangle, \quad \text{for all test functions } \varphi \text{, where}$$

$$(R^* \varphi)(x) = \int_{L \ni x} \varphi(L) d\mu(L), \quad \text{or}$$

$$(R^* \varphi)(x) = \int_{S^{n-1}} \varphi(\omega, x \cdot \omega) d\omega,$$

where $d\omega$ is surface measure on S^{n-1}.

Locally integrable functions $g(\omega, p)$ on $S^{n-1} \times \mathbb{R}$ are identified with distributions by means of the definition

$$\langle g, \varphi \rangle = \iint_{S^{n-1} \times \mathbb{R}} g(\omega, p) \varphi(\omega, p) d\omega \, dp, \quad \varphi \in C_c^\infty(S^{n-1} \times \mathbb{R}).$$

Radon transform supported on a hypersurface

Let f_0 be the function in the plane defined by

$$f_0(x) = \frac{1}{\pi} \frac{1}{\sqrt{1 - |x|^2}} \quad \text{for } |x| < 1$$

and $f = 0$ for all other $x = (x_1, x_2)$. An easy calculation shows that

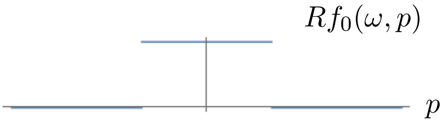
$$Rf_0(\omega, p) = \int_{x \cdot \omega = p} f_0(x)\, ds = 1 \quad \text{for } |p| < 1,$$

and obviously $Rf_0(\omega, p) = 0$ for $|p| \geq 1$.

Let f be the distribution $f = \Delta f_0 = (\partial_{x_1}^2 + \partial_{x_2}^2) f_0$.

Now use the formula $R(\Delta h)(\omega, p) = \partial_p^2 Rh(\omega, p)$ with $h = f_0$.

Note that $p \mapsto Rf_0(\omega, p)$ is piecewise constant:



It follows that

$$Rf(\omega,p) = \partial_p^2 Rf_0(\omega,p) = \delta'(p+1) - \delta'(p-1),$$

where $\delta(p)$ denotes the Dirac measure at the origin.

This means that the distribution $f = \Delta f_0$ has the property that its Radon transform, a distribution on the manifold of lines in the plane, must be supported on the set of tangents to the unit circle.

$f(x)$ is a smooth function in the unit disk, but tends fast to infinity at the boundary:

$$f(x) = \frac{1+2|x|^2}{\pi(1-|x|^2)^{5/2}}, \quad |x| < 1.$$

By means of an affine transformation we can easily construct a similar example where D is an ellipse.

QUESTION: Can one do the same for other domains than ellipses?

The answer is NO:

Theorem 1 (JB 2020). Let $D \subset \mathbb{R}^n$ be a bounded, convex domain. Assume that there exists a distribution $f \neq 0$, supported in $\overline{D}$, such that Rf is supported in the set of supporting planes to ∂D. Then the boundary of D is an ellipsoid.

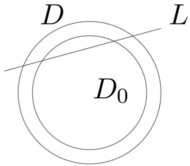
If ∂D is C^1 smooth, the supporting planes for D are of course tangent planes to ∂D.

And why did I ask the question above?

On Region of Interest reconstruction

Let D_0, the region of interest, be a proper subset of D. One would like to reconstruct a function supported in $\overline{D}$ from measurements of $Rf(L)$ only for lines that intersect D_0.

But this is in general not possible.



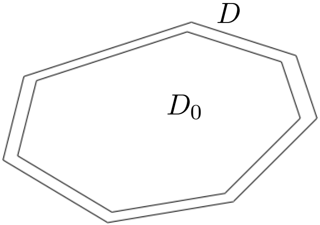
In fact, given two disks D and $\overline{D_0} \subset D$ there exist functions f with support *equal* to $\overline{D}$ such that

$$Rf(L) = 0 \quad \text{for all lines } L \text{ that meet } D_0.$$

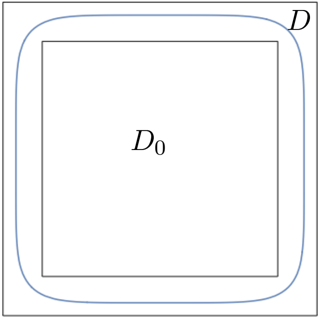
If D and D_0 are concentric and centered at the origin, one can take f radial, that is, $f(x) = f(r)$ with $r = |x|$, which makes the problem 1-dimensional.

