Inverse Scattering for Semilinear Wave Equations

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I thank the organizers for the opportunity to give this talk which is based on joint work with Gunther Uhlmann and Yiran Wang.

I will concentrate on scattering for the following question:

Consider the equation,

$$\left(\partial_t^2 - \Delta + h(|u|^2)\right) u = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^3,$$
$$u(0,x) = \varphi(x), \quad \partial_t u(0,x) = \psi(x),$$

where $h : [0,\infty) \longmapsto \mathbb{R}$ and $h(s) \sim s^2$, as $s \sim 0$ and $s \sim \infty$, $h \in C^\infty$ and $\int_0^u h(|s|^2)s \, ds \geq 0$. Can one recover h from scattering data?

The reason for looking at this case first is that there is a well defined scattering operator for fine energy initial data. This is due to Shatah and Struwe (1993,1994), Bahouri and Shatah (1998), Bahouri and Gérard (1999). This is called the energy critical defocusing case in 3-d. I will comment on other cases later on.

Global Finite Energy Solutions and Scattering:

Theorem (Shatah and Struwe complemented by Bahouri and Gérard)

For any initial data $(\varphi, \psi) \in \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, *there exists a unique* $u \in X(\mathbb{R};\mathbb{R}^3)$ *satisfying the semilinear wave equation,*

$$X(\mathbb{R};\mathbb{R}^3) = C^0(\mathbb{R};\dot{H}^1(\mathbb{R}^3)) \cap C^1(\mathbb{R};L^2(\mathbb{R}^3)) \cap L^5(\mathbb{R};L^{10}(\mathbb{R}^3)), \text{ and}$$

$$\dot{H}^1(\mathbb{R}^3) = \{v: \ ||v||^2_{\dot{H}^1} = \int_{\mathbb{R}^3} |\nabla_z v|^2 \, dx < \infty\}, \text{ and}$$

$$L^p(\mathbb{R}_t; L^q(\mathbb{R}^3)) = \{u(t,x) : ||u||^p_{L^p;L^q} = \int_{\mathbb{R}} \left(\int_{\mathbb{R}^3} |u(t,x)|^q \, dx \right)^{\frac{p}{q}} dt < \infty \ \}.$$

This defines a group $U(t)(\varphi, \psi) = (u(t,x), \partial_t u(t,x))$.

The Asymptotic Completeness of the Shatah-Struwe Solutions:

Bahouri and Gérard also proved that the maps

$$\Omega_\pm(\varphi,\psi) = \lim_{t\to\pm\infty} U(-t)U_0(t)(\varphi,\psi),$$

where U_0 and U are the free and semilinear (Shatah-Struwe) wave groups, are isometries

$$\Omega_\pm : \dot{H}^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3) \longrightarrow \dot{H}^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3) \tag{0.1}$$

The key to this is a result of Bahouri and Shatah (1998): If $u \in X(\mathbb{R},\mathbb{R}^3)$, then

$$\lim_{t\to\infty}\int_{\mathbb{R}^3} G(u(t,x))\,dx = 0, \quad G'(u) = h(|u|^2)u.$$

The scattering operator

As usual, the nonlinear Møller semilinear scattering operator is then defined as the map

$$\mathcal{M}: \dot{H}^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3) \longrightarrow \dot{H}^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$$
$$\mathcal{M} = \Omega_+ \circ \Omega_-^{-1}.$$

The Main Theorem, version 1

Theorem

(Joint work with Uhlmann and Wang, 2020) Let $f_j(u)h_j(|u|^2)u$, $j = 1, 2$ *be as stated above. Moreover suppose that the sets* $\{u \in \mathbb{R} : f_j^{(4)}(u) = 0\}$, $j = 1, 2$ *are countable. Let* $\mathcal{M}_j$ *be the nonlinear Møller scattering operator associated with* f_j*. If* $\mathcal{M}_1 = \mathcal{M}_2$*, then* $f_1(u) = f_2(u)$ *for all* $u \in \mathbb{R}$*.*

The scattering operator gives a huge amount of information and one wants to determine a function of one variable. But still, this is not obvious.

