On the number of Pollicott–Ruelle resonances for open hyperbolic systems and Axiom A flows

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Axiom A flows

- Smale, "Differentiable dynamical system", BAMS, 1967.

Nonwandering point for a flow

$\mathcal{M}$ is a C^∞ compact manifold without boundary, $\varphi^t = \exp(tX)$ is a flow generated by a C^∞ vector field X. $x \in \mathcal{M}$ is nonwandering if

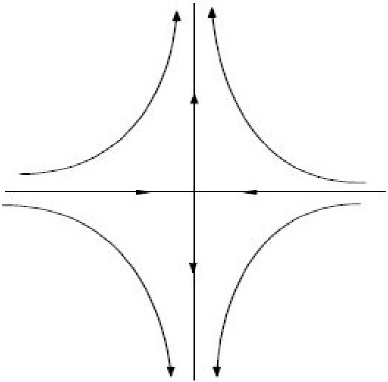
$$\forall V = \text{nbd}(x), T > 0, \exists t \in \mathbb{R}, \quad \text{s.t.} \quad |t| \geq T, \varphi^t(V) \cap V \neq \emptyset.$$

Axiom A flows

- the set of nonwandering point $\Omega = \mathcal{F} \sqcup \mathcal{K}$, $\mathcal{F}$ the set of fixed points, $\mathcal{K}$ closure of the union of all closed orbits;
- every fixed point is hyperbolic;
- $\mathcal{K}$ is a hyperbolic invariant set.

Hyperbolic fixed point

A fixed point $x \in \mathcal{M}$ (i.e $X(x) = 0$) is hyperbolic if the differential $\nabla X(x)$ has no eigenvalues on the imaginary axis. (x could be purely attractive or repulsive.)



Hyperbolic invariant set

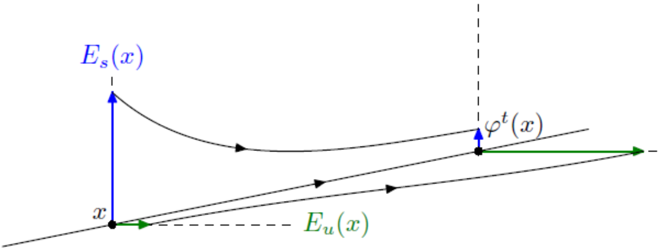
A compact invariant set $\mathcal{K}$ (with no fixed point) is hyperbolic if there is a continuous decomposition

$$\forall x \in \mathcal{K}, \quad T_x\mathcal{M} = E_0(x) \oplus E_s(x) \oplus E_u(x),$$

$$d\varphi^t(x)E_\bullet(x) = E_\bullet(\varphi^t(x)), \ \ \bullet = s, u, \ \ E_0(x) = \mathbb{R}X(x),$$

$$|d\varphi^t(x)v|_{\varphi^t(x)} \leq Ce^{-\theta|t|}|v|_x, v \in E_u(x), \quad t < 0$$

$$|d\varphi^t(x)v|_{\varphi^t(x)} \leq Ce^{-\theta|t|}|v|_x, v \in E_s(x), \quad t > 0,$$



Examples: Morse–Smale flows

Morse–Smale flows

- Ω is the disjoint union of finitely many hyperbolic "elementary critical element", i.e. fixed point or closed orbit.
- Transversal condition: For any elementary critical elements Λ and Λ',

$$\forall x \in W_u(\Lambda) \cap W_s(\Lambda'), \quad T_x M = T_x W_u(\Lambda) + T_x W_s(\Lambda').$$

Here $W_s(\Lambda')$ and $W_u(\Lambda)$ are stable/unstable manifolds:

$$W_s(\Lambda') = \{x \in \mathcal{M} : \varphi^t(x) \to \Lambda', t \to +\infty\};$$

$$W_u(\Lambda) = \{x \in \mathcal{M} : \varphi^t(x) \to \Lambda, t \to -\infty\}.$$

Morse–Smale flows

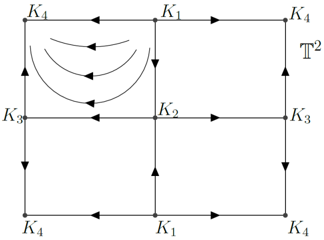


Figure: Meddane '21: Illustration of flows on torus with transversal condition(right) or without (left)

Examples: Anosov flows

Anosov flows

A flow $\varphi^t = \exp(tX)$ with nonvanishing X on a compact manifold $\mathcal{M}$ is Anosov, if $\mathcal{M}$ itself is a hyperbolic invariant set.

