On the Range conditions of the X-ray transform of planar symmetric tensors

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Outline

2D Tensor tomography: a brief overview

Range characterization in terms of the Fourier coefficients on the lattice $\mathbb{Z} \times \mathbb{Z}$

The even order tensors

The odd order tensors

Idea of proof

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Why bother with the Range?

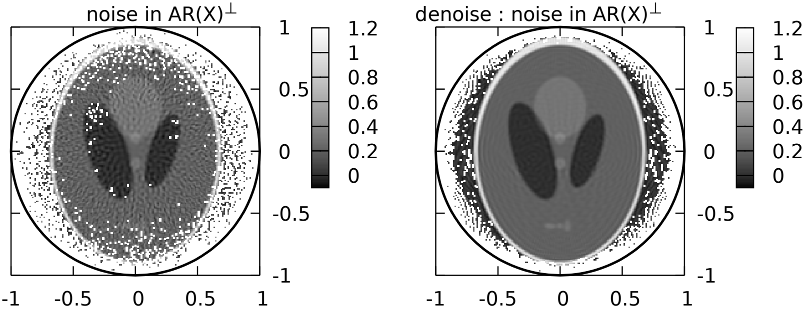


Figure: Courtesy: H. Fujiwara (Kyoto University)

Symmetric m-tensors in the plane

$\boldsymbol{\theta} = \langle \theta^1, \theta^2 \rangle = \langle \cos\theta, \sin\theta \rangle \in \mathbb{S}^1$, $\boldsymbol{\theta}^m := \underbrace{\boldsymbol{\theta} \otimes \boldsymbol{\theta} \otimes \cdots \otimes \boldsymbol{\theta}}_{m}$

$\mathbf{f} = (f_{i_1 i_2 \dots i_m}) \in L^1(S^m(\mathbb{R}^2); \mathbb{R}^2)$

$$\langle \mathbf{f}(x), \boldsymbol{\theta}^m \rangle = \sum_{i_1,\dots,i_m \in \{1,2\}} f_{i_1 \cdots i_m}(x) \theta^{i_1} \cdot \theta^{i_2} \cdots \theta^{i_m}, \quad x \in \mathbb{R}^2,$$

Symmetry $\Longrightarrow f_{i_1 \dots i_m} = f_{\underbrace{1 \cdots 1}_{m-k} \underbrace{2 \cdots 2}_{k}} =: \tilde{f}_k$.

$$\langle \mathbf{f}, \boldsymbol{\theta}^m \rangle = \begin{cases} \displaystyle\sum_{k=0}^{q} f_{2k} e^{-i(2k)\theta} + \sum_{k=1}^{q} f_{-2k} e^{i(2k)\theta}, & \text{if } m = 2q, \\ \displaystyle\sum_{k=0}^{q} f_{2k+1} e^{-i(2k+1)\theta} + f_{-(2k+1)} e^{i(2k+1)\theta}, & \text{if } m = 2q+1, \end{cases}$$

$$f_k \longleftrightarrow \tilde{f}_k \longleftrightarrow f_{i_1 \cdots i_m}$$

X-ray transform of symmetric m-tensors in the plane

For $\mathbf{f} = (f_{i_1 i_2 \dots i_m}) \in L^1(S^m(\mathbb{R}^2); \mathbb{R}^2)$

$$X\mathbf{f}(z, \theta) := \int_{-\infty}^{\infty} \langle \mathbf{f}(z + t\theta), \theta^m \rangle dt,$$

Tomography: $X\mathbf{f} \Longrightarrow \mathbf{f}$? ($m \geq 1$: *=solenoidal part of **f**).

$m = 0$ Radon (re-parametrization of lines)
$m = 1$ Doppler
$m \geq 2$ For the non-Euclidean case (linearized rigidity problem).

$$X\mathbf{f}(z, \theta) := \int \mathbf{f}(\gamma_{z,\theta}(t), \dot{\gamma}_{z,\theta}(t)) dt,$$

