Free boundary methods in inverse scattering

Mikko Salo
University of Jyväskylä

Geometric inverse problems, Linz, 8 November 2022

Finnish Centre of Excellence in
Inverse Modelling and Imaging
2018-2025

European Research Council
Established by the European Commission

Outline

1. Single measurement problems
2. Main results
3. Methods

Calderón problem

Schrödinger equation

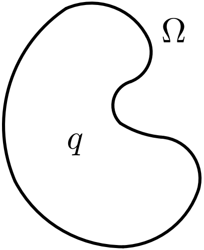
$$\begin{cases} (-\Delta + q)u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ bounded domain and $q \in L^\infty(\Omega)$ (potential).

Boundary measurements given by the
Dirichlet-to-Neumann (DN) map

$$\Lambda_q : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega), \; f \mapsto \partial_\nu u|_{\partial\Omega}.$$

Inverse problem: given Λ_q, recover q.



Calderón problem

Infinitely many measurements (know $\Lambda_q f$ for all f):
[Sylvester-Uhlmann 1987, Bukhgeim 2007, Guillarmou-Tzou 2011, ...]

Finitely many measurements (know $\Lambda_q f_1, \ldots, \Lambda_q f_N$):
[Alberti-Santacesaria 2019, Harrach 2019, ...]

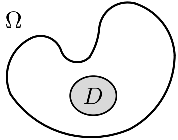
We are interested in the single measurement case, where we know $\Lambda_q f$ for a single function $f \not\equiv 0$.

Single measurement case

The data $\Lambda_q f$ depends on $n-1$ variables, and q depends on n variables. The inverse problem is formally underdetermined.

Consider a penetrable obstacle D with

$$q = h\chi_D$$



where $\overline{D} \subset \Omega$, and the contrast $h(x)$ satisfies $|h| \geq c > 0$ near ∂D.

Inverse problem: given $\Lambda_q f$ for fixed f, recover D.

Invisible obstacles

The obstacle D is invisible (for data $f \not\equiv 0$) if $\Lambda_q f = \Lambda_0 f$ for some contrast h. Such an obstacle looks like empty space, and cannot be recovered from $\Lambda_q f$.

Question: which obstacles D can be invisible (for some f and for some contrast h)?

(All this extends to inverse scattering in $\mathbb{R}^n$, i.e. to

$$(-\Delta - k^2\eta(x))u = 0 \qquad \text{(acoustic)}$$
$$(-\Delta - k^2 + V(x))u = 0 \qquad \text{(quantum)}$$

where $k > 0$ is a fixed frequency, $\eta = 1 + h\chi_D$, and $V = h\chi_D$.
Cf. non-scattering energies and interior transmission eigenvalues.)

Some earlier results

Corners always scatter, i.e. D is never invisible if

- D has a 90° corner [Blåsten-Päivärinta-Sylvester 2014]
- $n = 2$ and D has a $< 90^\circ$ corner [Päivärinta-S-Vesalainen 2017]
- D has high curvature points [Blåsten-Liu 2021, ...]

Based on CGO solutions + Laplace transforms on cones.
Implies single measurement results.

Another approach based on BVPs in corner domains:

- $n = 2, 3$ and D has a curvilinear corner or edge [Elschner-Hu 2018]
- $n = 2$ and ∂D is piecewise analytic [Li-Hu-Yang 2021]

Some earlier results

On the other hand, D can be invisible if

- D is a ball [Colton-Monk 1988]
- D is a union of balls [Gell-Redman-Hassell 2012]

Based on these results, the question of which obstacles D can or cannot be invisible remained mysterious.

Outline

1. Single measurement problems
2. Main results
3. Methods

Free boundary methods

New observation: the boundary of an invisible obstacle is a free boundary [Cakoni-Vogelius 2021, S-Shahgholian 2021].
Also true in the Calderón problem [Alessandrini-Isakov 1996].
Powerful methods from free boundary literature can be applied.

Assumptions

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $k \geq 0$ a fixed frequency, and D (the obstacle) an open set with $\overline{D} \subset \Omega$ such that

- D and $\mathbb{R}^n \setminus \overline{D}$ connected, $\mathrm{int}(\overline{D}) = D$ (solid domain),
- Dirichlet problem for $\Delta + k^2$ in D is well-posed.

This excludes holes in D and obstacles like the slit disk.

We let $u_0 = u_0^f$ (incident wave) be the solution of

$$\begin{cases} (\Delta + k^2) u_0 = 0 & \text{in } \Omega, \\ u_0 = f & \text{on } \partial\Omega. \end{cases}$$

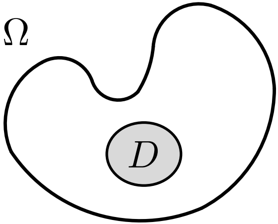
If $\Omega = \mathbb{R}^n$, we let u_0^f be the Herglotz wave[1] for $f \in L^2(S^{n-1})$.