Examples

- Geodesic flow on the sphere bundle of a negatively curved manifold;
- Suspension of Anosov maps.
- Frank–Williams '80: "Anomalous" Anosov flows with $\Omega \neq \mathcal{M}$ (even in dim 3).

Correlation, power spectrum and resonances

- We fix some Lebesgue measure dx on $\mathcal{M}$.
- For $f, g \in C^\infty(\mathcal{M})$, the *correlation function* for φ^t is defined as
$$\rho_{f,g}(t) = \int_{\mathcal{M}} f(\varphi_{-t}(x))g(x)dx.$$
- The *power spectrum* is the inverse Fourier-Laplace transform
$$\widehat{\rho}_{f,g}(\lambda) := \int_0^\infty \rho_{f,g}(t)e^{i\lambda t}dt, \quad \operatorname{Im}\lambda > 0.$$
- Pollicott '85, Ruelle '86, '87: The *Pollicott–Ruelle resonances* of the Axiom A flow φ^t are the poles of the meromorphic extension of $\widehat{\rho}_{f,g}(\lambda)$ to (possible a subset of) $\mathbb{C}$.

Decay of correlation

The distribution of resonances are also related to mixing property of the flow: If dx is an invariant probability measure for φ^t, then

- The flow φ^t is called *mixing* if

$$\lim_{t\to\infty} \rho_{f,g}(t) = \int_{\mathcal{M}} f dx \int_{\mathcal{M}} g dx.$$

- *exponential mixing* if for some $c > 0$,

$$\rho_{f,g}(t) = \int_{\mathcal{M}} f dx \int_{\mathcal{M}} g dx + \mathcal{O}(e^{-ct}), \quad t > 0.$$

Remark

- Bowen–Ruelle conjecture: for Anosov flows, (topologically) mixing $\Rightarrow$ exponential mixing. Tsujii–Zhang '20, 3-dim case.

Dynamical ζ functions

The Pollicott–Ruelle resonances are also related to the zeroes and poles of various dynamical ζ functions, e.g. *Ruelle ζ-function*:

$$\zeta(\lambda) = \prod_{\gamma^\#} (1 - e^{-\lambda T_{\gamma^\#}}).$$

- The product is taken over primitive closed orbits $\gamma^\#$ (excluding fixed points);
- $T_{\gamma^\#}$ is the length of the orbit $\gamma^\#$.
- A rough estimate on the number of closed orbits:

  $$\#\{\gamma : T_\gamma \leq T\} \leq Ce^{CT}$$

  shows that the product converges for $\operatorname{Re} \lambda \gg 1$.

Smale conjecture

$$\zeta(\lambda) = \prod_{\gamma^\#}(1 - e^{-\lambda T_{\gamma^\#}}), \quad \operatorname{Re}\lambda \gg 1.$$

- Smale '67: $\zeta(\lambda)$ extends meromorphically to $\lambda \in \mathbb{C}$?
- Giulletti–Liverani–Pollicott '13 for Anosov flows,
- Dyatlov–Zworski '16, microlocal proof, use Melrose's radial estimate.
- Dyatlov–Guillarmou '16, '18 for Axiom A flows (with orientable stable/unstable space).