Uniqueness*, inversion and stability in 2D tensor tomography

- Euclidean
  - Inversion: $m = 0$ Radon (1917),
  - Inversion: $m \geq 0^*$ Bukhgeim&Kazantsev (2004)
- Non-Euclidean (s. Riemannian)
  - Stability $m = 0$: Mukhometov (general) (1977),
  - Uniqueness $m = 1^*$: Anikonov& Romanov (1997)
  - Stability $m \geq 0^*$: Sharafutdinov (1994, bdd. curv.), Stefanov& Uhlmann (2004)
  - Inversion $m = 0, 1^*$: Pestov& Uhlmann (2004), $m = 1^*$ Krishnan (2010, const. curv.)
- Euclidean & Attenuated
  - Inversion $m = 0$: Arbuzov, Kazantsev&Bukhgeim (1998), Novikov (2001)
  - Inversion $m = 1^*$: Bukhgeim& Kazantsev (2007), T. (2007), $m \geq 0$ Monard (2016)
- s.Riemannian &Attenuated
  - Inversion $m = 0$: Bal (2005, hyperbolic)
  - Uniqueness $m = 0$: Salo& Uhlmann (2011)
  - Uniqueness $m \geq 1^*$: Paternain, Salo& Uhlmann (2013)
  - Inversion $m \geq 0^*$: Monard (2015, 2016)

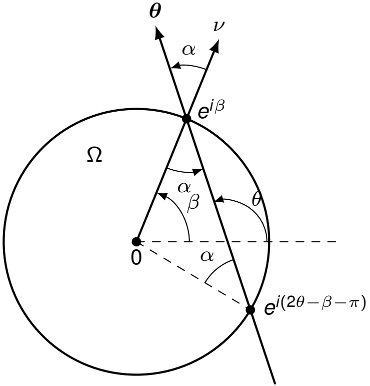
(2D) *X*-ray range characterization

- $m = 0$ Gelfand-Graev (1960), Helgason, Ludwig (1966), Pantyukhina (1990, $m \geq 1$): $L_{x,\boldsymbol{\theta}} := \{x + t\boldsymbol{\theta} : \ t \in \mathbb{R}\}$, $\boldsymbol{\theta} = \langle \cos\theta, \sin\theta \rangle$, $x \in \boldsymbol{\theta}^{\perp}$.
  $g = X\mathbf{f}$ *for some* $\mathbf{f} \in \mathcal{S}(S^m\mathbb{R}^2; \mathbb{R}^2)$ *iff*
  1. $g(x, -\boldsymbol{\theta}) = (-1)^m g(x, \boldsymbol{\theta})$
  2. $\forall n \geq 0$ *and* $0 \leq i \leq m$, $\exists P_i^n(x)$= *homog. polyn. deg. n,*

$$\int_{-\infty}^{\infty} g(s\boldsymbol{\theta}^{\perp}, \boldsymbol{\theta}) s^n ds = \sum_{i=0}^{m} P_i^n(\boldsymbol{\theta}) \cos^{m-i}(\theta) \sin^i(\theta),$$

- $m = 0, 1$: (s.Riemannian) *Pestov & Uhlmann* (2004) (Hilbert t. on the unit bundle).
- $m \geq 0$: Monard (2016, 2017) connects GGHL with PU
- $m = 0, 1$: *Sadiq& T.* (2014, '15), Sadiq, Scherzer& T, (2016, $m = 2$) (Bukhgeim-Hilbert transform)
- $m \geq 0$ Krishnan, Manaa, Sahoo& Sharafutdinov (2022) (Momenta Ray-transform)
- $m \geq 0$: Sadiq& T. (2021, 2022) (functions on the torus)

$X\mathbf{f}$ as a function on the torus $\Gamma \times \mathbb{S}^1$



Symmetry: $X\mathbf{f}(e^{i\beta}, e^{i\theta}) = (-1)^m X\mathbf{f}(e^{i\beta}, e^{i(\theta+\pi)}) = (-1)^m X\mathbf{f}(e^{i(2\theta-\beta-\pi)}, e^{i(\theta+\pi)})$

Outflux bdry: $\Gamma_+ := \left\{(e^{i\beta}, e^{i(\beta+\alpha)}) \in \Gamma \times \mathbb{S}^1 : \beta \in (-\pi, \pi],\ |\alpha| < \frac{\pi}{2}\right\}$

Influx bdry: $\Gamma_- := \left\{(e^{i\beta}, e^{i(\beta+\alpha)}) \in \Gamma \times \mathbb{S}^1 : \beta \in (-\pi, \pi],\ \frac{\pi}{2} < |\alpha| \leq \pi\right\}$

Scattering rel: $\Gamma_+ \ni (e^{i\beta}, e^{i\theta}) \leftrightarrow (e^{i(2\theta-\beta-\pi)}, e^{i\theta}) \in \Gamma_-$