Previously known results were for real analytic non-linearities. Examples:

1. C.S. Morawetz and W.A. Strauss. *On a nonlinear scattering operator.* Comm. Pure Appl. Math. 26 (1973), 47–54. **For the Klein-Gordon equation.**
2. A. Bachelot. *Inverse scattering problem for the nonlinear* **Klein-Gordon equation***.* Contributions to nonlinear partial differential equations (Madrid, 1981), 7–15, Res. Notes in Math., 89, Pitman, Boston, MA, 1983.
3. Weder, R. Multidimensional inverse scattering for the nonlinear **Klein-Gordon equation** with a potential. J. Differential Equations 184 (2002), no. 1, 62–77.
4. R. Carles and I. Gallagher *Analyticity of the scattering operator for semilinear dispersive equations.* Comm. Math. Phys. 286 (2009), no. 3, 1181–1209. **For the wave equation, n=3, n=4.**

The main point in those papers is the following: If f is real analytic, the Møller scattering operator is real analytic in the following sense:

If $w_0, w \in \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, $\varepsilon > 0$ is small enough, and $\mathcal{M}$ is the scattering operator

$$\mathcal{M}(w_0 + \varepsilon w) = \mathcal{M}(w_0) + \sum_{j=1}^{\infty} \varepsilon^j \theta_j, \quad \theta_j \in \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3),$$

where the series converges strongly, and from these coefficients one can determine the Taylor series of $h(|u|^2)u$ at $u = 0$.

More recent results, which came after our paper

Killip, Murphy and Visan [ArXiv, 2022] proved a similar result for the Schrödinger equation, but their methods should apply to this and more general nonlinearities.

The main idea of the proof of the Theorem

We use linearization and propagation of singularities (after Kurylev, Lassas, Uhlmann) to prove the following:

Let $f_j(u) = h_j(|u|^2)u$ $j = 1, 2$ be as above and let u_j be the solution to

$$\begin{gathered} \Box u_j + f_j(u_j) = 0, \text{ on } \mathbb{R} \times \mathbb{R}^3, \\ \text{with backward radiation field, (which will be defined shortly)} \\ \mathcal{N}_- u_j = \Upsilon(s) \in C_0^\infty(\mathbb{R}) \text{ is radial} \end{gathered} \tag{0.2}$$

If u_j, $j = 2, 1$ have the same forward radiation field then the third and fourth derivatives of f_1 and f_2 satisfy

$$f_1^{(3)}(u_1(t,x)) = f_2^{(3)}(u_2(t,x)) \text{ and } f_1^{(4)}(u_1(t,x)) = f_2^{(4)}(u_2(t,x)), \tag{0.3}$$

for all $(t, x) \in \mathbb{R} \times \mathbb{R}^3$.

The main idea of the Proof of the Theorem

Once one has proved this result, then in particular, for $t = 0$, $u_j(0, s) = \varphi_j(s)$, $j = 1, 2$,

$$f_1^{(3)}(\varphi_1(s)) = f_2^{(3)}(\varphi_2(s)), \text{ and } f_1^{(4)}(\varphi_1(s)) = f_2^{(4)}(\varphi_2(s)), \text{ for all } s \in \mathbb{R}_+.$$

By differentiating the first equation with respect to s we obtain

$$f_1^{(4)}(\varphi_1(s))\varphi_1'(s) = f_2^{(4)}(\varphi_2(s))\varphi_2'(s), \text{ for all } s \in \mathbb{R}, \tag{0.4}$$

and we conclude that

$$\varphi_1'(s) = \varphi_2'(s) \text{ for all } s \text{ such that } f_1^{(4)}(\varphi_1(s)) = f_2^{(4)}(\varphi_2(s)) \neq 0.$$

$$\mathcal{L}_1 = \{r \in \mathbb{R} : f_1^{(4)}(r) = 0\}, \text{ and } \mathcal{L}_2 = \{r \in \mathbb{R} : f_2^{(4)}(r) = 0\},$$

and so we conclude that

$$\varphi_1'(s) = \varphi_2'(s) \text{ for all } s \in \mathcal{O} = \{s \in \mathbb{R} : \varphi_1(s) \notin \mathcal{L}_1\} \cap \{s \in \mathbb{R} : \varphi_2(s) \notin \mathcal{L}_2\}.$$