Pollicott-Ruelle resonances via anisotropic Sobolev spaces

Let $P = -iX : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$, then $f(\varphi_{-t}(x)) = e^{-itP} f(x)$ and

$$\widehat{\rho}_{f,g}(\lambda) = \int_0^\infty \langle e^{-it(P-\lambda)} f, g \rangle_{L^2} dt = i\langle (P-\lambda)^{-1} f, g \rangle_{L^2}.$$

Pollicott–Ruelle resonances λ are

- Poles of the meromorphic continuation of the resolvent $(P-\lambda)^{-1} : C^\infty(\mathcal{M}) \to \mathscr{D}'(\mathcal{M})$;
- the eigenvalues of P on certain *anisotropic Sobolev spaces* (or microlocally weighted Sobolev spaces) $H_G(\mathcal{M})$, the "microlocal" weight function G is often called "escape function".

The set of resonances is denoted by $\mathrm{Res}(P)$.

Phase space structure for Anosov flows

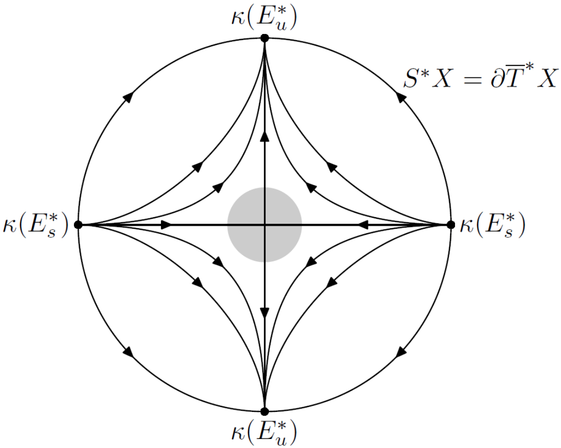


Figure: Dyatlov–Zworski '16: the dynamics on the part $E_u^* \oplus E_s^*$ of phase space $T^*\mathcal{M}$ for Anosov flows. The anisotropic Sobolev spaces H_{sG} is like H^s hear $\kappa(E_s^*)$ and H^{-s} near $\kappa(E_u^*)$

Pollicott-Ruelle resonances via anisotropic Sobolev spaces

Microlocal construction of anisotropic Sobolev spaces and the meromorphic continuation of the resolvent

$$(P-\lambda)^{-1} : C^\infty(\mathcal{M}) \to \mathscr{D}'(\mathcal{M});$$

- Faure–Sjöstrand '11 for Anosov flows;
- Dang–Rivière '19, '20 for Morse–Smale flows;
- Meddane '21 for Axiom A flows with Strong Transversal Condition.

Some history

- Early work by Ruelle, Pollicott, Bowen, Fried, Rugh,... are based on the ideas to code the flow to a symbolic dynamical system via a Markov partition.
- The anisotropic Hölder space approach: Blank–Keller–Liverani, Baillif, Baladi, Butterley, Liverani, Tsujii, Dolgopyat, Guilietti–Liverani–Pollicott, Gouëzel–Liverani,....
- The use of anisotropic Sobolev spaces at least dates back to the work on scattering resonances of Heffler, Sjöstrand, etc. For the study of Pollicott–Ruelle resonances, Faure–Roy, Faure–Roy–Sjöstrand , Faure–Sjöstrand, Baladi–Tsujii, Faure–Tsujii, Dyatlov–Zworski, Dyatlov–Guillarmou, Dang–Rivière, Weich–Küster, Hadfield, Bonthonneau, Jézéquel, Cekic, Lefeuvre, Paternain, Chen–Hu, Meddane,....
- Alternative definition by Dyatlov–Zworski, Dorout,...: Limits of the eigenvalues of $P + i\epsilon\Delta$ as $\epsilon \to 0+$

Estimates on the number of Pollicott–Ruelle resonances: Upper bound

Theorem (J.-Tao '22 in progress)

Let $n = \dim \mathcal{M}$, $\beta > 0$ *fixed, then for Axiom A flows with Strong Transversal Condition*,

$$\# \operatorname{Res}(P) \cap \{\lambda : |\operatorname{Re}\lambda - E| \leq 1, \operatorname{Im}\lambda > -\beta\} = \mathcal{O}(E^n).$$