Range question on the lattice $\mathbb{Z} \times \mathbb{Z}$

Given $g \in L^1(\Gamma \times \mathbb{S}^1)$ determine some necessary and sufficient conditions for

$$g = \begin{cases} X\mathbf{f}, & \text{on } \Gamma_+, \\ -X\mathbf{f}, & \text{on } \Gamma_-, \end{cases} \quad \text{for some } \mathbf{f},$$

in terms of

$$g_{n,k} := \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} g(e^{i\beta}, e^{i\theta}) e^{-in\theta} e^{-ik\beta} d\theta d\beta, \;\; n, k \in \mathbb{Z}.$$

First index n= angular mode
Second index k= boundary mode
Remark:

- If m= EVEN, then $g(e^{i\beta}, \cdot)$ is ODD $\Rightarrow g_{even,k} = 0$.
- If m= ODD, then $g(e^{i\beta}, \cdot)$ is EVEN $\Rightarrow g_{odd,k} = 0$

Range description for the even order m-tensor

Necessity:

Let $\mathbf{f} \in L^1(\mathbf{S}^m; \Omega)$ real valued and $g \in L^1_{\text{sym,odd}}(\Gamma \times \mathbb{S}^1)$ with

$$g = X\mathbf{f} \text{ on } \Gamma_+ (\text{ and } g = -X\mathbf{f} \text{ on } \Gamma_-).$$

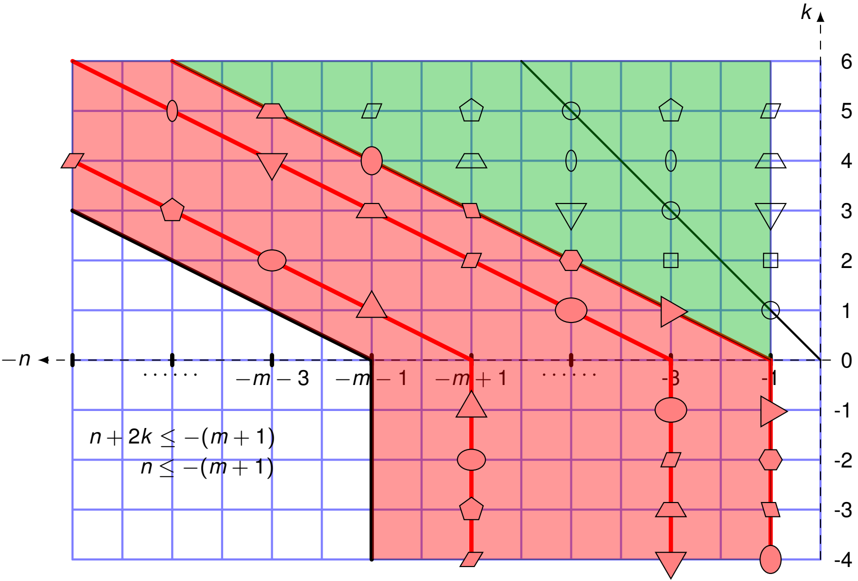
Then the Fourier coefficients $\{g_{n,k}\}_{n,k\in\mathbb{Z}}$ satisfy:

$$
\begin{aligned}
&\textit{Oddness}: && g_{n,k} = 0, && \forall \text{even } n \in \mathbb{Z}, \forall k \in \mathbb{Z};\\
&\textit{Conjugacy}: && g_{-n,-k} = \overline{g_{n,k}}, && \forall n,k \in \mathbb{Z};\\
&\textit{Symmetry}: && g_{n,k} = (-1)^{n+k} g_{n+2k,-k}, && \forall n,k \in \mathbb{Z};\\
&\textbf{Moments}: && \mathbf{g_{n,k} = (-1)^k g_{n+2k,-k}}, && \forall \text{odd } \mathbf{n \le -(m+1)}, \forall \mathbf{k \le 0}.
\end{aligned}
$$

Remarks:

- Only **Moments** are intrinsic to X-ray of some $\mathbf{f}$.
- Change in sign in Symmetry vs. Moments.
- Modulo some decay they are also sufficient

Range conditions for the even *m*-order tensor



White = Moments + Symmetry, Red = Symmetry, Green = Symmetry + Conjugacy

Reduced constraints for even order tensors

For $\mathbf{f} \in L^1(\mathbf{S}^m; \Omega)$ let

$$g = \begin{cases} X\mathbf{f}, & \text{on } \Gamma_+, \\ -X\mathbf{f}, & \text{on } \Gamma_-, \end{cases} \quad \text{for some } \mathbf{f},$$

