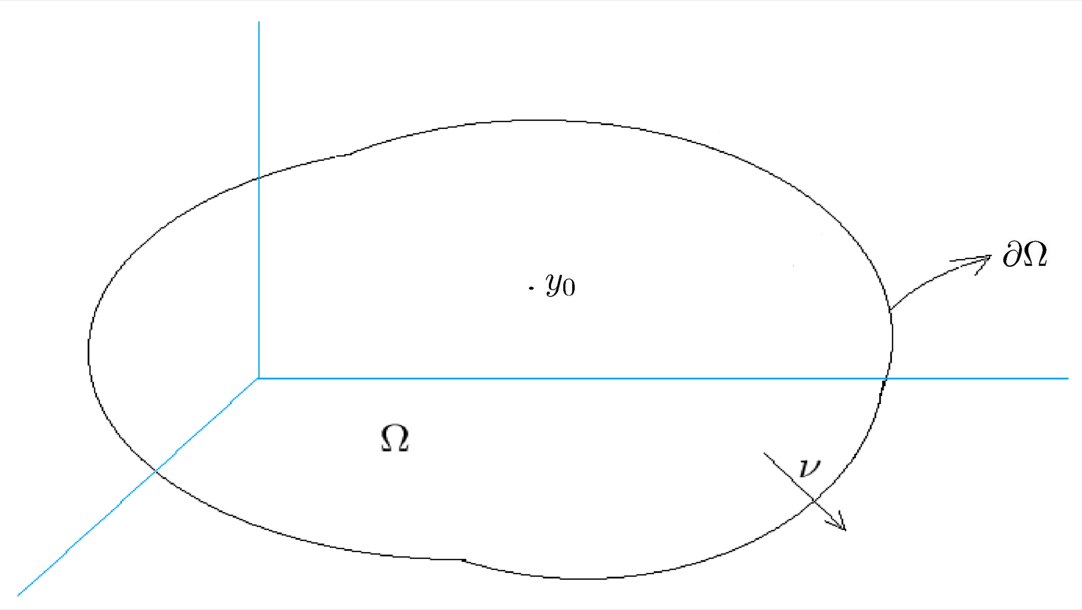
On the origin of Minnaert resonances

Andrea Mantile

a joint work with A. Posilicano and M. Sini

Acoustic resonant frequencies generated by a micro-bubble

$\Omega \subset \mathbb{R}^3 \quad s.t. \quad \partial\Omega \in \mathcal{C}^2$



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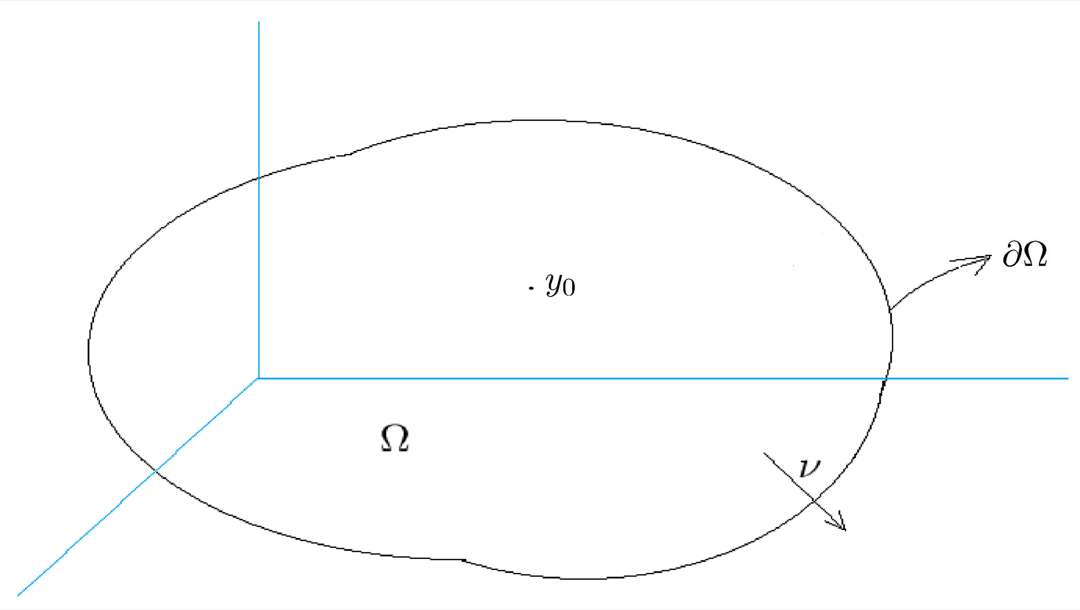
$$x(y) := y_0 + \varepsilon\,(y - y_0)$$



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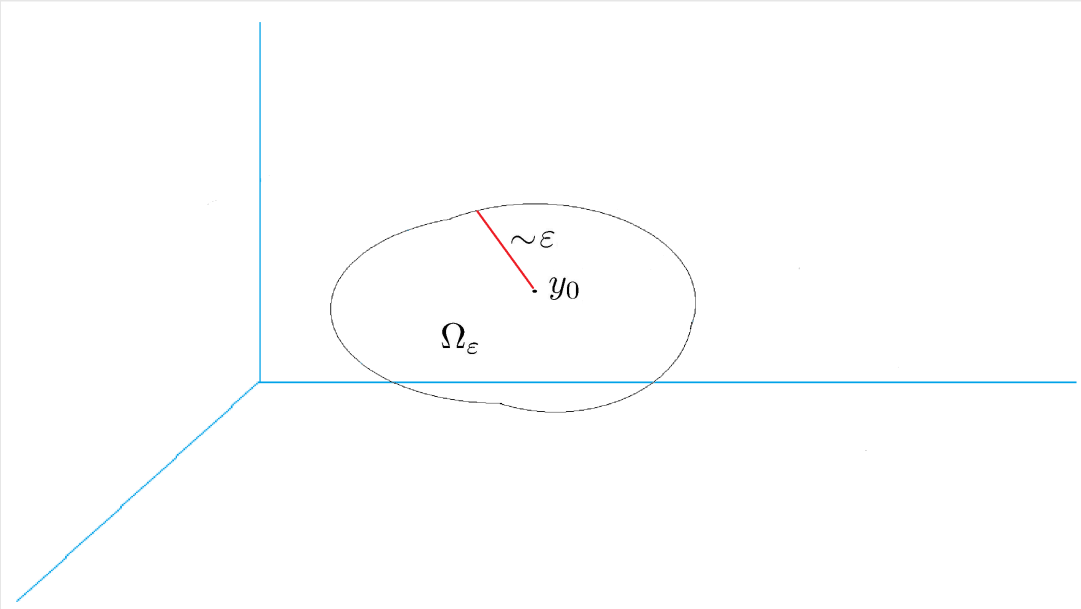