It is natural to replace the disks by arbitrary convex sets.

Conjecture. Let D and D_0 be bounded convex domains in the plane with $\overline{D_0} \subset D$. Then there exists a smooth function f with $\operatorname{supp} f = \overline{D}$, such that its Radon transform $Rf(L)$ vanishes for every line L that intersects D_0.



Proof idea: find a compactly supported distribution f whose Radon transform is supported on the set of tangents to the blue curve.

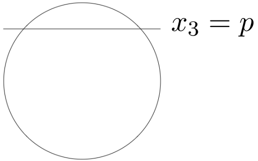


Then a regularization of f, $f_1 = f * \phi$, will solve our problem, because $Rf_1 = g_1$ will be a smooth function (on the manifold of lines) that is supported in a neighborhood of the set of tangents to the curve.

Theorem 1 shows that this idea must fail.

Arnold's Conjecture

Example:



The volume of the part of the unit ball in $\mathbb{R}^3$ that lies above the plane $x_3 = p$ is

$$\int_p^1 \pi(\sqrt{1-t^2})^2 dt = \int_p^1 \pi(1-t^2) dt = \frac{\pi}{3}(p^3 - 3p + 2).$$

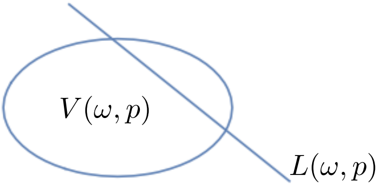
Similar for all odd dimensions and for ellipses instead of balls.

In even dimensions the volume function is not algebraic.

Arnold's Conjecture, cont.

Problem 1987-14 in *Arnold's Problems* reads:

Do there exist smooth hypersurfaces in $\mathbb{R}^n$ (other than the quadrics in odd-dimensional spaces), for which the volume of the segment cut by any hyperplane from the body bounded by them is an algebraic function of the hyperplane?



Vassiliev 1988: There exist no convex algebraically integrable bounded domains in even dimensions.

V. A. Vassiliev: *Applied Picard - Lefschetz Theory*, AMS 2002.

Case of odd dimension still unsolved.

Arnold's Conjecture, cont.

Special case: assume n is odd and the volume function $p \mapsto V(\omega, p)$ is *polynomial* for all ω. Prove that the boundary of D is an ellipsoid.

Solved by Koldobsky, Merkurjev, and Yaskin 2017.

Mark Agronovsky (2919) obtained the same conclusion with the weaker assumption that $p \mapsto V(\omega, p)$ is algebraic under an additional condition on the boundary of D.

Moreover:

Theorem 1 implies the result of Koldobsky, Merkurjev, and Yaskin.

Because if $p \mapsto V(\omega, p)$ is a polynomial of degree $\leq N$ for all ω, then the Radon transform, $p \mapsto R\chi_D(\omega, p)$, of the characteristic function for the domain D is a polynomial of degree $\leq N$ (for p in some interval that depends on ω). Hence the Radon transform of $\Delta^m \chi_D$ is supported on the set of tangent planes, if $m > N/2$. By Theorem 1 the boundary of D must then be an ellipsoid.

On the proof of Theorem 1

Strategy of proof ($n = 2$):

1. Write down an expression for an arbitrary distribution $g(\omega, p)$ on the manifold of lines in $\mathbb{R}^2$ that is supported on the set of tangents to the boundary of D.

2. Write down the condition on $g(\omega, p)$ for g to be the Radon transform of a distribution f on $\mathbb{R}^2$ (with sufficient decay at infinity).