Then, for all *odd* $n \leq -1$,

$$g_{n,k} = \begin{cases} 0, & \text{if } (n,k) \in W, \\ (-1)^{1+k}\overline{g_{-n-2k,k}}, & \text{if } (n,k) \in G^+, \\ (-1)^{1+k} g_{n+2k,-k}, & \text{if } (n,k) \in R. \end{cases}$$

Remark: $g_{-n,n} \in \mathbb{R}$.

Preparation for sufficiency (even order tensors)

Decay: For some $\mu > 1/2$,

$$\sum_{\substack{n \leq -1 \\ n = \text{odd}}} \langle n \rangle^2 \sum_{k=-\infty}^{\infty} |g_{n,k}| < \infty, \quad \text{and} \quad \sum_{k=-\infty}^{\infty} \langle k \rangle^{1+\mu} \sum_{\substack{n \leq -1 \\ n = \text{odd}}} |g_{n,k}| < \infty,$$

Non-uniqueness class

On Γ define $g_{-n} := \sum_{k=-\infty}^{\infty} g_{-n,k} \, e^{ik\beta}$, for odd $n \leq -1$.

For the constructed $m/2$ functions $g_{-1}, g_{-3}, \cdots, g_{-(m-1)} \in L^1(\Gamma)$:

$$\Psi_g^{\text{even}} := \left\{ \left(\psi_{-1}, \psi_{-3}, \cdots \psi_{-(m-1)}\right) \in \left(W^{1,1}(\Omega; \mathbb{C}) \right)^{\frac{m}{2}} : \right.$$

$$\left. \psi_{-(2j-1)}\big|_{\Gamma} = g_{-(2j-1)}, \; 1 \leq 2j-1 \leq m \right\}.$$

Sufficiency (even order tensors)

If $\{g_{n,k} : \text{odd } n \leq -1,\ k \in \mathbb{Z}\}$ satisfy:
Symmetry, Moments, and **Decay**, then

$$\exists \mathbf{f} \in L^1(\mathbf{S}^m; \Omega),\ \text{real valued, s.t.}$$

$$(\Gamma \times \mathbb{S}^1) \ni (e^{i\beta}, e^{i\theta}) \longmapsto 2\,\mathbb{R}\mathrm{e}\left\{ \sum_{\substack{n\leq -1\\ n=\text{odd}}} \sum_{k\in\mathbb{Z}} g_{n,k} e^{in\theta} e^{ik\beta} \right\}$$

defines a function in $L^1_{\text{sym,odd}}(\Gamma \times \mathbb{S}^1)$, which coincides with

$$X\mathbf{f} \text{ on } \Gamma_+ \text{ (and } -X\mathbf{f} \text{ on } \Gamma_-\text{).}$$

- If $m \geq 2$: $\mathbf{f}$ is uniquely determined by an element in Ψ_g^{even}.
- If $m = 0$, $\mathbf{f}$ is unique.

Remark: The method of proof is constructive.

Outline

Odd-order tensors: Changes induced by the flip in parity

For m= odd,

$$g = \begin{cases} X\mathbf{f}, & \text{on } \Gamma_+, \\ -X\mathbf{f}, & \text{on } \Gamma_-. \end{cases} \text{ is}$$

- *skew-symmetric*:

$$g(e^{i\beta}, e^{i\theta}) = -g(e^{i(2\theta-\beta-\pi)}, e^{i(\theta+\pi)}), \text{ for } (e^{i\beta}, e^{i\theta}) \in \Gamma \times \mathbb{S}^1,$$

- *angularly even*:

$$g(e^{i\beta}, e^{i\theta}) = g(e^{i\beta}, e^{i(\theta+\pi)}),$$

In particular $g_{odd,k} = 0$, for all $k \in \mathbb{Z}$.