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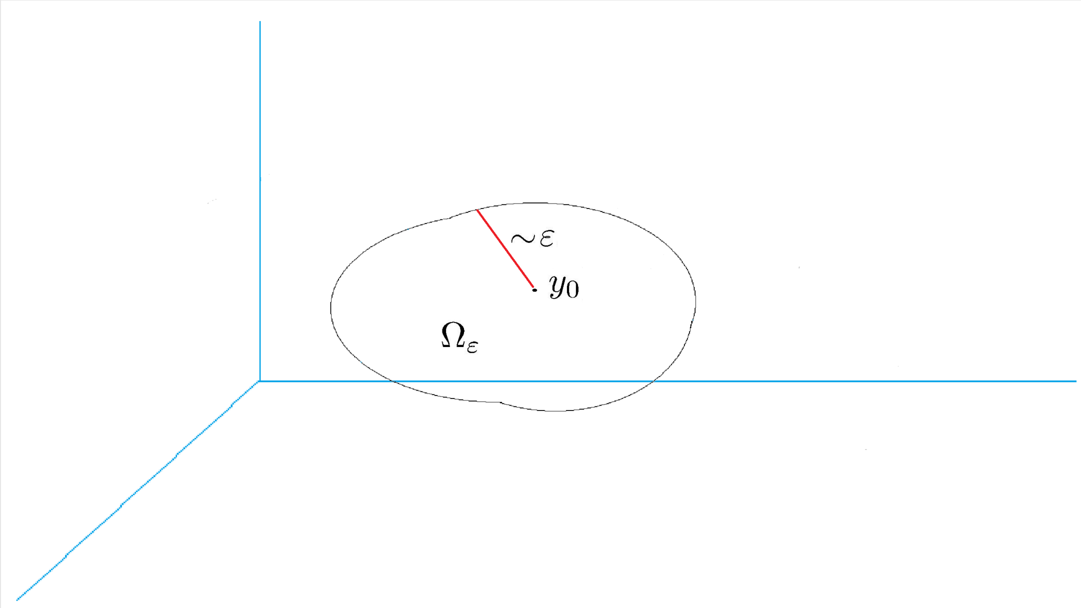
$1/\rho =$ *acoustic density*

$1/\kappa =$ *acoustic bulk*



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$$\rho = \rho_\varepsilon \mathbf{1}_{\Omega_\varepsilon} + \rho_0 \mathbf{1}_{\mathbb{R}^3 \setminus \Omega_\varepsilon}$$

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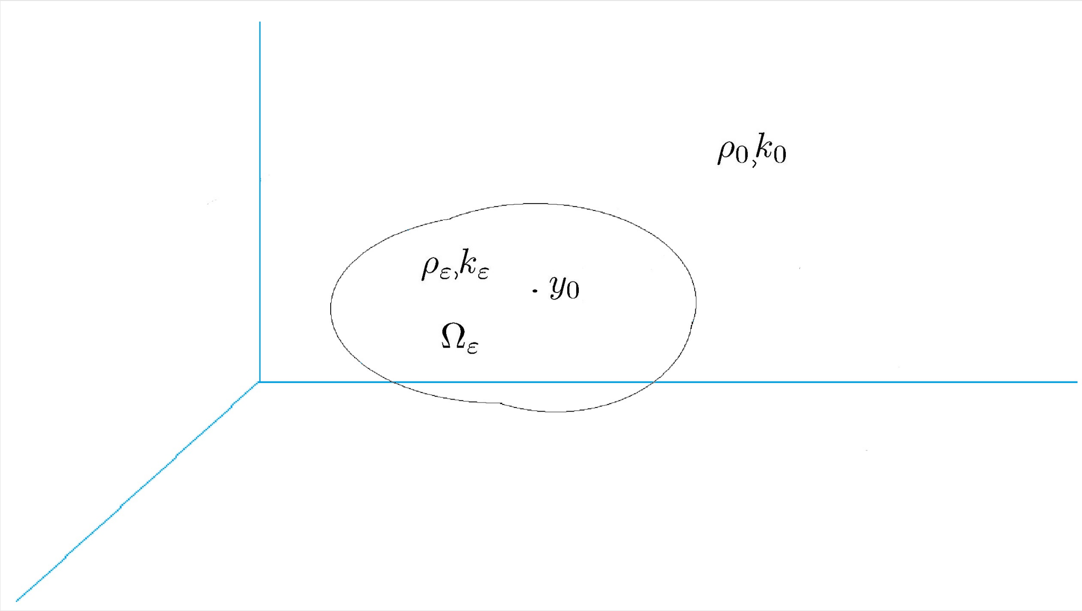

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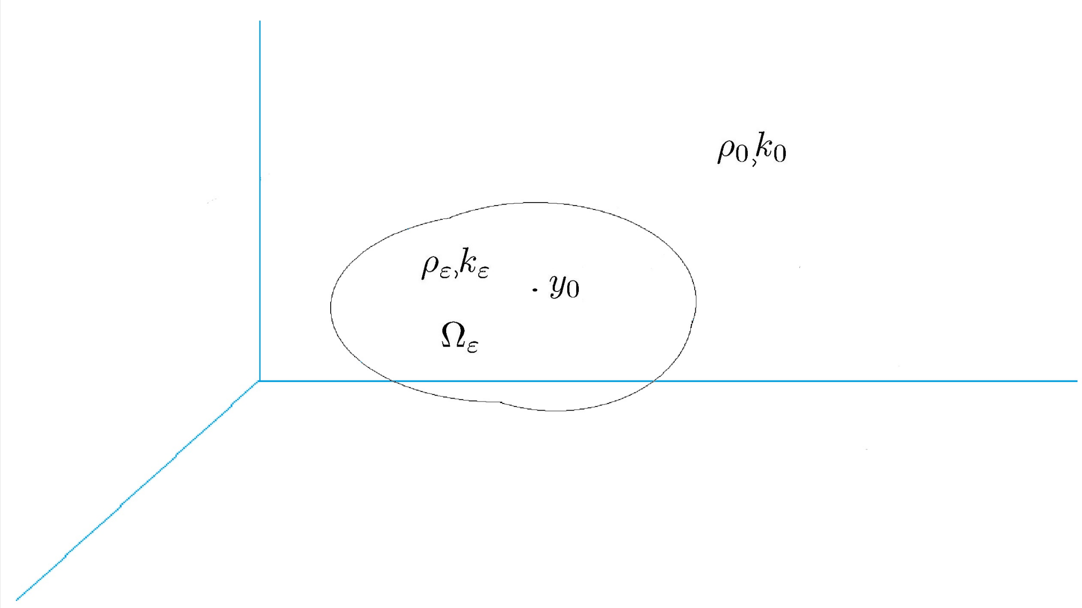

$$\begin{cases} \left(\nabla \cdot \frac{1}{\rho}\nabla + \frac{1}{\kappa}\omega^2\right) u = 0 & \text{in } \mathbb{R}^3\setminus\partial\Omega_\varepsilon\,, \\ u|_{in} = u|_{ex}\,, \quad \frac{1}{\rho_\varepsilon}\nu\cdot\nabla u\Big|_{in} = \frac{1}{\rho_0}\nu\cdot\nabla u\Big|_{ex}\,, & \text{on } \partial\Omega_\varepsilon\,, \\ u = u^{sc} + u^{in}\,, \quad \left(\Delta + \frac{\rho_0}{\kappa_0}\omega^2\right) u^{in} = 0\,, & \text{in } \mathbb{R}^3\,, \\ \frac{\partial u^{sc}}{\partial|x|} - i\omega\sqrt{\frac{\rho_0}{k_0}}\, u^{sc} = o\left(\frac{1}{|x|}\right)\,, & \text{as } |x| \to +\infty\,, \end{cases}$$