- Faure–Sjöstrand '11: For Anosov flows,

$$\# \operatorname{Res}(P) \cap \{\lambda : |\operatorname{Re}\lambda - E| \leq \sqrt{E}, \operatorname{Im}\lambda > -\beta\} = o(E^{n-\frac{1}{2}}).$$

- Datchev–Dyatlov–Zworski '13: For Anosov flows with C^∞ contact form, sharp upper bound

$$\# \operatorname{Res}(P) \cap \{\lambda : |\operatorname{Re}\lambda - E| \leq 1, \operatorname{Im}\lambda > -\beta\} = \mathcal{O}(E^{\frac{n-1}{2}}).$$

Fractal Weyl Law

In general, Fractal Weyl Law conjectures

$$\# \mathrm{Res}(P) \cap \{\lambda : |\operatorname{Re}\lambda - E| \leq 1, \operatorname{Im}\lambda > -\beta\} = \mathcal{O}(E^{\frac{\dim K - 1}{2}}).$$

where K is the dimension of "trapped set" in phase space.

- Faure–Tsujii '17: For Anosov flows, with $\alpha_0 \in (0,1]$ the Hölder exponent of $E_u \oplus E_s$

$$\# \mathrm{Res}(P) \cap \{\lambda : |\operatorname{Re}\lambda - E| \leq 1, \operatorname{Im}\lambda > -\beta\} = \mathcal{O}(E^{\frac{n-1}{1+\alpha_0}}).$$

- Faure–Tsujii '13, '21, (Cekić–Guillarmou '20 for dim 3): Band structure and Weyl Law in a band for contact Anosov flows with certain pinching conditions on minimal/maximal expansion rates.

Estimates on the number of Pollicott–Ruelle resonances: Lower bound

Theorem (J.–Tao '22 in progress)

For Axiom A flows with Strong Transversal Condition and $\mathcal{K} \neq \emptyset$*, for every* $\delta \in (0,1)$ *there exists a constant* $\beta = \beta(\delta) > 0$ *such that*

$$\#(\mathrm{Res}(P) \cap \{\lambda : |\lambda| \leq E, \operatorname{Im} \lambda > -\beta\}) \neq \mathcal{O}(E^{\delta}).$$

- J.–Zworski '17 (+J.–Tao '22): the same for Anosov flows.
- Jézéquel '21: Global lower bound for "ultradifferntial" Anosov flows.
- Upper bound on essential spectral gap: there are infinitely many resonances in the strip $\operatorname{Im} \lambda > -A$ for A sufficiently large.
- Tsujii, '10, '12, Nonnenmacher–Zworski '15: Lower bound on essential spectral gap for contact Anosov flows.

Special cases: Suspension of Axiom A map

Let $f : M \to M$ be an Axiom A map, $e^{-i\lambda_k} \in B_{\mathbb{C}}(0,1)$ be the resonances for f, then the resonances of its suspension flow $\varphi^t : \mathcal{M} \to \mathcal{M}$, $\mathcal{M} = M \times [0,1]/\sim$ are given by

$$\text{Res}(P) = \{2j\pi + \lambda_k, \quad j \in \mathbb{Z}\}.$$

- The "weak" lower bound above is almost optimal: linear growth.
- Jézéquel '19 gives an example of Axiom A map on $\mathbb{S}^4$ (with some C^∞ weight function) such that

$$\sum e^{-ip\lambda_k}$$

does not converge for all $p \geq 1$. This suggests that in general a global trace formula does not work.