The main idea of the Proof of the Theorem

The set $\mathcal{O} \subset \mathbb{R}$ is open, and so

$$\mathcal{O} = \bigcup_{j=1}^{\infty} I_j, \;\; I_j = (a_j, b_j), \quad I_j \cap I_k = \emptyset \text{ if } j \neq k,$$

and therefore

$$\varphi_1(s) - \varphi_2(s) = c_j \text{ on } I_j, \quad j = 1, 2, 3, \ldots$$

Since by assumption $\mathcal{L}_1$ and $\mathcal{L}_2$ are countable, we conclude that

$$D(s) = \varphi_1(s) - \varphi_2(s)$$

is a C^∞ function which has a countable range. Therefore it has to be a constant. But $D(s) = 0$ for $|s|$ large, so $D(s) = 0$ and we conclude that $\varphi_1 = \varphi_2 = \varphi$ and $f_1^{(3)}(\varphi(s)) = f_2^{(3)}(\varphi(s))$. Therefore

$$\varphi_1(s) = \varphi_2(s).$$

But then $f_1^{(3)}(\varphi(s)) = f_2^{(3)}(\varphi(s))$ for all $\varphi \in C_0^\infty(\mathbb{R})$ and so $f_1^{(3)}(s) = f_2^{(3)}(s)$ but then $f_1(s) = f_2(s)$.

Remark:

Since we recover $f(\varphi(s))$ pointwise, we can reach the same conclusion if we assume that $\varphi(|x|)$ has small $\dot{H}^1(\mathbb{R}^3)$ norm. In fact if $\varphi(0) = 1$ and $\varphi_m(x) = m^{-\frac{1}{2}}\varphi(\frac{x}{m})$, (the dimension is 3) with $m > 0$, then $||\nabla_x \varphi_m(x)||_{L^2} = ||\nabla_x \varphi(x)||_{L^2}$.

Since scattering for small energy initial data holds for a larger class of semilinear equations of the form

$$(\Box + h(|u|^2))u = 0,$$

one expects that these methods should apply there as well. In particular to the focusing case, where $\int_0^u h(|s|^2)s\, ds < 0$, and we do not have a conserved positive energy. However, since we use propagation of singularities, we need h to be C^∞.

Our Plan

1. We prove an analogue of the expansion

$$\mathcal{M}(w_0 + \varepsilon w) = \mathcal{M}(w_0) + \sum_{j=1}^{\infty} \varepsilon^j \theta_j, \quad \theta_j \in \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3),$$

to finite order only when $h(|u|^2)$ is not real analytic.

2. We then apply the methods of Kurylev, Lassas and Uhlmann and Lassas, Uhlmann and Wang and use the singularities of some of the terms of the expansion to obtain information about f.

However, the Møller operator is not quite suitable for tracking the propagation of singularities.

The semilinear scattering operator in terms of Friedlander radiation fields.

In the case of the linear wave equation (i.e. $f(u) = 0$,) the forward and backward radiation fields for the wave equation with a forcing term $F(t,x) \in C_0^\infty(\mathbb{R} \times \mathbb{R}^3)$,

$$\begin{gathered}\left(\partial_t^2 - \Delta\right) v(t,x) = F(t,x), \\ v(0,x) = \varphi(x), \;\; \partial_t v(0,x) = \psi(x), \;\;\; \varphi, \psi \in C_0^\infty(\mathbb{R}^3),\end{gathered}$$

the forward and backward radiation fields are defined to be respectively

$$\begin{gathered}\mathcal{R}_+(\varphi,\psi,F)(s,\theta) = \lim_{r\to\infty} r(\partial_t v)(s+r, r\theta), \\ \mathcal{R}-(\varphi,\psi,F)(s,\theta) = \lim_{r\to\infty} r(\partial_t v)(s-r, r\theta),\end{gathered}$$

where $r = |x|$ and $\theta = \frac{x}{|x|}$.