Acoustic resonant frequencies generated by a micro-bubble

$\Omega \subset \mathbb{R}^3 \quad s.t. \quad \partial\Omega \in \mathcal{C}^2$

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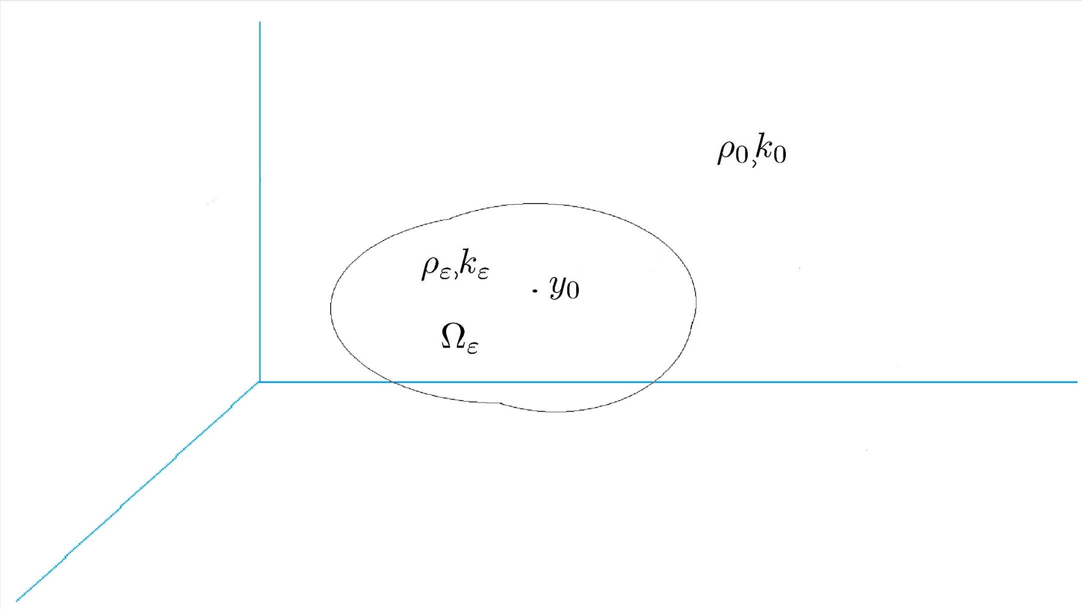

...*denoting with* R_ω *the Green's function of* $\left(-\Delta - \omega^2\right)^{-1}$ *and*

$$\alpha_\varepsilon := \frac{1}{\rho_\varepsilon} - \frac{1}{\rho_0}, \qquad \beta_\varepsilon := \frac{1}{\kappa_\varepsilon} - \frac{1}{\kappa_0}, \qquad \text{contrast coefficients}$$

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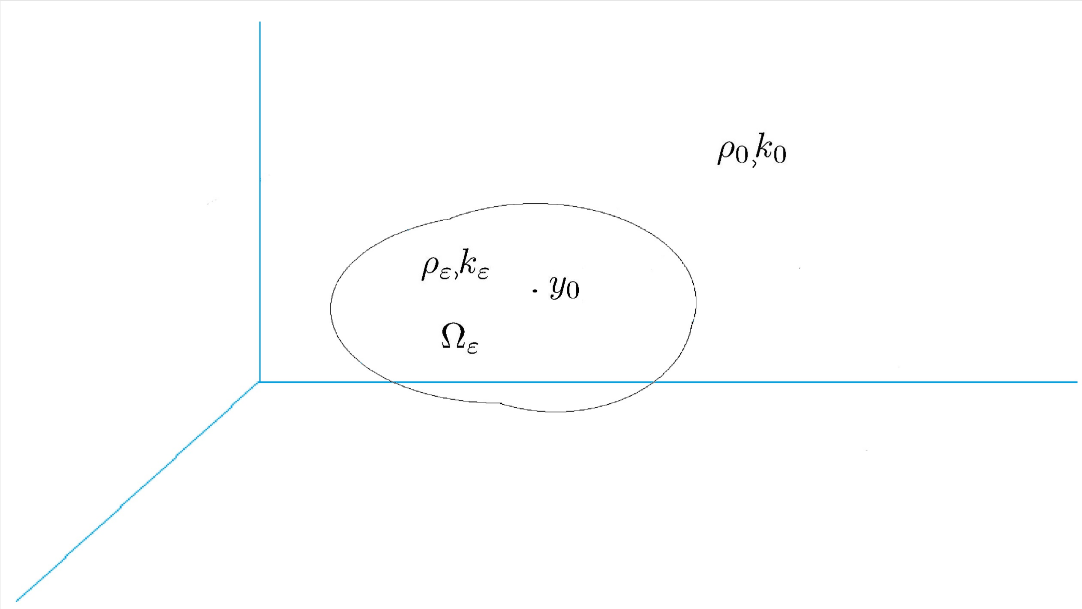

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$$\alpha_\varepsilon := \frac{1}{\rho_\varepsilon} - \frac{1}{\rho_0}, \qquad \beta_\varepsilon := \frac{1}{\kappa_\varepsilon} - \frac{1}{\kappa_0}, \qquad \textit{contrast coefficients}$$

the Lippmann-Schwinger representation and integration by parts yield

$$u(x) - \left(\beta_\varepsilon - \alpha_\varepsilon \frac{\rho_\varepsilon}{\kappa_\varepsilon}\right)\omega^2 \int_{\Omega_\varepsilon} R_{\sqrt{\frac{\rho_0}{\kappa_0}}\omega}(x-y)u(y)dy + \alpha_\varepsilon \int_{\partial\Omega_\varepsilon} R_{\sqrt{\frac{\rho_0}{\kappa_0}}\omega}(x-y)\frac{\partial u}{\partial \nu}(y)dy = u^{\mathrm{in}}(x)$$