The condition is that

$$\omega = (\omega_1, \omega_2) \mapsto \int_{\mathbb{R}} g(\omega, p) p^k dp \quad \text{is a homogeneous polynomial}$$

$$\text{of degree } k \text{ for every } k.$$

3. Prove that those conditions imply that the boundary curve is an ellipse.

On the proof of Theorem 1, cont.

Let $\rho_D(\omega) = \rho(\omega)$ be the supporting function for D

$$\rho(\omega) = \sup\{x \cdot \omega;\, x \in D\}.$$

The line $L(\omega, p)$ is tangent to ∂D iff

$$p = \rho(\omega) \quad \text{or} \quad p = \rho(-\omega).$$

Assume first that D is symmetric, $D = -D$. Then we may assume that g is even with respect to ω and p separately. If g is of order 0, then for some density $q(\omega)$

$$g(\omega, p) = q(\omega)\big(\delta(p - \rho(\omega)) + \delta(p + \rho(\omega))\big).$$

Here $\delta(\cdot)$ denotes the Dirac measure.

Use range conditions to deduce information on $\rho(\omega)$.

Case $D = -D$ and $Rf = g$ is a distribution of order 0

$k = 0$:

$\int_{\mathbb{R}} g(\omega, p)p^0 dp = 2\, q(\omega)$ must be constant, $q(\omega) = q$.

$k = 2$:

$\int_{\mathbb{R}} g(\omega, p)p^2 dp = 2q\, \rho(\omega)^2$ must be polynomial of degree 2, hence

$\rho(\omega)^2 = \rho(\omega_1, \omega_2)^2$ is a homogeneous polynomial of degree 2.

If $D = -D$, then ∂D is an ellipsoid iff $\rho(\omega)^2$ is a (quadratic) polynomial.

It follows that ∂D is an ellipse.

If the distribution $g(\omega, p)$ is of order 1, then

$$g(\omega, p) = q_0(\omega)\big(\delta(p - \rho(\omega)) + \delta(p + \rho(\omega))\big) + q_1(\omega)\big(\delta'(p - \rho(\omega)) - \delta'(p + \rho(\omega))\big).$$

The minus sign is needed to make g even, $g(-\omega, -p) = g(\omega, p)$.

Note that for instance

$$\int_{\mathbb{R}} \delta'(p - \rho(\omega))p^k dp = -k\rho(\omega)^{k-1}.$$

The range conditions imply that there must exist polynomials p_0, p_2, p_4 etc., where $p_k(\omega)$ is homogeneous of degree k, such that

$$\begin{aligned} q_0 &= p_0 \\ q_0\rho^2 + 2\, q_1\rho &= p_2 \\ q_0\rho^4 + 4\, q_1\rho^3 &= p_4 \\ q_0\rho^6 + 6\, q_1\rho^5 &= p_6 \\ \ldots . \end{aligned}$$

Eliminating q_0 and q_1 we easily see that ρ^2 must be a rational function. Here is an efficient way to eliminate q_0 and q_1.

In matrix form the equations read

$$\begin{pmatrix} 1 & 0 \\ \rho^2 & 2\rho \\ \rho^4 & 4\rho^3 \\ \rho^6 & 6\rho^5 \\ \dots & \dots \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \end{pmatrix} = \begin{pmatrix} p_0 \\ p_2 \\ p_4 \\ p_6 \\ \dots \end{pmatrix}.$$

This means that

$$\begin{pmatrix} 0 & 1 \\ \rho^2 & 2\rho \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \end{pmatrix} = \begin{pmatrix} p_0 \\ p_2 \end{pmatrix}, \quad \begin{pmatrix} \rho^2 & 2\rho \\ \rho^4 & 4\rho^3 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \end{pmatrix} = \begin{pmatrix} p_2 \\ p_4 \end{pmatrix}, \text{ etc.}$$