Why the Friedlander radiation fields are suitable for studying propagation of singularities:

Written in the variables (s, r, ω), $s = t - r$, as $r \to \infty$, the operator $P = r\Box r^{-1}$ becomes

$$P = \partial_r(2\partial_s - \partial_r) - \frac{1}{r^2}\Delta_\omega,$$

where Δ_ω is the Laplacian on $\mathbb{S}^2$. If one sets $x = r^{-1}$, then

$$-x^{-2}P = \mathcal{P} = \partial_x(2\partial_s + x^2\partial_x) + \Delta_\omega.$$

This is a compactification of $\mathbb{R}^n \setminus 0$, which has a natural extension to a neighborhood of $\{x = 0\}$.
The Hamilton vector field of $\varrho = -\sigma_2(\mathcal{P}) = 2\mu\xi + x^2\xi^2 + H(\omega, \varkappa)$, where ξ is the dual variable to x, μ is the dual to s, and $\varkappa$ is the dual to ω, is given by

$$H_\varrho = 2(\mu + x^2\xi)\partial_x + 2\xi\partial_s + H_h.$$

Why the Friedlander radiation fields are suitable for studying propagation of singularities:

On the characteristic variety of $\Box$, and away from the zero section of $T^*\mathbb{R}^4$, $\tau^2 \neq 0$, $\tau = \mu$, and so H_ϱ is transversal to $\{x = 0\}$. So singularities that travel along the integral curves of H_ϱ will hit $\{x = 0\}$ transversally and can be observed.

The Semilinear Friedlander Radiation Fields

The Friedlander radiation fields can also be defined for the semilinear wave equation. This was shown by Grillakis (1990) for initial data $\varphi, \psi \in C_0^\infty$, but one can show that the maps

$$\begin{aligned} \mathcal{L}_+(\varphi,\psi)(s,\theta) &= \lim_{r\to\infty} r(\partial_t u)(s+r, r\theta), \\ \mathcal{L}_-(\varphi,\psi)(s,\theta) &= \lim_{r\to\infty} r(\partial_t u)(s-r, r\theta), \end{aligned} \tag{0.5}$$

where $u(t,x)$ is the Shatah-Struwe solution of the semilinear wave equation with initial data $(\varphi,\psi) \in C_0^\infty(\mathbb{R}^3) \times C_0^\infty(\mathbb{R}^3)$, extend to nonlinear isomorphisms

$$\begin{gathered} \mathcal{L}_\pm : \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \to L^2(\mathbb{R} \times \mathbb{S}^2), \\ \mathcal{L}_\pm(\phi,\psi) = \mathcal{R}_\pm(\phi,\psi,-h(|u|^2)u), \end{gathered}$$

but now with the semilinear energy norm

$$E(\varphi,\psi) = \frac{1}{2}\int_{\mathbb{R}^3} \left[(\partial_t u(t,x))^2 + |\nabla_x u(t,x)|^2 + G(u(t))\right] \, dx, \;\; G'(u) = h(|u|^2)u.$$

The Semilinear Friedlander Radiation Fields

Therefore the Friedlander semilinear scattering operator

$$\mathcal{A} = \mathcal{L}_+ \circ \mathcal{L}_-^{-1} : L^2(\mathbb{R} \times \mathbb{S}^2) \longmapsto L^2(\mathbb{R} \times \mathbb{S}^2) \tag{0.6}$$

is an isometry and it is not hard to see that Friedlander and the Møller scattering operators are related by

$$\mathcal{A} = \mathcal{R}_{0+} \circ \mathcal{M} \circ \mathcal{R}_{0-}^{-1},$$

where $\mathcal{R}_{0,\pm} = \mathcal{R}_\pm(\varphi, \psi, 0)$.

The Main Theorem, version 2.