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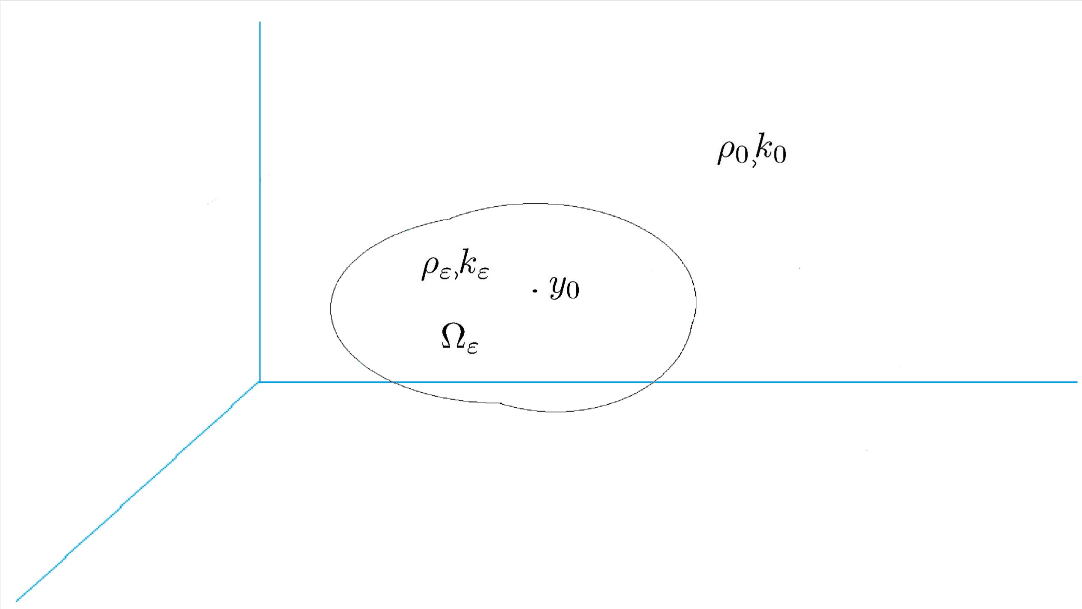

$\Rightarrow$ *total acoustic field in* $R^3 \setminus \overline{\Omega_\varepsilon}$ *is fully computable from:* $u \restriction \Omega_\varepsilon$ *and* $\partial_\nu u|_{in}$

Acoustic resonant frequencies generated by a micro-bubble

...setting

$$N_\omega(\varepsilon)\, u := \int_{\Omega_\varepsilon} R_{\sqrt{\frac{\rho_0}{\kappa_0}}\omega}(\cdot - y) u(y) dy \,,$$ *Newton potential operator*

$$SL_\omega(\varepsilon)\, \varphi := \int_{\partial\Omega_\varepsilon} R_{\sqrt{\frac{\rho_0}{\kappa_0}}\omega}(\cdot - y) \varphi(y) dy \,,$$ *Single layer potential operator*

$$K_\omega(\varepsilon)\, \varphi := \int_{\partial\Omega_\varepsilon} \frac{\partial}{\partial \nu_y} R_{\sqrt{\frac{\rho_0}{\kappa_0}}\omega}(\cdot - y) \varphi(y) dy \,,$$ *Neumann-Poincaré operator*

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$$SL_\omega(\varepsilon)\, \varphi := \int_{\partial\Omega_\varepsilon} R_{\sqrt{\frac{\rho_0}{\kappa_0}}\omega}(\cdot - y) \varphi(y) dy\,, \qquad SL_\omega(\varepsilon) \in \mathrm{B}\left(L^2(\Omega_\varepsilon)\right)$$

$$K_\omega(\varepsilon)\, \varphi := \int_{\partial\Omega_\varepsilon} \frac{\partial}{\partial \nu_y} R_{\sqrt{\frac{\rho_0}{\kappa_0}}\omega}(\cdot - y) \varphi(y) dy\,, \qquad K^*_\omega(\varepsilon) \in \mathrm{B}\left(H^{1/2}(\partial\Omega_\varepsilon)\right)$$

Acoustic resonant frequencies generated by a micro-bubble

our problem rephrases as

$$\left[I - \left(\beta_\varepsilon - \alpha_\varepsilon \frac{\rho_\varepsilon}{\kappa_\varepsilon}\right)\omega^2 N_\omega(\varepsilon)\right] u + \alpha_\varepsilon SL_\omega(\varepsilon)\, \partial_\nu u = u^{\mathsf{in}}(x)\,, \qquad \text{in } \Omega_\varepsilon\,,$$

$$-\frac{1}{\alpha_\varepsilon}\left(\beta_\varepsilon - \alpha_\varepsilon \frac{\rho_\varepsilon}{\kappa_\varepsilon}\right)\omega^2 \partial_\nu N_\omega(\varepsilon)\, u + \left[\frac{1}{\alpha_\varepsilon} + \frac{1}{2} + K^*_\omega(\varepsilon)\right] \partial_\nu u = \frac{1}{\alpha_\varepsilon}\partial_\nu u^{\mathsf{in}}\,, \qquad \text{on } \partial\Omega_\varepsilon\,.$$

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Denoting with: $N_\omega, SL_\omega, K_\omega$ *the operators related to* Ω*, results*

$$N_\omega(\varepsilon) \sim \varepsilon^2 N_0 \qquad SL_\omega(\varepsilon) \sim \varepsilon^2 SL_0$$

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...and, for a certain value *of the incident frequency...*

$$N_\omega(\varepsilon) \sim \varepsilon^2 N_0 \qquad SL_\omega(\varepsilon) \sim \varepsilon^2 SL_0 \qquad K^*_\omega(\varepsilon) \sim -1/2 + \varepsilon^2 \mathcal{O}(1)\,.$$

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<u>In this case</u>: the critical ratio condition

$$\frac{\rho_\varepsilon}{\rho_0} \sim \frac{\kappa_\varepsilon}{\kappa_0} \sim \varepsilon^2 \implies \begin{cases} I - \left(\beta_\varepsilon - \alpha_\varepsilon \frac{\rho_\varepsilon}{\kappa_\varepsilon}\right) \omega^2 N_\omega(\varepsilon) \sim I - \varepsilon^2\,, \\ \alpha_\varepsilon SL_\omega(\varepsilon) \sim SL_0\,, \\ \frac{1}{\alpha_\varepsilon} + \frac{1}{2} + K^*_\omega(\varepsilon) \sim \varepsilon^2\,, \end{cases}$$

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...implies an enhancement of u_{sc}.

Acoustic resonant frequencies generated by a micro-bubble

...following:

H. Ammari, B. Fitzpatrick, D. Gontier, H. Lee, H. Zhang. Minnaert resonances for acoustic waves in bubbly media. *Ann. I.H.P. Anal. Non Linéaire*, **35**, no. 7, 2018.

F Feppon, H Ammari. Modal decompositions and point scatterer approximations near the Minnaert resonance frequencies. *Studies in Applied Mathematics*, 2022.