Outline

Reduction to the transport model

The unique solution $u(z,\boldsymbol{\theta})$ to the boundary value problem

$$\boldsymbol{\theta} \cdot \nabla u(z,\boldsymbol{\theta}) = 2\langle \mathbf{f}(z), \boldsymbol{\theta}^m \rangle$$
$$u|_{\Gamma_-} = -X\mathbf{f}|_{\Gamma_-}$$

has the trace $u|_{\Gamma\times\mathbb{S}^1} = \begin{cases} X\mathbf{f}, & \text{on } \Gamma_+, \\ -X\mathbf{f}, & \text{on } \Gamma_-. \end{cases}$

- For $m = even$:
  - $-X\mathbf{f}$ is the only data Γ_- yielding $u|_{\Gamma\times\mathbb{S}^1}$ angularly odd.
  - NO data on Γ_- would yield $u|_{\Gamma\times\mathbb{S}^1}$ angularly even.
- For $m = odd$:
  - $-X\mathbf{f}$ is the only data on Γ_- yielding $u|_{\Gamma\times\mathbb{S}^1}$ angularly even.
  - NO data on Γ_- would yield $u|_{\Gamma\times\mathbb{S}^1}$ angularly odd.
- $m = even \Longrightarrow \boldsymbol{\theta} \mapsto u(z,\boldsymbol{\theta})$ is *odd* $\Longrightarrow X\mathbf{f}$ affects only angularly *odd* modes of $u(z,\cdot)$.
- $m = odd \Longrightarrow \boldsymbol{\theta} \mapsto u(z,\boldsymbol{\theta})$ is *even* $\Longrightarrow X\mathbf{f}$ affects only angularly *even* modes of $u(z,\cdot)$.

Transport equation ⇔ Bukhgeim-Beltrami system ($m = 2q$=even)

$$\boldsymbol{\theta} \cdot \nabla u(z, \boldsymbol{\theta}) = 2\langle \mathbf{f}(z), \boldsymbol{\theta}^m \rangle, \quad (z, \boldsymbol{\theta}) \in \Omega \times \mathbb{S}^1.$$

$$\left[e^{-i\theta}\overline{\partial} + e^{i\theta}\partial \right] u(z, \boldsymbol{\theta}) = \sum_{k=-q}^{q} f_{2n}(z) e^{-2in\theta}$$

- $u(z, \theta) \longleftrightarrow \mathbf{u}_{odd}(z) := (u_{-1}(z), u_{-3}(z), ...)$
- Shift $L\mathbf{u} = L(u_{-1}, u_{-3}, ...) = (u_{-1}, u_{-3}, ...)$
- $L^q\mathbf{u} = L^q(u_{-1}, u_{-3}, ...) = (u_{-2q-1}, u_{-2q-3}, ...)$

$$\overline{\partial} u_{-(2n-1)}(z) + \partial u_{-(2n+1)}(z) = f_{2n}(z), \qquad 0 \leq n \leq q,$$
$$\overline{\partial} u_{-(2n-1)}(z) + \partial u_{-(2n+1)}(z) = 0, \qquad n \geq q+1.$$

The tail $\mathbf{v} := L^q\mathbf{u}$ is L-analytic (Bukhgeim '95): $\overline{\partial}\mathbf{v} + L\partial\mathbf{v} = 0$

Characterization of traces of L-analytic

- Bukhgeim-Cauchy integral operator

$$(\mathcal{B}\mathbf{g})_{-n}(z) = \frac{1}{2\pi i}\int_{\Gamma}\frac{g_{-n}(\zeta)}{\zeta - z}d\zeta + \frac{1}{2\pi i}\int_{\Gamma}\left\{\frac{d\zeta}{\zeta - z} - \frac{d\overline{\zeta}}{\overline{\zeta} - \overline{z}}\right\}\sum_{j=1}^{\infty} u_{-n-j}(\zeta)\left(\frac{\overline{\zeta - z}}{\zeta - z}\right)^{j}.$$

- Bukhgeim-Hilbert transform

$$(\mathcal{H}\mathbf{g})_{-n}(z) = \frac{1}{\pi i}\int_{\Gamma}\frac{g_{-n}(\zeta)}{\zeta - z}d\zeta + \frac{1}{\pi i}\int_{\Gamma}\left\{\frac{d\zeta}{\zeta - z} - \frac{d\overline{\zeta}}{\overline{\zeta} - \overline{z}}\right\}\sum_{j=1}^{\infty} u_{-n-j}(\zeta)\left(\frac{\overline{\zeta - z}}{\zeta - z}\right)^{j}.$$