With shorter notation

$$M_0 Q = P_0, \quad M_1 Q = P_1, \quad M_2 Q = P_2, \quad \text{etc.}$$

Here I have set

$$M_0 = \begin{pmatrix} 1 & 0 \\ \rho^2 & 2\rho \end{pmatrix}, \quad M_1 = \begin{pmatrix} \rho^2 & 2\rho \\ \rho^4 & 4\rho^3 \end{pmatrix}, \quad \text{etc., and}$$

$$Q = \begin{pmatrix} q_0 \\ q_1 \end{pmatrix}, \quad P_0 = \begin{pmatrix} p_0 \\ p_2 \end{pmatrix}, \quad P_1 = \begin{pmatrix} p_2 \\ p_4 \end{pmatrix} \quad \text{etc.}$$

The matrices $M_0, M_1, M_2, \dots$ form a geometric series

$$M_k = S^k M_0 \quad \text{or all } k \geq 0, \quad \text{where}$$

$$S = M_1 M_0^{-1} = \begin{pmatrix} 0 & 1 \\ \rho^2 & -2\rho \end{pmatrix}.$$

This makes it easy to eliminate Q. Because for instance

$$SP_0 = SM_0Q = M_1Q = P_1.$$

And similarly

$$SP_k = P_{k+1} \quad \text{for all } k.$$

In other words

$$\begin{pmatrix} 0 & 1 \\ \rho^2 & -2\rho \end{pmatrix} \begin{pmatrix} p_0 \\ p_2 \end{pmatrix} = \begin{pmatrix} p_2 \\ p_4 \end{pmatrix} \quad \text{and}$$

$$\begin{pmatrix} 0 & 1 \\ \rho^2 & -2\rho \end{pmatrix} \begin{pmatrix} p_2 \\ p_4 \end{pmatrix} = \begin{pmatrix} p_4 \\ p_6 \end{pmatrix} \quad \text{etc.}$$

The last two equations can be combined to the matrix equation

$$\begin{pmatrix} 0 & 1 \\ \rho^2 & -2\rho \end{pmatrix} \begin{pmatrix} p_0 & p_2 \\ p_2 & p_4 \end{pmatrix} = \begin{pmatrix} p_2 & p_4 \\ p_4 & p_6 \end{pmatrix}.$$

By the product law for determinants these equations shows that ρ^2 must be a rational function. However, the fact that $SM_k = M_{k+1}$ shows that more generally

$$\begin{pmatrix} 0 & 1 \\ \rho^2 & -2\rho \end{pmatrix}^k \begin{pmatrix} p_0 & p_2 \\ p_2 & p_4 \end{pmatrix} = \begin{pmatrix} p_{2k} & p_{2k+2} \\ p_{2k+2} & p_{2k+4} \end{pmatrix}$$

for every k. And this shows that ρ^{2k} must be a rational function *with the same denominator* for every k. Hence ρ^2 must be a polynomial.

More generally: g is a distribution of order m

If $m = 3$ we get the system

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \rho & 1 & 0 & 0 \\ \rho^2 & 2\rho & 2 & 0 \\ \rho^3 & 3\rho^2 & 6\rho & 6 \\ \rho^4 & 4\rho^3 & 12\rho^2 & 24\rho \\ \rho^5 & 5\rho^4 & 20\rho^3 & 60\rho^2 \\ \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ \dots \end{pmatrix}.$$

With the same notation as before this can be written

$$M_0 Q = P_0, \quad M_1 Q = P_1, \quad M_2 Q = P_2, \quad \text{etc.}$$

The important point is that

$$M_k = S^k M_0 = M_0 T^k \quad \text{for all } k \text{, where}$$

$$S = M_1 M_0^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\rho^4 & 4\rho^3 & -12\rho^2 & 24\rho \end{pmatrix}$$

and

$$T = M_0^{-1} M_1 = \begin{pmatrix} \rho^2 & 1 & 0 & 0 \\ 0 & \rho^2 & 2 & 0 \\ 0 & 0 & \rho^2 & 3 \\ 0 & 0 & 0 & \rho^2 \end{pmatrix}.$$