[1] $u_0^f(x) = \int_{S^{n-1}} e^{ikx\cdot\omega} f(\omega)\, d\omega$

Main results

Theorem 1 (S-Shahgholian 2021)

Let D have real-analytic boundary. Then D is invisible for any Dirichlet data f such that $u_0^f|_{\partial D} > 0$ (with some contrast h).



Note: h depends of f and may have varying sign.

Main results

Theorem 2 (Kow-Larson-S-Shahgholian 2022)

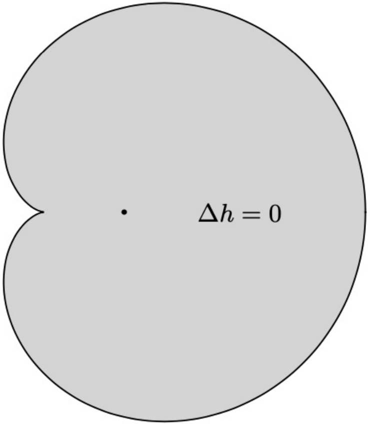
Let D be a quadrature domain for $\Delta + k^2$. Then D is invisible for any Dirichlet data f such that $u_0^f|_{\partial D} > 0$.

D is a quadrature domain if there is $\mu \in \mathscr{E}'(D)$ such that

$$\int_D w\, dx = \langle \mu, w \rangle, \quad w \in L^1(D),\ (\Delta + k^2)w = 0.$$

Basic case: $D = B_r$ and $\mu = c_{k,r}\delta_0$ (mean value theorem).
Includes domains with inward cusps and double points.

Quadrature domain



$$\int_\Omega h = \frac{3\pi}{2} h(0) + \frac{\pi}{2} \partial_x h(0)$$

Figure from [Petrosyan-Shahgholian-Uraltseva 2012].

Main results

Theorem 3 (S-Shahgholian 2021)

Let D be invisible for f. Near each $x_0 \in \partial D$ with $u_0^f(x_0) \neq 0$, either

(a) D is C^1 near x_0, or

(b) $\mathbb{R}^n \setminus D$ is thin near x_0 (e.g. inward cusp).

If h is $C^{k,\alpha}$/analytic and (a) holds, then D is $C^{k+1,\alpha}$/analytic.

This follows from free boundary regularity results.
Proved earlier for Lipschitz domains D [Cakoni-Vogelius 2021].
Related results: [Cakoni-Vogelius-Xiao 2022, . . .]

Main results

Compared to earlier approaches, free boundary methods

- apply to general sets D (not just curvilinear polyhedra)
- apply in any dimension (not just $n = 2, 3$)
- characterize what invisible obstacles can look like

However, they require the positivity condition $u_0^f|_{\partial D} > 0$.
Since real solutions to $(\Delta + k^2)u = 0$ have many zeros, this condition may be nontrivial. In this direction:

Theorem 4 (Kow-S-Shahgholian 2022)

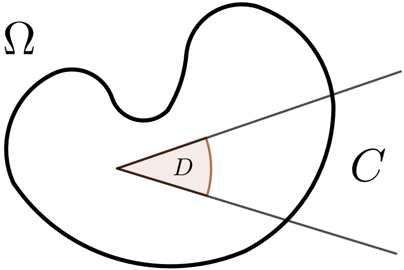
Suppose D is a bounded Lipschitz domain. There is a Herglotz wave satisfying $u_0^f|_{\partial D} > 0$.

Outline

1. Single measurement problems
2. Main results
3. Methods

We first explain methods in [Blåsten-Päivärinta-Sylvester 2014], [Päivärinta-S-Vesalainen 2017] based on CGO solutions and Laplace transforms.

Let $k = 0$, and let D have a corner at $0 \in \partial D$ modelled by a cone C.



1. If $\Lambda_q f = \Lambda_0 f$ for fixed $f \not\equiv 0$, Alessandrini identity and $q = h\chi_D$ give
$$\int_D h u_0^f w \, dx = 0.$$
Here u_0^f is fixed, but can vary w solving $(-\Delta + q)w = 0$.

2. Choose w as a CGO solution with $\rho \in \mathbb{C}^n$, $\rho \cdot \rho = 0$:
$$w = e^{-\rho \cdot x}(1 + r)$$
Since $\Delta u_0^f = 0$, expand $u_0^f = H + O(|x|^{N+1})$ near 0 where $H = H_N \not\equiv 0$ is a harmonic homog. polynomial. Get
$$0 = \int_D h u_0^f w \, dx = \underbrace{\int_D e^{-\rho \cdot x} h(0) H(x) \, dx}_{\text{main term}} + \text{error terms}.$$

3. Homogeneity (blowup $x \to x/|\rho|$) $\implies$ main term contains Laplace transform $(\chi_C H)\hat{}(\rho/|\rho|)$. Prove

$$(\chi_C H)\hat{}(\rho/|\rho|) \equiv 0 \qquad (*)$$

for certain ρ, by showing $|$error terms$| \lesssim |\rho|^{-\delta}|$main term$|$ via L^p estimates for CGO solutions.