Acoustic resonant frequencies generated by a micro-bubble

...the scattering problem recasts as

$$u(x) = \begin{cases} \left(\widetilde{SL}_\omega(\varepsilon)\ \phi\right)(x)\ ,\ \text{in } \Omega_\varepsilon\ , \\ \\ u^{\text{in}}(x) + \left(SL_\omega(\varepsilon)\ \psi\right)(x)\ ,\ \text{in } \mathbb{R}^3 \backslash \Omega_\varepsilon\ , \end{cases}$$

$$\widetilde{SL}_\omega(\varepsilon)\,\varphi := \int_{\partial\Omega_\varepsilon} R_{\sqrt{\frac{\rho_\varepsilon}{\kappa_\varepsilon}}\omega}(\cdot - y)\varphi(y)dy\,, \qquad \tilde{K}_\omega(\varepsilon)\,\varphi := \int_{\partial\Omega_\varepsilon} \frac{\partial}{\partial\nu_y} R_{\sqrt{\frac{\rho_\varepsilon}{\kappa_\varepsilon}}\omega}(\cdot - y)\varphi(y)dy\,,$$

$$A_\omega(\varepsilon)\begin{pmatrix}\phi \\ \psi\end{pmatrix} = \begin{pmatrix} u^{\text{in}} \\ \varepsilon\,\partial_\nu u^{\text{in}}\end{pmatrix}\,, \qquad A_\omega(\varepsilon) = \begin{pmatrix} \tilde{S}_\omega(\varepsilon) & S_\omega(\varepsilon) \\ -\frac{1}{2} + \tilde{K}_\omega(\varepsilon) & -\varepsilon\left(\frac{1}{2} + K_\omega(\varepsilon)\right)\end{pmatrix}\,.$$

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In this framework: ω is a resonance if $A_{\omega}(\varepsilon)$ is not inversible.

Acoustic resonant frequencies generated by a micro-bubble

A point scatterer *approximation* $(\varepsilon\,,\ |x|^{-1} << 1)$

$$(u^{\mathsf{sc}}_\omega)\,(x) = \left(SL_\omega\,(\varepsilon)\,\left(A_\omega^{-1}\,(\varepsilon)\begin{pmatrix} u^{\mathsf{in}} \\ \varepsilon\,\partial_\nu u^{\mathsf{in}}\end{pmatrix}\right)_2\right)(x)$$

$$\sim -\frac{\varepsilon\,\omega_M^2}{\omega^2 - \omega_M^2 + i\,\varepsilon\,\frac{\omega\ \omega_M^2\,C_\Omega}{4\pi}\sqrt{\frac{\kappa_0}{\rho_0}}}\ u^{\mathsf{in}}_\omega(y_0)\,R_{\sqrt{\frac{\rho_0}{\kappa_0}}\omega}(x-y_0) + \mathcal{O}\,(\varepsilon) + \mathcal{O}\left(|x|^{-1}\right)\ ,$$

$$\omega_M^2 := \frac{\rho_\varepsilon}{\kappa_\varepsilon}\frac{C_\Omega}{|\Omega|}\,, \qquad C_\Omega := \left\langle 1, S_0^{-1}1\right\rangle_{\partial\Omega}\,.$$

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Subwavelenght resonances have tremendous applications...

- *Imaging using contrast agents*
- *Material sciences (effective medium theory)*

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Subwavelenght resonances have tremendous applications...

A. Dabrowski, A. Ghandriche, M. Sini, Mathematical analysis of the acoustic imaging modality using bubbles as contrast agents at nearly resonating frequencies, 2021.

H. Ammari, D. P. Challa, A. P. Choudhury, and M. Sini. The equivalent media generated by bubbles of high contrasts: Volumetric metamaterials and metasurfaces. 2020.

An equivalent frequency-dependent Schrödinger operator

<u>The purpose of our work</u>:

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• *An equivalent formulation of the acoustic scattering problemy as a generalized eigenvalue problem for a frequency dependent Schrödinger operator.*

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An equivalent frequency-dependent Schrödinger operator

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- *An equivalent formulation of the acoustic scattering problemy as a generalized eigenvalue problem for a frequency dependent Schrödinger operator.*

- *Resolvent analysis in the small-scale limit.*

- *Uniform-in-space error estimates for the scattering expansion.*

An equivalent frequency-dependent Schrödinger operator

Notation: *referred to* Ω

$$\gamma_{in/ex} \quad u := u \restriction \partial\Omega\,, \quad \gamma'_{in/ex} \quad u := \gamma_{in/ex} \quad (\nu \cdot \nabla u)\ ,$$ *lateral traces on* $\partial\Omega$

$$\gamma \quad := \tfrac{1}{2}\left(\gamma_{in/ex} \quad + \gamma_{in/ex} \quad\right), \quad [\gamma \quad] := \gamma_{ex} - \gamma_{in}\,,$$

$$\gamma' \quad := \tfrac{1}{2}\left(\gamma'_{in/ex} \quad + \gamma'_{in/ex} \quad\right), \quad [\gamma' \quad] := \gamma'_{ex} - \gamma'_{in}\,,$$

global traces and jumps on $\partial\Omega$

$$R_k := \left(-\Delta - k^2\right)^{-1}\ , \qquad \mathrm{Im}\, k \geq 0\,,$$ *unperturbed resolvent*

$$SL_k \quad := R_k \gamma^* \quad , \qquad S_k \quad := \gamma \quad R_k \gamma^* \quad ,$$ *Single layer*

$$DL_k \quad := R_k\,(\gamma' \quad)^*, \quad K_k \quad := \gamma \quad R_k\,(\gamma' \quad)^*,$$ *Neumann-Poincaré*

$$DN_k \quad \varphi := \gamma'_{in} \quad u\,, \quad \begin{cases} \left(\Delta + k^2\right) u = 0 & \text{in } \Omega\ , \\ \gamma_{in} \quad u = \varphi & \text{on } \partial\Omega\ . \end{cases}$$ *Dirichlet-to-Neumann*

An equivalent frequency-dependent Schrödinger operator

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Neumann-Poincaré

$$DN_k(\varepsilon)\,\varphi := \gamma'_{in}(\varepsilon)\, u\,, \qquad \begin{cases} \left(\Delta + k^2\right) u = 0 & \text{in } \Omega_\varepsilon\,, \\ \gamma_{in}(\varepsilon)\, u = \varphi & \text{on } \partial\Omega_\varepsilon\,. \end{cases}$$

Dirichlet-to-Neumann

An equivalent frequency-dependent Schrödinger operator

A simplified setting

$$\rho = \kappa := \varepsilon^2 1_{\Omega_\varepsilon} + 1_{\mathbb{R}^3 \setminus \Omega_\varepsilon},$$

An equivalent frequency-dependent Schrödinger operator

A simplified setting

$$\rho = \kappa := \varepsilon^2 1_{\Omega_\varepsilon} + 1_{\mathbb{R}^3 \backslash \Omega_\varepsilon} \,,$$

$$\Longrightarrow \begin{cases} \left(\Delta + \omega^2\right) u = 0 \,, & \text{in } \mathbb{R}^3 \backslash \partial\Omega_\varepsilon \,, \\ \\ [\gamma\,(\varepsilon)]\, u = 0 \,, \quad \left(\gamma'_{ex}\,(\varepsilon) - \varepsilon^{-2} \gamma'_{in}\,(\varepsilon)\right) u = 0 \,, & \text{on } \partial\Omega_\varepsilon \,, \end{cases}$$