- **Theorem** (Sadiq&T. 2014)Let $1/2 < \mu < 1$.
(i) If $\mathbf{g} \in l_{\infty}^{1,1}(\Gamma) \cap C^{\mu}(\Gamma; l_1)$ is the boundary value of an $\mathcal{L}$-analytic function, then $\mathcal{H}\mathbf{g} \in C^{\mu}(\Gamma; l_{\infty})$ and

$$(I + i\mathcal{H})\mathbf{g} = \mathbf{0}.$$

(ii) If $\mathbf{g} \in Y_{\mu}(\Gamma)$ with $(I + i\mathcal{H})\mathbf{g} = 0$, then the extension $\mathbf{u} := \mathcal{B}\mathbf{g} \in C^{1,\mu}(\Omega; l_1) \cap C^{\mu}(\overline{\Omega}; l_1)$ is $\mathcal{L}$-analytic and

$$\mathbf{u}|_{\Gamma} = \mathbf{g}.$$

Key mapping property of Bukhgeim-Hilbert transform

For $\mathbf{g} = \langle g_0, g_{-1}, g_{-2} \dots \rangle$, let $g_{-n,k} := \frac{1}{2\pi} \int_{-\pi}^{\pi} g_{-n}\left(e^{i\beta}\right) e^{-ik\beta} d\beta$, $n \geq 0, k \in \mathbb{Z}$.
Bukhgeim-Hilbert t. $\mathcal{H}\mathbf{g} = \langle (\mathcal{H}\mathbf{g})_0, (\mathcal{H}\mathbf{g})_{-1}, (\mathcal{H}\mathbf{g})_{-2}, \dots \rangle$

$$(\mathcal{H}\mathbf{g})_{-n}(z) = \frac{1}{\pi i} \int_{\Gamma} \frac{g_{-n}(\zeta)}{\zeta - z} d\zeta + \frac{1}{\pi i} \int_{\Gamma} \left\{ \frac{d\zeta}{\zeta - z} - \frac{d\overline{\zeta}}{\overline{\zeta} - \overline{z}} \right\} \sum_{j=1}^{\infty} u_{-n-j}(\zeta) \left(\frac{\overline{\zeta - z}}{\zeta - z} \right)^j .$$

$$(\mathcal{H}\mathbf{g})_{-n,k} := \frac{1}{2\pi} \int_{-\pi}^{\pi} (\mathcal{H}\mathbf{g})_{-n}\left(e^{i\beta}\right) e^{-ik\beta} d\beta, \text{ for } n \geq 0, k \in \mathbb{Z}.$$

Theorem(Sadiq&T. 2022) If $\langle g_0, g_{-1}, g_{-2} \dots \rangle \in l_{\infty}^{1,1}(\Gamma) \cap C^{\mu}(\Gamma; l_1)$, $0 < \mu < 1$, then $\mathcal{H}\mathbf{g} \in C^{\mu}(\Gamma; l_{\infty})$, and

$$(-i)(\mathcal{H}\mathbf{g})_{-n,k} = \begin{cases} g_{-n,k} & \text{if } k \geq 0, \\ -g_{-n,k} + 2(-1)^k g_{-n+2k,-k} & \text{if } k \leq -1. \end{cases}$$

Remark: $\mathcal{H} : L^2(\Gamma \times \mathbb{S}^1) \to L^2(\Gamma \times \mathbb{S}^1)$ is an isomorphism and $\mathcal{H}^2 = -I$.

X-ray tomography using denoising by projection on the Range

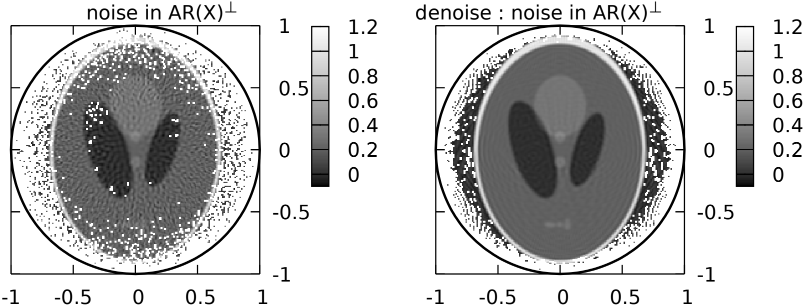


Figure: Raw data $= Xf^{\text{exact}} + \delta$ has 10.0% relative L^2- error. Left: reconstruction from raw data has 45.2% relative L^2-error. Right: reconstruction via the proposed denoising method has 19.4% relative L^2- error.

Thank you!

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