Special cases: Morse–Smale flows

Dang–Rivière '20: For Morse–Smale flows (with certain assumptions), there exists a sequence $\mu_k \in \mathbb{C}$ with

$$\operatorname{Im}\mu_k \leq 0, \quad \lim_{k\to\infty} \operatorname{Im}\mu_k = -\infty$$

such that

$$\mathrm{Res}(P) = \bigcup_{\Lambda,k} (\sigma_\Lambda + \mu_k)$$

where Λ runs over all fixed points and closed orbits:

- For fixed point, $\sigma_\Lambda = \{0\}$;
- For closed orbit,

$$\sigma_\Lambda = \{2\pi(m + \varepsilon_\Lambda + \gamma_j^\Lambda)/T_\Lambda : 1 \leq j \leq N, m \in \mathbb{Z}\}$$

  - $\varepsilon_\Lambda = 0$ or $\frac{1}{2}$ is the "twist index";
  - $e^{2\pi i \gamma_j^\Lambda}$, $1 \leq j \leq N$ are eigenvalues of "monodromy matrix";
  - T_Λ is the (minimal) period of the closed orbit.

Special cases: Geodesic flow on hyperbolic manifolds

- Dyatlov–Faure–Guillarmou '15, for compact hyperbolic manifolds M, the resonances for geodesic flow on $\mathcal{M} = SM$ is related to the eigenvalues of Laplacian on M, e.g. for surfaces (also by Faure–Tsujii '13)

$$\begin{aligned}\mathrm{Res}(P) \setminus \{-(1+\frac{k}{2})i, k \in \mathbb{N}\} \\ &= \{(-m-1+s)i : m \in \mathbb{N}, \\ &\qquad s \in [0,1] \cup (\frac{1}{2} + i\mathbb{R}), s(1-s) \in \mathrm{Spec}(\Delta_M)\}.\end{aligned}$$

  In particular, Weyl law for Δ_M gives Weyl law of $\mathrm{Res}(P)$.
- Guillarmou–Hilgert–Weich '18, Hadfield '20: Convex co-compact case. $\mathrm{Res}(\Delta_M)$ instead of $\mathrm{Spec}(\Delta_M)$.

Spectral decomposition

Let $\varphi^t = \exp(tX)$ be an Axiom A flow on a compact manifold $\mathcal{M}$,

- $\Omega = \mathcal{F} \sqcup \mathcal{K}$, $\mathcal{F}$ the set of fixed points, $\mathcal{K}$ closure of the union of all closed orbits;
- The spectral decomposition: $\mathcal{K}$ can be uniquely decomposed into a finite disjoint union

$$\mathcal{K} = K_1 \sqcup K_2 \sqcup \cdots \sqcup K_N$$

  of "basic hyperbolic set".
- K is a basic hyperbolic set if
  - K is compact and hyperbolic invariant set;
  - K is locally maximal: there exists a neighborhood V of K such that $K = \bigcap_{t\in\mathbb{R}} \varphi^t(V)$;
  - φ^t is topologically transitive on K, i.e. contains a dense closed orbit;
  - K is the closure of the union of all closed orbits in K.

Open hyperbolic systems

- Guillarmou–Mazzucchelli–Tzou '17, Dyatlov–Guillarmou '18, (based on Conley–Easton '71): Every locally maximal hyperbolic set K can be "isolated into" an "open hyperbolic system" with suitable modification near boundary.
- Open hyperbolic system (Dyatlov–Guillarmou '16):
  - A compact manifold with boundary $\overline{\mathcal{U}}$ and a nonvanishing vector field X;
  - The boundary $\partial\mathcal{U}$ is strictly convex with respect to $\varphi^t = \exp(tX)$: for any boundary defining function $\rho \in C^\infty(\overline{\mathcal{U}})$, ($\rho > 0$ on $\mathcal{U}$, $\rho = 0$ and $d\rho \neq 0$ on $\partial\mathcal{U}$),

$$x \in \partial\mathcal{U}, X\rho(x) = 0 \quad \Rightarrow X^2\rho(x) < 0.$$

  - The "trapped set" $K = \bigcap_{t\in\mathbb{R}} \varphi^t(\overline{\mathcal{U}})$ is a hyperbolic invariant set for φ^t.