Theorem- Joint work with Uhlmann and Wang

Let $f_j(u) = h_j(|u|^2)u$, $j = 1, 2$, be as above and suppose that the sets $\{u : f_j^{(4)}(u) = 0\}$ are countable, $j = 1, 2$. Let $\mathcal{A}_j$ be the Friedlander scattering operator associated with f_j. If $\mathcal{A}_1 = \mathcal{A}_2$, then $f_1(u) = f_2(u)$ for all $u \in \mathbb{R}$.

The Linearization

We establish the following linearization in terms of the Friedlander scattering operator

Theorem- Joint work with Uhlmann and Wang

Suppose that f is as above, and let $\Upsilon_j \in L^2(\mathbb{R} \times \mathbb{S}^2)$, $j = 0, 1, 2, 3, 4$, where and $\varepsilon_j > 0$, $j = 1, 2, 3, 4$, are is small enough, and let $\mathcal{A}$ is the Friedlander scattering operator, then

$$\mathcal{A}(\Upsilon_0 + \sum_{j=1}^{4} \varepsilon_j \Upsilon_j) = \mathcal{A}(\Upsilon_0) + \sum_{|\alpha| \leq 4} (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)^\alpha \, \Theta_\alpha +$$

$$O(|(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)|^5) \quad \text{where } \Theta_\alpha \in L^2(\mathbb{R} \times \mathbb{S}^3).$$

The choice of Υ_0

We choose $\Upsilon_0(s,\omega) = \Upsilon_0(s) \in C_0^\infty(\mathbb{R})$ to be radial.

The choice of Υ_j, $j = 1, 2, 3, 4$.

Planes waves are not very convenient when dealing with radiation fields, so we pick Υ_j to be linear radiation fields of spherical waves. Let $(s_j, z_j) \in \mathbb{R}^4$, and Υ_j, $j = 1, 2, 3, 4$ of the form

$$\Upsilon_j(s,\omega) = \partial_s \left[(s - s_j - \langle \omega, z_j \rangle)_+^m \chi(s - s_j - \langle \omega, z_j \rangle) \right], \; j = 1, 2, 3, 4, \quad m > 0,$$
$$\chi \in C_0^\infty(\mathbb{R}) \text{ supported in } [-1, 1],$$

Indeed, notice that

$$\zeta_j^-(t,x) = |x - z_j|^{-1}(t - s_j + |x - z_j|)_+^m \chi(t - s_j + |x - z_j|),$$

for $j = 1, 2, 3, 4$, satisfy $\Box \zeta_j^- = 0$.

Also notice that if $x = r\omega$,

$$|x - z_j| = \left(r^2 - 2r\langle\omega, z_j\rangle + |z_j|^2\right)^{\frac{1}{2}} = r + O(1),$$

$$t - s_j + |x - z_j| = t - s_j + \left(r^2 + |z_j|^2 - 2r\langle\omega, z_j\rangle\right)^{\frac{1}{2}} =$$

$$t - s_j + r\left(1 - \frac{2}{r}\langle\omega, z_j\rangle + \frac{|z_j|^2}{r^2}\right)^{\frac{1}{2}} =$$

$$t - s_j + r - \langle\omega, z_j\rangle + O(r^{-1}),$$

and therefore, for $j = 1, 2, 3$,

$$\lim_{r\to\infty} r\partial_s\zeta_j(s - r, r\omega) = \partial_s\left((s - s_j - \langle\omega, z_j\rangle)_+^m \chi(s - s_j - \langle\omega, z_j\rangle)\right) = \Upsilon_j(s, \omega).$$

The linearization

Let u satisfy $\Box u = -f(u)$ and we write $u = u_0 + (u - u_0)$, and so

$$\Box u = \Box u_0 + \Box(u - u_0) = -f(u_0 + (u - u_0)) = -f(u_0) - f'(u_0)(u - u_0) - \frac{1}{2!} f^{(2)}(u_0)(u - u_0)^2 - \frac{1}{3!} f^{(3)}(u_0)(u - u_0)^3 - \frac{1}{4!} f^{(4)}(u_0)(u - u_0)^4 - G(u, u_0)(u - u_0)^5,$$

$$\text{where } G(u, u_0) = \frac{1}{4!} \int_0^1 f^{(5)}((1 - t)u_0 + tu)(1 - t)^4 dt.$$