An equivalent frequency-dependent Schrödinger operator

A simplified setting

$$\rho = \kappa := \varepsilon^2 1_{\Omega_\varepsilon} + 1_{\mathbb{R}^3 \backslash \Omega_\varepsilon},$$

$$\Longrightarrow \begin{cases} \left(\Delta + \omega^2\right) u = 0\,, & \text{in } \mathbb{R}^3 \backslash \partial\Omega_\varepsilon\,, \\ \left[\gamma\left(\varepsilon\right)\right] u = 0\,, \quad \left[\gamma'\left(\varepsilon\right)\right] u = \left(\varepsilon^{-2} - 1\right) \gamma'_{in}\left(\varepsilon\right) u\,, & \text{on } \partial\Omega_\varepsilon\,, \end{cases}$$

An equivalent frequency-dependent Schrödinger operator

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Using

$$\gamma'_{in}(\varepsilon)\, u = DN_\omega(\varepsilon)\, \gamma(\varepsilon)\, u \,,$$

we get

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An equivalent frequency-dependent Schrödinger operator

A common approach to the scale limit problem: unitary dilations

$$(U_\varepsilon u)(x) := \varepsilon^{-3/2} u\left(y_0 + \tfrac{x-y_0}{\varepsilon}\right), \qquad \left(U_\varepsilon^{-1} u\right)(y) = \varepsilon^{3/2} u\left(y_0 + \varepsilon\left(y - y_0\right)\right),$$

S. Albeverio, F. Gesztesy, R. Høegh-Krohn. The low energy expansion in non-relativistic scattering theory. *Ann.IHP Sec. A*, **37** n$^\circ$1, 1-28, 1982.

S. Shimada. Resolvent convergence of sphere interactions to point interactions. *J. Math. Phys*. **44**, 990-1005, 2003.

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This suggests to define the equivalent frequency-dependent Schrödinger operator

$$\operatorname{dom}\left(H_\omega(\varepsilon)\right) := \left\{u \in L^2_\Delta\left(\mathbb{R}^3\backslash\partial\Omega\right) \cap H^1\left(\mathbb{R}^3\right) : [\gamma]\,u = 0\,,\ [\gamma']\,u = \left(\varepsilon^{-2} - 1\right) DN_{\varepsilon\omega}\gamma u\right\} ,$$

$$H_\omega(\varepsilon)\,u := -\Delta u\,, \quad \text{in } \mathbb{R}^3\backslash\partial\Omega\,,$$

An equivalent frequency-dependent Schrödinger operator

$H_\omega(\varepsilon)$ = *selfadjoint below semibounded operator associated to the quadratic form*

$$\mathrm{dom}\,(\mathsf{F}_\omega(\varepsilon)) = H^1\left(\mathbb{R}^3\right), \qquad \mathsf{F}_\omega(\varepsilon)\,u := \|\nabla u\|_2^2 + \left\langle \gamma u, \left(\varepsilon^{-2} - 1\right) DN_{\varepsilon\omega}\gamma u\right\rangle_{\partial\Omega}.$$

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$$H_\omega(\varepsilon) = -\Delta + \left(\varepsilon^{-2} - 1\right) DN_{\varepsilon\omega}\delta_{\partial\Omega}.$$

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Recall: $\alpha \in \mathsf{B}\left(H^s(\partial\Omega), H^{-s}(\partial\Omega)\right)$

$$H_\alpha := -\Delta + \alpha\delta_{\partial\Omega}, \qquad \alpha\delta_{\partial\Omega}u := \int_{\partial\Omega} d\sigma\, \alpha\, \gamma u.$$

Scattering theory for (H_α, H_0) *has been provided under the condition:* $s < 1/2$, *and* $\partial\Omega$ *a* d*-set, with* $d \in (1, 3)$.

A. Mantile, A. Posilicano, Asymptotic completeness and S-matrix for singular perturbations, *J. Math. Pures Appl.*, **130**, 36-67, 2019.

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$$\Longrightarrow \qquad \left(H_\alpha - k^2\right)^{-1} := R_k - SL_k\left(1+\alpha S_k\right)^{-1}\alpha\gamma R_k.$$

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$$\left(H_\omega(\varepsilon) - k^2\right)^{-1} := R_k - SL_k \left(1 + \left(\varepsilon^{-2} - 1\right) DN_{\varepsilon\omega} S_k\right)^{-1} \left(\varepsilon^{-2} - 1\right) DN_{\varepsilon\omega}\gamma R_k$$

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<u>Remark</u>: $DN_{\varepsilon\omega} \in \mathsf{B}\left(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\right)$ does not fulfill the assumptions

$\Longrightarrow$ the existence of $\left(1+\left(\varepsilon^{-2}-1\right)DN_{\varepsilon\omega}S_k\right)^{-1}$ has to be proved !

Asymptotic expansions

By: $DN_{\varepsilon\omega} = S_{\varepsilon\omega}^{-1}\left(1/2 + K_{\varepsilon\omega}\right)$,

$$\left(1 + \left(\varepsilon^{-2} - 1\right) DN_{\varepsilon\omega} S_k\right) = \varepsilon^{-2} S_{\varepsilon\omega}^{-1} \left(\varepsilon^2 + \left(1 - \varepsilon^2\right)\left(\frac{1}{2} + K_{\varepsilon\omega}\right) S_k S_{\varepsilon\omega}^{-1}\right) S_{\varepsilon\omega} \, .$$

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Recall

$$S_z = S_0 + \sum_{n \geq 1} S_0^{(n)} z^n \, , \qquad K_z = K_0 + z^2 K_0^{(2)} + \sum_{n \geq 3} z^n K_0^{(n)} \, ,$$

K_0 *is compact and selfadjoint on* $H^{1/2}(\partial\Omega)$ *w.r.t.:* $\langle \zeta, \chi \rangle_{\partial\Omega} := \left\langle \zeta, S_0^{-1} \chi \right\rangle_{H^{1/2}, H^{-1/2}}$

$$\sigma(K_0) \subseteq [-1/2, 1/2) \, , \qquad K_0(1) = -1/2 \, .$$

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Let: $K_0 P_0 = -1/2 P_0$. *By the Hilbert-Schmidt theorem*

$$K_0 = -1/2 P_0 + (1 - P_0) K_0 (1 - P_0) \, ,$$

$$H^{1/2}(\partial\Omega) \sim \operatorname{ran}(P_0) \oplus \operatorname{ran}(1 - P_0) \, .$$

Asymptotic expansions

Let: $$\omega_M^2 := \frac{C_\Omega}{|\Omega|}, \qquad C_\Omega := \int_{\partial\Omega} S_0^{-1}(1)(x)\, d\sigma_x\, .$$