So

$$\det S = \det T = (\rho^2)^4 = \rho^8.$$

General case: D not assumed symmetric

Then we have to consider matrices with two variables

$$M = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ \rho & 1 & 0 & \tau & 1 & 0 \\ \rho^2 & 2\rho & 2 & \tau^2 & 2\tau & 2 \\ \rho^3 & 3\rho^2 & 6\rho & \tau^3 & 3\tau^2 & 6\tau \\ \rho^4 & 4\rho^3 & 12\rho^2 & \tau^4 & 4\tau^3 & 12\tau^2 \\ \rho^5 & 5\rho^4 & 20\rho^3 & \tau^5 & 5\tau^4 & 20\tau^3 \\ \rho^6 & 6\rho^5 & 30\rho^4 & \tau^6 & 6\tau^5 & 30\tau^4 \\ \rho^7 & 7\rho^6 & 42\rho^5 & \tau^7 & 7\tau^6 & 42\tau^5 \\ \dots & \dots & \dots & \dots & \dots & \end{pmatrix}.$$

As before we introduce the successive $2m \times 2m$ submatrices, here 6×6 submatrices, M_0, M_1, M_2, etc. Then

$$M_k = S^k M_0, \quad \text{and}$$
$$M_k = M_0 T^k,$$

where

$$T = \begin{pmatrix} T_\rho & 0 \\ 0 & T_\tau \end{pmatrix}$$

with $$T_\rho = \begin{pmatrix} \rho & 1 & 0 \\ 0 & \rho & 2 \\ 0 & 0 & \rho \end{pmatrix} \quad \text{and} \quad T_\tau = \begin{pmatrix} \tau & 1 & 0 \\ 0 & \tau & 2 \\ 0 & 0 & \tau \end{pmatrix},$$

and the matrix S can be written (for $m = 3$)

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ s_0 & s_1 & s_2 & s_3 & s_4 & s_5 \end{pmatrix}.$$

Here s_j is (up to sign) the elemntary symmetric polynomial of degree $6 - j$ in 6 variables, evaluated at $(\rho, \rho, \rho, \tau, \tau, \tau)$.

Local questions

Assume that there exists a distribution f with support in $\overline{D}$, a tangent plane L_0, a point $x^0 \in L_0 \cap \operatorname{supp} f$, and a neighborhood V of L_0 in the manifold of hyperplanes, such that the restriction of Rf to V is supported on the set of tangent planes to ∂D in V. Does it follow that ∂D is a quadric in some neighborhood of x^0?

We don't know.

Singularities of a distribution and of its support

It turned out that the arguments in the proof of Theorem 1 could prove an apparently completely unrelated theorem that connects singularities of a distribution with singularities of its support.

First a couple of introductory remarks.

If a distribution f is supported on a hypersurface, we can talk about singularities of f (in terms of wave fronts), and we can talk about the singularities of the surface (wave front set of the defining function).

What is the relationship?

The same if f is the characteristic function for a region. What is the relationship between $WF_A(f)$ and the singularities of the boundary of the region?

Let us look at some examples.

The following is an easy consequence of the definition of $WF(f)$:

If f is a C^∞ density on a C^∞ hypersurface Σ, then $WF(f)$ is contained in the set $N^*(\Sigma)$ of conormals to Σ,

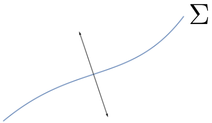
$$N^*(\Sigma) = \{(x, \xi);\ x \in \Sigma,\ \text{and } \xi \text{ conormal to } \Sigma \text{ at x}\}.$$

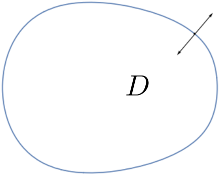
If f is a real analytic density on a real analytic hypersurface Σ, then

$$WF_A(f) \subset N^*(\Sigma).$$

And if f is the characteristic function for a domain D with real analytic boundary, then

$$WF_A(f) = N^*(\partial D).$$





Similarly, for distributions of higher order:

Let Σ be a hypersurface in R^{n+1} defined by $y = \Psi(x)$ and f be the distribution

$$\begin{aligned}\langle f, \varphi \rangle &= \sum_{j=0}^{m-1} \int_{\Sigma} q_j \, \partial_y^j \varphi \, dx \\ &= \sum_{j=0}^{m-1} \int_{\mathbb{R}^n} q_j(x) \, (\partial_y^j \varphi)(x, \Psi(x)) dx, \quad \varphi \in C_c^\infty(U).\end{aligned}$$

If Ψ and all q_j are real analytic, then $WF_A(f) \subset N^*(\Sigma)$.

