The Unruh state for Dirac fields on Kerr spacetime

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Scattering and inverse scattering, Linz November 3 2022

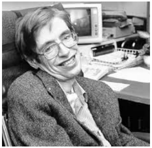
I. Introduction

QFT on curved spacetimes

Second quantized solutions of linear $\not{D}\hat{\phi} = 0$ on (fixed) blackhole spacetime (M, g). What is the physical solution $\hat{\phi}$?

1. Locally, $\hat{\phi}$ has to look the same as vacuum solution on Minkowski. (high frequency singularities in the sense of microlocal analysis)
   In particular $\hat{\phi}$ shouldn't acquire extra singularities when crossing horizon (infinite accumulation of quantum energy).
2. But black hole collapse situation enforces asymptotic symmetries, therefore global conditions (scattering theory, low frequency analysis)

$\Rightarrow$ Quantum effects on curved spacetimes!



Stephen Hawking

William Unruh

Stephen A. Fulling

Robert Wald

II. Setting and main result

Massless Dirac operator

Let (M, g) Lorentzian spacetime with global frame (e_0, e_1, e_2, e_3) of TM.

With the e_i we can construct a spinor bundle $S \xrightarrow{\pi} M$ and

(1) $\gamma : C^\infty(M; TM) \to C^\infty(M; End(S))$ such that

$$\gamma(X)\gamma(Y) + \gamma(Y)\gamma(X) = 2X \cdot gY\mathbf{1}, \quad X, Y \in C^\infty(M; TM),$$

(2) non-degenerate $\beta \in C^\infty(M; End(S, S^*))$

$$\gamma(X)^*\beta = -\beta\gamma(X), \quad i\beta\gamma(e) > 0, \ e \text{ timelike, future directed.}$$

(3) a connection ∇^S such that:

$$\nabla^S_X(\gamma(Y)\psi) = \gamma(\nabla_X Y)\psi + \gamma(Y)\nabla^S_X\psi.$$

$$\not{D} = g^{\mu\nu}\gamma(e_\mu)\nabla^S_{e_\nu}$$

The advantage of $\not{D}$ (massless) over $\not{D} + \lambda$ with $\lambda \neq 0$ (massive) is conformal invariance: $g \to c^2 g$ corresponds to $\not{D} \to c^{-2}\not{D}c$.

Second-quantized Dirac fields

$$(\text{second-quantized } \not{D}\hat{\phi} = 0) \leftrightarrow (\text{specific } \textbf{state } C^{\pm})$$

A **state** is a pair $C^{\pm} : \mathrm{Ker}_{L^2} \not{D} \to \mathrm{Ker}_{L^2} \not{D}$ such that

$$C^{+} + C^{-} = \mathbf{1}, \quad C^{\pm} \geq 0.$$

$C^{\pm}$ is pure if $(C^{\pm})^2 = C^{\pm}$.

Examples if (M, g) has time-like Killing vector field ∂_t:

(1) $C^{\pm} = \mathbf{1}_{\mathbb{R}^{\pm}}(D_t)$ is the vacuum w.r.t. ∂_t (it is pure)

(2) $C^{\pm} = (1 + e^{\mp \beta D_t})^{-1}$ is the thermal state at temperature $T = \beta^{-1}$ w.r.t. ∂_t (it is mixed)

Both (1) and (2) are Hadamard states. These are states which look microlocally like vacuum states on Minkowski, they also permit to renormalize the quantum energy momentum tensor.

Hadamard states

Let

$$\mathcal{N} := \{(x,\xi) \in T^*M \setminus o \,:\, \xi \cdot g^{-1}(x)\xi = 0\},$$
$$\mathcal{N}^{\pm} := \mathcal{N} \cap \{(x,\xi) \in T^*M \setminus o \,:\, \pm v\cdot\xi > 0 \;\forall v \in T_x^+M\},$$
T_x^+M : future directed timelike vectors

Proposition

Suppose that for all $\phi \in \mathrm{Ker}_{L^2} \not{D}$, $\mathrm{WF}(C^{\pm}\phi) \subset \mathcal{N}^{\pm}$. Then $C^{\pm}$ is a Hadamard state.

Remark

1. *Condition on L^2 solutions rather than on distributional bisolutions as in the definition given by Radzikowski.*
2. *Proof relies on the use of oscillatory test functions.*
3. *Non existence theorems by Kay, Wald and Pinamonti, Sanders, Verch for a Hadamard state invariant by a Killing field that is not everywhere timelike.*

Kerr spacetime (black hole exterior)

Kerr spacetime (M, g) solves vacuum Einstein equations and models rotating black hole.

exterior region : $\mathrm{M_I} = \mathbb{R}_t \times]r_+, +\infty[_r \times \mathbb{S}^2_{\theta,\varphi}$ with

$$g = -\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{4aMr\sin^2\theta}{\rho^2}\, dt\, d\varphi + \frac{\rho^2}{\Delta}\, dr^2 + \rho^2 d\theta^2 + \frac{\sigma^2}{\rho^2}\sin^2\theta\, d\varphi^2.$$

where

$$\Delta = r^2 - 2Mr + a^2, \quad \rho^2 = r^2 + a^2\cos^2\theta,$$

$$\sigma^2 = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta = (r^2 + a^2)\rho^2 + 2a^2Mr\sin^2\theta,$$

and $r_\pm = M \pm \sqrt{M^2 - a^2}$ roots of $\Delta(r)$, and small rotation $a > 0$.