On: $\mathrm{ran}\,(P_0) \oplus \mathrm{ran}\,(1 - P_0)$

$$\left(\varepsilon^2 + \left(1 - \varepsilon^2\right)\left(\frac{1}{2} + K_{\varepsilon\omega}\right) S_k S_{\varepsilon\omega}^{-1}\right) = \begin{pmatrix} M_{00}(\varepsilon) & M_{01}(\varepsilon) \\ M_{10}(\varepsilon) & M_{11}(\varepsilon) \end{pmatrix},$$

$$M_{00}(\varepsilon) = \varepsilon^2 P_0 \left(|\Omega| \left(\omega_M^2 - \omega^2\right) + \varepsilon \left(\omega^3 K_0^{(3)} + \omega^2 K_0^{(2)} (k - \omega) S_0^{(1)} S_0^{-1}\right) + \mathcal{O}\left(\varepsilon^2\right)\right) P_0$$

$$M_{01}(\varepsilon) = P_0 \mathcal{O}\left(\varepsilon^2\right)(1 - P_0)$$

$$M_{10}(\varepsilon) = (1 - P_0)\, \mathcal{O}\left(\varepsilon^2\right) P_0$$

$$M_{11}(\varepsilon) = (1 - P_0)\left(1/2 + K_0 + \mathcal{O}\left(\varepsilon^2\right)\right)(1 - P_0)$$

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Since: $\sigma(K_0) \subseteq [-1/2, 1/2) \Rightarrow M_{11}^{-1}(\varepsilon) \in \mathrm{B}\left(H^{1/2}(\partial\Omega)\right)$

$$C_{00}(\varepsilon) = P_0 \left(\left(1 - \frac{\omega^2}{\omega_M^2}\right)\varepsilon^2 + \left(-i\,\omega^3 \frac{|\Omega|}{4\pi} + \omega^2 (z - \omega) K_{(2)} S_{(1)} S_0^{-1}\right)\varepsilon^3 + O(\varepsilon^4)\right) P_0\,.$$

(*Schur's complement of* $M_{11}(\varepsilon)$)

Asymptotic expansions

This yields

$$\left(\varepsilon^2 + \left(1-\varepsilon^2\right)\left(\frac{1}{2} + K_{\varepsilon\omega}\right) S_k S_{\varepsilon\omega}^{-1}\right)^{-1} =$$

$$= \begin{pmatrix} C_{00}^{-1}(\varepsilon) & -C_{00}^{-1}(\varepsilon) M_{01}(\varepsilon) M_{11}^{-1}(\varepsilon) \\ -M_{11}^{-1}(\varepsilon) M_{10}(\varepsilon) C_{00}^{-1}(\varepsilon) & M_{11}^{-1}(\varepsilon) + M_{11}^{-1}(\varepsilon) M_{10}(\varepsilon) C_{00}^{-1}(\varepsilon) M_{01}(\varepsilon) M_{11}(\varepsilon) \end{pmatrix},$$

with:

$$C_{00}^{-1}(\varepsilon) = \frac{1}{\varepsilon^2}\left(\frac{\omega_M^2}{\omega_M^2 - \omega^2} P_0 + P_0 O(\varepsilon) P_0\right), \qquad \textit{if } \omega \neq \omega_M ,$$

$$C_{00}^{-1}(\varepsilon) = \frac{1}{\varepsilon^3}\left(\frac{4\pi}{c_\Omega}\,\frac{i}{k} P_0 + P_0 O(\varepsilon) P_0\right), \qquad \textit{if } \omega = \omega_M .$$

Asymptotic expansions

In conclusion

$$\varepsilon^2 \left(\varepsilon^2 + \left(1-\varepsilon^2\right)\left(\frac{1}{2} + K_{\varepsilon\omega}\right) S_k S_{\varepsilon\omega}^{-1}\right)^{-1} = \qquad \text{if } \omega \neq \omega_M \,,$$

$$= \begin{pmatrix} \frac{\omega_M^2}{\omega_M^2-\omega^2} P_0 + P_0 O(\varepsilon) P_0 & P_0 O(\varepsilon^2)\left(1-P_0\right) \\ \\ \left(1-P_0\right) O(\varepsilon^2) P_0 & \left(1-P_0\right) O(\varepsilon^2)\left(1-P_0\right) \end{pmatrix} ,$$

$$\varepsilon^3 \left(\varepsilon^2 + \left(1-\varepsilon^2\right)\left(\frac{1}{2} + K_{\varepsilon\omega}\right) S_k S_{\varepsilon\omega}^{-1}\right)^{-1} \qquad \text{if } \omega = \omega_M \,,$$

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where: $\left\|O(\varepsilon^j)\right\|_{\mathrm{B}\left(H^{1/2}(\partial\Omega)\right)} \leq C_{r_0}\,\varepsilon^j$ *uniformly w.r.t.* $|k| < r_0$ *if* $\omega \neq \omega_M$,

Asymptotic expansions

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...or $\left\| O(\varepsilon^j) \right\|_{\mathsf{B}(H^{1/2}(\partial\Omega))} \leq C_{r_0,r_1}\, \varepsilon^j$ *uniformly w.r.t.* $r_1 < |k| < r_0$ *if* $\omega = \omega_M$.

Back to the physical system

Let introduce the *physical operator*

$$\text{dom}\,(h_\omega(\varepsilon)) :=$$
$$\left\{u \in L^2_{\Delta}\left(\mathbb{R}^3\backslash\partial\Omega_\varepsilon\right) \cap H^1\left(\mathbb{R}^3\right) : [\gamma(\varepsilon)]\,u = 0\,,\ [\gamma'(\varepsilon)]\,u = \left(\varepsilon^{-2} - 1\right) DN_\omega(\varepsilon)\,\gamma(\varepsilon)\,u\right\},$$

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It results

$$\left(h_\omega(\varepsilon) - k^2\right)^{-1} = \varepsilon^2 U_\varepsilon \left(H_\omega(\varepsilon) - \varepsilon^2 k^2\right)^{-1} U_\varepsilon^{-1}\,.$$

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$$\left(h_\omega(\varepsilon) - k^2\right)^{-1} = \varepsilon^2 U_\varepsilon \left(H_\omega(\varepsilon) - \varepsilon^2 k^2\right)^{-1} U_\varepsilon^{-1}\,.$$

The resolvent formula, validated for $\varepsilon << 1$, *implies*

$$\left(h_\omega(\varepsilon) - k^2\right)^{-1} =$$
$$= R_k - \left(1 - \varepsilon^2\right) U_\varepsilon S L_{\varepsilon k} S_{\varepsilon\omega}^{-1} \left(\varepsilon^2 + \left(1 - \varepsilon^2\right)\left(\frac{1}{2} + K_{\varepsilon\omega}\right) S_k S_{\varepsilon\omega}^{-1}\right)^{-1} S_{\varepsilon\omega} DN_{\varepsilon\omega} \gamma R_{\varepsilon k} U_\varepsilon^{-1}$$

Back to the physical system

...plugging-in the asymptotic expansions above (and doing some further computations)