Next we write $y = (t, x)$ and $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$, and so

$$u - u_0 = w_{\vec{\varepsilon}} + \mathcal{Z}(y, \vec{\varepsilon}), \text{ where } w_{\vec{\varepsilon}} = \sum_{|\alpha| \leq 4} \vec{\varepsilon}^{\,\alpha} w_\alpha, \tag{0.7}$$

We substitute this expression into the equation, match powers of $\vec{\varepsilon}$ up to order four, and obtain global bounds for the remainder.

The terms of the linearization satisfy a number of linear equations

The terms of order one

The terms in $\vec{\varepsilon}^{\alpha}$ with $|\alpha| = 1$ are denoted by

$$w_1 = w_{1,0,0,0}, \quad w_2 = w_{0,1,0,0}, \quad w_3 = w_{0,0,1,0} \text{ and } w_4 = w_{0,0,0,1},$$

and satisfy

$$\begin{gathered} \Box w_j = -f'(u_0) w_j, \\ \text{with backward radiation field} \\ \mathcal{N}_- w_j = \Upsilon_j, \quad j = 1, 2, 3, 4. \end{gathered} \tag{0.8}$$

The terms of order two

To compute the terms in $\vec{\varepsilon}^{\alpha}$ with $|\alpha| = 2$, we write $\alpha = \beta_1 + \beta_2$, with $|\beta_1| = |\beta_2| = 1$, and we have

$$\Box w_\alpha = -f'(u_0) w_\alpha - \frac{1}{2!} f^{(2)}(u_0) \sum_{\alpha=\beta_1+\beta_2, |\beta_1|=|\beta_2|=1} w_{\beta_1} w_{\beta_2}, \tag{0.9}$$

with backward radiation field $\mathcal{N}_- w_\alpha = 0$.

The terms of order three

To compute the terms of order three, we write

$$\alpha = \beta_1 + \beta_2, \ \ |\beta_1| = 1, \ |\beta_2| = 2 \text{ and } \alpha = \gamma_1 + \gamma_2 + \gamma_3, \ |\gamma_j| = 1, \\ j = 1, 2, 3.$$

and we find

$$\Box w_\alpha = -f'(u_0) w_\alpha - \frac{1}{2!} f^{(2)}(u_0) \sum_{\alpha=\beta_1+\beta_2, |\beta_1|=1, |\beta_2=2} w_{\beta_1} w_{\beta_2} - \\ \frac{1}{3!} f^{(3)}(u_0) \sum_{\alpha=\gamma_1+\gamma_2+\gamma_3, |\gamma_j|=1} w_{\gamma_1} w_{\gamma_2} w_{\gamma_3}, \tag{0.10}$$

with backward radiation field $\mathcal{N}_- w_\alpha = 0$.

The top singularity of the term $w_{1,1,1}$ comes from the term

$$\frac{1}{3!}f^{(3)}(u_0)w_{1,0,0}w_{0,1,0}w_{0,0,1}.$$

The waves w_{γ_j}, $|\gamma_j| = 1$, are elliptic conormal distributions with respect to the cones $\{t - s_j + |x - z_j| = 0\}$, $j = 1, 2, 3$. These cones intersect at a codimension one submanifold Γ.

We track the singularities of $\Box^{-1}\left[f^{(3)}(u_0)w_{1,0,0}w_{0,1,0}w_{0,0,1}\right]$ which will emanate from Γ. We read them from the radiation filed and we show that this determines $f^{(3)}(u_0(t,x))$ for every $(t,x) \in \Gamma$. By varying s_j and z_j we can show that this determines $f^{(3)}(u_0(t,x))$ for every $(t,x) \in \mathbb{R}^4$.

We repeat the argument with four waves and show that one can determine $f^{(4)}(u_0(t,x))$ for every $(t,x) \in \mathbb{R}^4$.

THANK YOU!