Resonances and zeta functions for open hyperbolic systems

- Pollicott–Ruelle resonances for $(\overline{\mathcal{U}}, X)$ is the poles of meromorphic continuation of

$$R(\lambda) = \mathbb{1}_{\mathcal{U}}(P-\lambda)^{-1}\mathbb{1}_{\mathcal{U}} : C_c^\infty(\mathcal{U}) \to \mathcal{D}'(\mathcal{U}), \quad \operatorname{Im}\lambda \gg 1.$$

  Here $P = -iX$. $\overline{\mathcal{U}}$ embedded in some $\mathcal{M}$.
- Ruelle zeta function for $(\overline{\mathcal{U}}, X)$ is

$$\zeta_K(\lambda) = \prod_{\gamma^\#}(1 - e^{-\lambda T_{\gamma^\#}}), \quad \operatorname{Re}\lambda \gg 1.$$

- Dyatlov–Guillarmou '16:
  - Anisotropic Sobolev spaces;
  - Meromorphic continuation of resolvent;
  - Meromorphic continuation of zeta function. (Dyatlov–Guillarmou '18, for Axiom A flows)

From open hyperbolic systems to Axiom A flows

For Axiom A flows with Strong Transversal Condition:

- If $K_\pm$ are basic hyperbolic set or hyperbolic fixed point, $x_\pm \in K_\pm$ and $x \in W_u(x_-) \cap W_s(x_+)$,

$$T_x W_{uo}(x_-) + T_x W_{so}(x_+) = T_x \mathcal{M}.$$

Here W_{so} and W_{uo} are "weakly stable/unstble manifolds"

- Smale '67: partial order on basic hyperbolic set and hyperbolic fixed point.
- Hirsch–Palis–Pugh–Shub '70, Robbin '71, arbitrary small unrevisited neighborhood around each basic set.
- Meddane '21: Escape function, anisotropic Sobolev spaces, meromorphic continuation of the resolvent.

From open hyperbolic systems to Axiom A flows

For Axiom A flows with Strong Transversal Condition:

- The Pollicott–Ruelle resonances are basically the union of resonances for the open hyperbolic systems/fixed points obtained by spectral decomposition and "isolation".
- The contribution of fixed points are finite in each strip $\{\operatorname{Im}\lambda > -\beta\}$.
- For open hyperbolic systems, we view it as a scattering problem, both in space and frequency variable.

Proof of upper bound: resolvent estimate

Let

- $p(x, \xi) = \langle X(x), \xi \rangle$ be the classical Hamiltonian for $P = -iX$
- $Q_\infty \in \Psi_h^1(\mathcal{M})$, $q_1 \in C^\infty(\mathcal{M})$ be complex absorbing potentials outside $\mathcal{U}$;
- $W \in \Psi_h^{\mathrm{comp}}(\mathcal{M})$ be a potential near the trapped set $p^{-1}(1) \cap E_0^*$, with rank $W = \mathcal{O}(h^{-n})$.

Lemma

For $M \gg 1$, $z \in [1-h, 1+h] + i[-\beta h, 1]$

$$\tilde{P} - z = hP - i(Q_\infty + q_1 + hMrW) - z : \mathcal{H}_h^r \to \mathcal{H}_h^r$$

is invertible and

$$\|(\tilde{P} - z)^{-1}\|_{\mathcal{H}_h^r \to \mathcal{H}_h^r} \leq \frac{C}{\max(h, \operatorname{Im} z - Ch)}.$$

Proof of upper bound

Theorem (J.-Tao '22 in progress)

Let $n = \dim \mathcal{M}$, $\beta > 0$ *fixed, then for an open hyperbolic system* $(\overline{\mathcal{U}}, X)$, $P = -iX$,

$$\# \operatorname{Res}(P) \cap \{\lambda : |\operatorname{Re}\lambda - E| \leq 1, \operatorname{Im}\lambda > -\beta\} = \mathcal{O}(E^n).$$

- A similar resolvent estimate already appears in Dyatlov–Guillarmou '16 focus on $\{p = 0\}$.
- The upper bound for resonances follows from standard Fredholm argument.
- Here we focus on energy shell $\{p = 1\}$ to get slightly better estimate.