I am interested in a strong converse to this statement. That is, assuming some regularity of the distribution f, I want to conclude that Ψ and all q_j are real analytic.

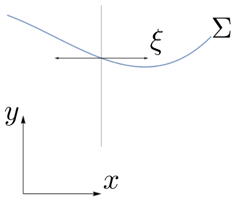
Theorem 2. Let f be the distribution above, supported on the C^1 surface $\Sigma : \; y = \Psi(x)$, $x \in U \subset \mathbb{R}^n$, that is

$$\langle f, \varphi \rangle = \sum_{j=0}^{m-1} \int_\Sigma q_j \, \partial_y^j \varphi \, dx.$$

Assume that $WF_A(f)$ contains no horisontal cotangent vectors $(\xi, \eta) = (\xi, 0)$, i.e. that

$$N^*(\gamma_x) \cap WF_A(f) = \emptyset,$$

for every line $\gamma_x : \; y \mapsto (x, y)$ for $x \in \mathbb{R}^n$. Then the surface Σ and all densities q_j are real analytic.



In particular, if $WF_A(f) \subset N^*(\Sigma)$, then the surface Σ and all densities q_j are real analytic.

Corollary. Let f be the characteristic function $\chi_D(x)$ for a domain D with C^1 boundary, or the product of $\chi_D(x)$ with a real analytic function, and let $x^0 \in \partial D$. Let v be a tangent vector that is transversal to the boundary at x^0. Assume that $(x^0, \xi) \notin WF_A(f)$ for all ξ that are conormal to v. Then the boundary of D is real analytic in a neighborhood of x^0.

There is in fact a coordinate free formulation of the theorem.

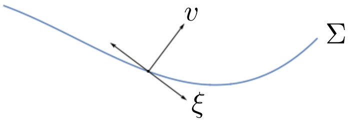
Theorem 2′. Let Σ be a C^1 hypersurface in a real analytic manifold M, let $f \in \mathcal{D}'(M)$ be supported in Σ, and let $z \in \operatorname{supp} f$. Assume that $v \in T_z(M)$ is a tangent vector to M at z that is transversal to Σ and that

$$(z, \xi) \notin WF_A(f) \text{ for every } \xi \text{ that is conormal to } v.$$

Then there exists a neighborhood U of z such that the surface Σ is real analytic in U and the distribution f has the form

$$\langle f, \varphi \rangle = \sum_{j=0}^{m-1} \int_{\mathbb{R}^n} q_j(x)\, (\partial_y^j \varphi)(x, \Psi(x)) dx, \quad \varphi \in C_c^\infty(U).$$

in suitable local coordinates in U with all q_j real analytic.



Theorems 1 and 2 appear unrelated, but proofs are very similar.

What is the relationship?

The *hypothesis* of Theorem 1 implies that

$$S^{n-1} \ni \omega \mapsto \int g(\omega, p) p^k dp \quad \text{is a polynomial for every } k.$$

The closely related condition

$$S^{n-1} \ni \omega \mapsto \int g(\omega, p) \varphi(\omega, p) dp \quad \text{is real analytic for every real analytic } \varphi$$

is equivalent to a microlocal regularity property of $g(\omega, p)$ (conormal of $p \mapsto (\omega, p)$ is disjoint from $WF_A(g)$).

The *conclusion* of Theorem 1 is a very strong regularity property of the supporting hypersurface of g (the surface is an ellipsoid).

Similarly, the assumption of Thm 2 is a microlocal regularity property of f, and the conclusion is that the supporting hypersurface is real analytic (and more).

References

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