Theorem *Let* $\omega > 0$, $k \in \mathbb{C}_+$ *and assume:* $\varepsilon \in (0, \varepsilon_0)$ *with* $\varepsilon_0 > 0$ *small enough.*

$$\left\| \left(h_\omega(\varepsilon) - k^2 \right)^{-1} - R_k \right\| = \mathcal{O}(\varepsilon) \ , \qquad \omega \neq \omega_M$$

$$\left\| \left(h_{\omega_M}(\varepsilon) - k^2 \right)^{-1} - R_k - \frac{i4\pi}{k} \left| R_k(\cdot, y_0) \right\rangle \left\langle R_k(\cdot, y_0) \right| \right\| = \mathcal{O}\left(\varepsilon^{1/2} \right) \ , \quad \omega = \omega_M$$

where $\left\| \mathcal{O}\left(\varepsilon^j \right) \right\|_{\mathsf{B}(L^2(\mathbb{R}^3))} \leq C_{r_0} \varepsilon^j$ *uniformly w.r.t.* $|k| < r_0$ *if* $\omega \neq \omega_M$,

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In particular

$\Longrightarrow \left(h_{\omega_M}(0) - k^2 \right)^{-1}$ is the resolvent of a point interaction supported in y_0.

Back to the physical system

Building on the abstract scattering theory for singular perturbations

- *For each* $\omega > 0 \in R$, $\exists\ \varepsilon_\omega > 0$ *small s.t.:* $\underline{h_\omega(\varepsilon) \text{ is selfadjoint}}$ *for* $\varepsilon \in (0, \varepsilon_\omega)$ *and*

$$\sigma_{ess}(h_\omega(\varepsilon)) = \sigma_{ac}(h_\omega(\varepsilon)) = [0, +\infty) , \qquad \sigma_{sc}(h_\omega(\varepsilon)) = \varnothing .$$

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- *If* $\omega = \pm\omega_M \Rightarrow$ *there exists* $\tilde{M} > 0$ *such that: the possible eigenvalues or resonances of* $h_\omega(\varepsilon)$ *belong to* $\left\{k^2 : |k| <\in \tilde{M}\varepsilon\right\}$ (provided that ε is small enough).

Back to the physical system

Our acoustic problem expresses in terms of generalized eigenfunctions

$$\left(h_\omega(\varepsilon) - \omega^2\right) u = 0\,,$$

where $h_\omega(\varepsilon)$ *is interpreted as an extension to* $L^2_{\Delta,loc}\left(\mathbb{R}^3 \backslash \partial\Omega\right) \cap H^1_{loc}\left(\mathbb{R}^3\right)$.

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Using the scattering theory for singular perturbations

$$u - u^{\mathrm{in}} =$$

$$-\left(1-\varepsilon^2\right) U_\varepsilon\, SL_{\varepsilon\omega}\, S^{-1}_{\varepsilon\omega}\left(\varepsilon^2 + \left(1-\varepsilon^2\right)\left(\frac{1}{2} + K_{\varepsilon\omega}\right) S_k S^{-1}_{\varepsilon\omega}\right)^{-1} S_{\varepsilon\omega} DN_{\varepsilon\omega}(\varepsilon\omega)\,\gamma U_\varepsilon^{-1} u^{\mathrm{in}}$$

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Combining the LAP estimates

$$SL_k = R_k \gamma \in \mathsf{B}\left(H^{-1/2}(\partial\Omega)\,, H^1_{-\alpha}\left(\mathbb{R}^3\right)\right)\,, \qquad \alpha > 1/2\,,$$

$$H^s_\alpha\left(\mathbb{R}^3\right) := \left\{u \in \mathcal{D}'\left(\mathbb{R}^3\right) : \left(1 + |x|^2\right)^{\alpha/2} u \in H^s\left(\mathbb{R}^3\right)\right\}$$

with the previous asymptotic expansions...

Back to the physical system

Let $\omega > 0$ *and* $u_\omega^{\mathsf{in}} \in H^2_{-\alpha}(R^3)$, $\alpha > 1/2$, *be a solution of the homogeneous Helmholtz equation*

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Theorem *For any* $\varepsilon > 0$ *sufficiently small, one has*

$$\omega \neq \omega_M \quad \Longrightarrow \quad \begin{cases} u^{\text{sc}}_\omega(\varepsilon) = \varepsilon\, \frac{\omega^2\, c_\Omega}{\omega_M^2 - \omega^2}\; u^{\text{in}}_\omega(y_0)\, \frac{e^{i\omega|x-y_0|}}{4\pi|x-y_0|} + r_\omega(\varepsilon)\,, \\ \\ \|r_\omega(\varepsilon)\|_{L^2_{-\alpha}(\mathbb{R}^3)} \leq c\, \varepsilon^{3/2}\,, \end{cases}$$

$$\omega = \omega_M \quad \Longrightarrow \quad \begin{cases} u^{\text{sc}}_\omega(\varepsilon) = \frac{4\pi i}{\omega}\; u^{\text{in}}_\omega(y_0)\, \frac{e^{i\omega|x-y_0|}}{4\pi|x-y_0|} + r_\omega(\varepsilon), \\ \\ \|r_\omega(\varepsilon)\|_{L^2_{-\alpha}(\mathbb{R}^3)} \leq c\, \varepsilon^{1/2}\,. \end{cases}$$

(uniformly with respect to the choice of the incoming wave u^{in}_ω*)*

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Let $\omega > 0$ *and* $u_\omega^{\text{in}} \in H^2_{-\alpha}(R^3)$, $\alpha > 1/2$, *be a solution of the homogeneous Helmholtz equation*

$$(\Delta + \omega^2) u_\omega^{\text{in}} = 0\,.$$

Theorem *Let* $c_M > 0$ *and* $I_M \subset R_+$ *be a bounded interval containing* ω_M. *For any* $\varepsilon > 0$ *sufficiently small, the expansion*

$$(u_\omega^{\text{sc}}(\varepsilon))\,(x) = \frac{\varepsilon\,\omega^2 c_\Omega}{\omega_M^2 - \omega^2 - i\varepsilon\frac{\omega^3 c_\Omega}{4\pi}}\; u_\omega^{\text{in}}(y_0)\,\frac{e^{i\omega|x-y_0|}}{4\pi|x-y_0|} + (r_\omega(\varepsilon))\,(x)\,,$$

$$\|r_\omega(\varepsilon)\|_{L^2_{-\alpha}(\mathbb{R}^3)} \leq c\,\frac{\varepsilon^{3/2}}{\omega_M^2 - \omega^2}\,, \qquad \alpha > 1/2\,.$$

holds uniformly with respect to both ω *in* $\{\omega \in \mathcal{I}_M : |\omega - \omega_M| \geq c_M\,\varepsilon\}$ *and* u_ω^{in}.

Conclusions

$\omega_M =$ the unique value of ω s.t. $h_\omega(\varepsilon)$ has a non-trivial resolvent limit as $\varepsilon \longrightarrow 0^+$

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Thanks for your attention