Proof of weak lower bound: local trace formula

Theorem

For any $A > 0$, there exists a distribution $F_A(t) \in \mathcal{S}'(\mathbb{R})$ supported in $[0, \infty)$ such that

$$\sum_{\mu \in \mathrm{Res}(P), \operatorname{Im}\mu > -A} e^{-i\mu t} + F_A(t) = \sum_{\gamma} \frac{T_\gamma^\sharp \delta(t - T_\gamma)}{|\det(I - \mathcal{P}_\gamma)|}, \quad t > 0$$

and the Fourier transform of F_A is analytic in the region $\{\operatorname{Im}\lambda < A\}$ with

$$\left|\widehat{F_A}(t)\right| = \mathcal{O}_{A,\epsilon}(\langle\lambda\rangle^{2n+1+\epsilon}), \quad \operatorname{Im}\lambda < A - \epsilon.$$

- The proof relies on the upper bound.

Motivation: Poisson formula for resonances

In the case of the resonances for a convex co-compact hyperbolic surface M, combining the wave Selberg trace formula

$$0 - \operatorname{tr} \cos t\sqrt{\Delta_M - \frac{1}{4}} = \frac{1}{2} \sum_{\gamma} \frac{T_\gamma^{\#} \delta(t - T_\gamma)}{|\det(I - \mathcal{P}_\gamma)|^{1/2}} + \Psi_{\text{top}}(t), \quad t \in \mathbb{R}$$

(where $\Psi_{\text{top}}(t)$ is the term comes from the topology of the surface) with the Melrose's Poisson formula (Guillopé–Zworski '97)

$$0 - \operatorname{tr} \cos t\sqrt{\Delta_M - \frac{1}{4}} = \frac{1}{2} \sum_{\lambda \in \operatorname{Res}(\Delta_M)} e^{i\lambda|t|}, \quad t \neq 0,$$

we also have a trace formula relates the spectrum with the geometry of the closed geodesics.

Lower bounds on scattering resonances

- The trace formula relates the spectrum with the geometry of the closed geodesics:

$$\sum_{\lambda \in \mathrm{Res}(\Delta_M)} e^{i\lambda|t|} = \sum_{\gamma} \frac{T_\gamma^{\#}\delta(t - T_\gamma)}{|\det(I - \mathcal{P}_\gamma)|^{1/2}} + \Psi_{\mathrm{top}}(t), \quad t \neq 0,$$

- From this trace formula, Guillopé–Zworski '99 proved for $0 < \epsilon < \frac{1}{2}$,

$$\#(\mathrm{Res}(\Delta_M) \cap \{\mu : |\mu| \leq r, \mathrm{Im}\,\mu > -\epsilon^{-1}\}) \neq \mathcal{O}(r^{1-\epsilon}).$$

- Jakobson–Naud '12 also use a variation of this trace formula to give an upper bound on the essential gap for the Laplacian Δ_M.

Proof of local trace formula

$$\sum_{\mu\in \mathrm{Res}(P), \operatorname{Im}\mu > -A} e^{-i\mu t} + F_A(t) = \sum_{\gamma} \frac{T_\gamma^\sharp \delta(t - T_\gamma)}{|\det(I - \mathcal{P}_\gamma)|}, \quad t > 0$$

- Here we use Atiyah–Bott–Guillemin trace formula:

$$\operatorname{tr}^\flat \chi e^{-itP}\chi = \sum_\gamma \frac{T_\gamma^\sharp \delta(t - T_\gamma)}{|\det(I - \mathcal{P}_\gamma)|}.$$