4. For $n = 2$, use that $H(re^{i\theta}) = r^N(ae^{iN\theta} + be^{-iN\theta})$ to derive a contradiction with $(*)$ unless C has angle 180°.

We proceed to free boundary methods. Assume, for simplicity, that ∂D is Lipschitz and $k = 0$. Let $q = h\chi_D$ and

$$\Lambda_q f = \Lambda_0 f. \qquad (*)$$

Let $u, u_0 \in H^1(\Omega)$ solve[1]

$$\begin{cases} (-\Delta + q)u = \Delta u_0 = 0 \text{ in } \Omega, \\ u = u_0 = f \text{ on } \partial\Omega, \\ \partial_\nu u = \partial_\nu u_0 \text{ on } \partial\Omega. \end{cases}$$

The last equality used $(*)$.

Since $q = 0$ outside D and $\mathbb{R}^n \setminus \overline{D}$ is connected, unique continuation implies $u = u_0$ outside $\overline{D}$.

---

[1] Cf. interior transmission problem, where $u, u_0 \in L^2(\Omega)$ instead.

Setting $w = u - u_0$ gives

$$\begin{cases} -\Delta w = hu\chi_D & \text{in } \Omega, \\ w = 0 & \text{in } \Omega \setminus \overline{D}. \end{cases}$$

Writing $f = hu$ and using the assumption that $h|_{\partial D} \neq 0$ and $u|_{\partial D} = u_0|_{\partial D} > 0$, this can be rewritten as

$$-\Delta w = f\chi_{\{w \neq 0\}} \quad \text{near } \partial D.$$

In this equation D has disappeared, and the solution w remains.

The equation

$$-\Delta w = f\chi_{\{w\neq 0\}} \quad \text{in } U$$

appears in (no-sign) obstacle problems. The free boundary is

$$\partial\{w = |\nabla w| = 0\} = \partial D.$$

Properties of w lead to regularity of ∂D. (Blowup analysis, [Caffarelli 1980, ..., Petrosyan-Shahgholian-Uraltseva 2012].)

Steps for studying the equation

$$-\Delta w = f\chi_{\{w\neq 0\}} \quad \text{in } U.$$

1. Optimal $C^{1,1}$ regularity of w. Nontrivial since RHS is only in L^∞, and Calderón-Zygmund estimates fail at $p=\infty$. Scale-invariant estimates

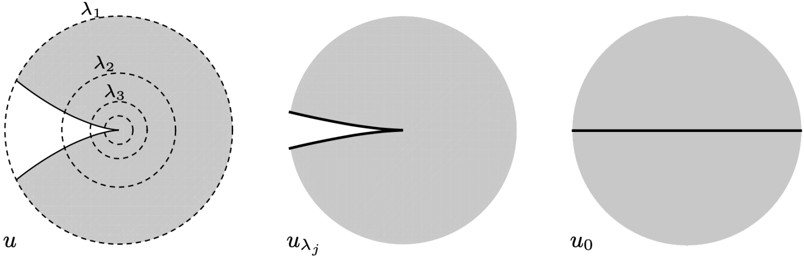
$$|\nabla^2 w| \leq C. \qquad (*)$$

2. Blowup analysis: if $0 \in \partial D$, consider

$$w_{\lambda_j}(x) = \frac{w(\lambda_j x)}{\lambda_j^2}, \qquad x \in B_1,$$

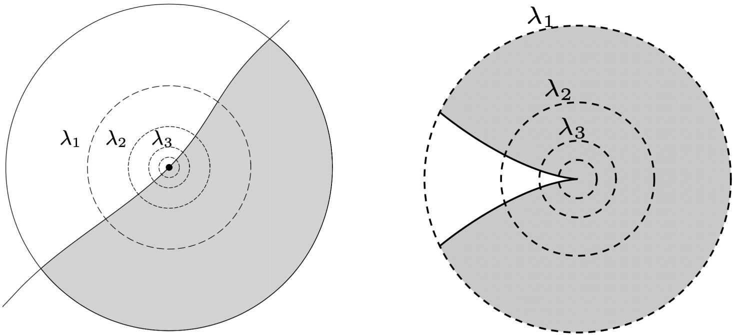
where $\lambda_j \to 0$. By compactness and $(*)$, there is a $C^{1,1}$ limit w_0 (blowup solution).

Blowup



Blowup near the cusp point of a cardioid domain.
Figure from [Petrosyan-Shahgholian-Uraltseva 2012].

3. Classification of blowup solutions. Two possibilities:



$w_0 = \frac{1}{2}(x_n)_+^2$ (regular point) $\qquad$ $w_0 = Ax \cdot x$ (singular point)

4. Directional monotonicity of blowups near regular points

$\implies$ Lipschitz / C^1 / analytic regularity of ∂D.

Requires $u_0^f|_{\partial D} > 0$. What happens in general?