- Another key ingredient is a wavefront set estimate to give quantitative flat trace estimate. (J.-Tao '22, complement to J.-Zworski '16 for Anosov flows.)
- Jézéquel '19: global trace formula for "ultradifferential" Anosov flows and global lower bound: $\exists \rho > 0$,

$$\sum_{\lambda\in \mathrm{Res}(P)} \frac{1}{1 + |\lambda|^\rho} = \infty.$$

Flat trace and wavefront set condition

Consider an operator $B : C^\infty(X) \to \mathcal{D}'(X)$ with

$$\mathrm{WF}'(B) \cap \Delta(T^*X) = \emptyset.$$

Then we can define the *flat trace* of B as

$$\operatorname{tr}^\flat B = \int_X K_B(x,x)dx = \int_X (\iota^* K_B)(x)dx := \langle \iota^* K_B, 1 \rangle$$

where $\iota : x \mapsto (x,x)$ is the diagonal map, K_B is the Schwartz kernel of X with respect to the density dx on X.

Semiclassical wavefront set and quantitative flat trace estimate

Let $\tilde{R}_h(z) = (\tilde{P} - z)^{-1}$, $z \in [-h^\varepsilon, h^\varepsilon] + i[-\beta h, 1]$, where

$$\tilde{P} - z = hP - i(Q_\infty + q_1 + hMrW) - z : \mathcal{H}_h^r \to \mathcal{H}_h^r.$$

Then

$$\mathrm{WF}'_h(\tilde{R}_h(z)) \cap S^*(\mathcal{U} \times \mathcal{U}) \subset \kappa(\Delta(T^*\mathcal{U}) \cup \Omega_+ \cup (E_+^* \times E_-^*) \setminus \{0\})$$

so

$$\mathrm{WF}'_h(e^{-it_0(\tilde{P}-z)/h}\tilde{R}_h(z)) \cap \Delta(S^*\mathcal{U}) = \emptyset.$$

and

$$\mathrm{tr}^\flat(\chi e^{-it_0(\tilde{P}-z)/h}\tilde{R}_h(z)\chi) = \mathcal{O}(h^{-2n-2}).$$

Proof of weak lower bound

Theorem (J.–Tao '22 in progress)

for an open hyperbolic system $(\overline{\mathcal{U}}, X)$, $P = -iX$, *for every* $\delta \in (0,1)$ *there exists a constant* $\beta = \beta(\delta) > 0$ *such that*

$$\#(\mathrm{Res}(P) \cap \{\lambda : |\lambda| \le E, \operatorname{Im} \lambda > -\beta\}) \ne \mathcal{O}(E^\delta).$$

- The idea is the same as Guillopé–Zworski '99, J.-Zworski '16.
- Use a test function $\varphi_{\ell,d}(t) = \varphi(\ell^{-1}(t-d))$ with center $d \in (0,\infty)$ and width $\ell < 1$

$$\sum_{\mu \in \mathrm{Res}(P), \operatorname{Im} \mu > -A} \widehat{\varphi}_{\ell,d}(\mu) + \langle F_A, \varphi_{\ell,d} \rangle = \sum_\gamma \frac{T_\gamma^\# \varphi_{\ell,d}(T_\gamma)}{|\det(I - \mathcal{P}_\gamma)|}.$$

Proof of weak lower bound

In the local trace formula tested with $\varphi_{\ell,d}$,

$$\sum_{\mu\in \mathrm{Res}(P), \mathrm{Im}\,\mu>-A} \widehat{\varphi}_{\ell,d}(\mu) + \langle F_A, \varphi_{\ell,d}\rangle = \sum_{\gamma} \frac{T_\gamma^{\#}\varphi_{\ell,d}(T_\gamma)}{|\det(I-\mathcal{P}_\gamma)|}.$$

- If there is upper bound $\mathcal{O}(E^\delta)$, the resonance term $\leq C\ell^{1-\delta}$;
- The error term $\leq C\ell^{-2n-2}e^{(d-\ell)(-A+\varepsilon)}$;
- If we choose $d = kT_{\gamma_0}$ where γ_0 is a primitive periodic orbit, $k \in \mathbb{N}$, the dynamical side $\geq ce^{-\theta d}$;
- With $\ell = e^{-Bd}$ there is a contradiction with A, B both large and $k \to \infty$.

Thanks for your attention!