Killing vector fields $v_{\mathscr{I}} = \partial_t$ et $v_{\mathscr{H}} = \partial_t + \Omega\partial_\varphi$.

⚠ none is everywhere time-like!

Kerr spacetime (black hole exterior & interior)

To future $\{r = r_+\}$ one glues to $\mathrm{M_I}$ the black hole interior $\mathrm{M_{II}}$.

Better coordinates (κ : surface gravity of the horizon).

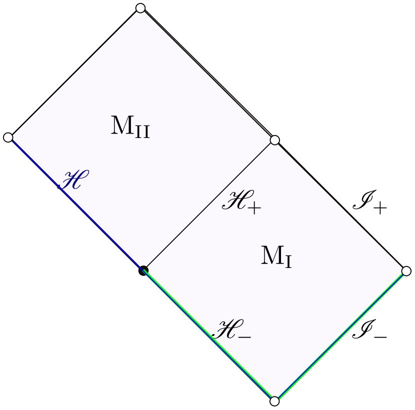
$$U = \mathrm{e}^{-\kappa\, {}^*t}, \; V = \mathrm{e}^{\kappa t^*}, \;\; \text{sur } \mathrm{M_I},$$
$$U = -\mathrm{e}^{-\kappa\, {}^*t}, \; V = \mathrm{e}^{\kappa t^*} \text{ sur } \mathrm{M_{II}},$$

with $t^* = t + r_*(r)$, $\dot{t}^* = 0$ along incoming principal null geodesics, ${}^*t = t - r_*(r)$, ${}^*\dot{t} = 0$ along outgoing principal null geodesics. Then,

$$v_{\mathscr{I}} = \partial_t = \kappa(-U\partial_U + V\partial_V) - \Omega\partial_\varphi,$$
$$v_{\mathscr{H}} = \partial_t + \Omega\partial_\varphi = \kappa(-U\partial_U + V\partial_V),$$

where $\Omega = \frac{a}{r_+^2 + a^2}$ (angular velocity of the horizon).

At past horizon $\{V = 0\}$, $\; v_{\mathscr{H}} = \partial_t + \Omega\partial_\varphi = -\kappa U\partial_U$.



(conformally rescaled) **Kerr spacetime**, $M_{I \cup II} = M_I \cup M_{II}$

M_I = black hole exterior

M_{II} = black hole interior

$\mathscr{H}_\pm$ = future/past horizon of M_I, $\mathscr{H}$ = extension of $\mathscr{H}_-$

$\mathscr{I}_\pm$ = null future/past infinity of M_I

Main result

Consider $\not{D}\phi = 0$. We define a pure state on $\mathrm{M}_{\mathrm{I}\cup\mathrm{II}}$ by taking:

on $\mathscr{H}$ we take $\mathbf{1}_{\mathbb{R}_\pm}(-D_U)$ ("Kay–Wald vacuum")
on $\mathscr{I}_-$ we take $\mathbf{1}_{\mathbb{R}_\pm}(D_{t^*})$ (asymptotic vacuum)

Theorem

For $|a| << 1$, the so-obtained Unruh state is pure and Hadamard in $\mathrm{M}_{\mathrm{I}\cup\mathrm{II}}$. Its restriction to M_{I} is asymptotically thermal with respect to $v_{\mathscr{H}}$ at the past horizon $\mathscr{H}_-$ with temperature equal to the Hawking temperature $T_{\mathrm{H}} = \frac{\kappa}{2\pi}$.

Remark: ∂_U is *not* Killing! Yet Hadamard condition and symmetries of the problem impose this choice. Recall Hadamard condition:

$$\mathrm{WF}(C^\pm\phi) \subset \mathcal{N}^\pm \quad \text{for all solutions of } \not{D}\phi = 0$$

Interpretation: $:\phi^2:$ doesn't blow up at $\mathscr{H}_+$, "smooth" extendability across $\mathscr{H}_+$.

Emergence of the Hawking temperature

- Take D_x in $L^2(\mathbb{R})$.
- Restrict $\mathbf{1}_{\mathbb{R}_+}(D_x)$ to $L^2(]0,+\infty[)$.
- Consider also $xD_x + D_x x$ in $L^2(]0,+\infty[)$.

There exists f such that on $L^2(]0,+\infty[)$:

$$\boxed{f(xD_x + D_x x) = \mathbf{1}_{\mathbb{R}_+}(D_x)} \; !$$

And this is $f(s) = (1 + \mathrm{e}^{-\pi s})^{-1}$ from the definition of a thermal state !

$\Rightarrow$ In our Kerr horizon situation, $\mathbf{1}_{\mathbb{R}_+}(-D_U)$ is thermal for the Killing vector that acts as $-\frac{\kappa}{2}(U\partial_U + \partial_U U)$ on $\mathscr{H}$.

Link to the Hawking effect

One considers the collapse of a star to a black hole. Suppose that at $t = 0$ the vector field ∂_t is timelike outside the surface of the star. We then can define a vacuum state in an unambiguous way. We impose totally reflecting boundary conditions on the surface of the star. In the dynamical situation, does there exist a limit state ?

Theorem (H '09)

The limit state exists and it is the Unruh state.

Remark

The effect was first observed by Hawking in '75. The first mathematical theorem is due to Bachelot '99 (spherical symmetry).

III. Scattering theory

Reductions

1. Weyl equation ($\mathscr{W}_{\mathrm{e}}$ even Weyl spinors).

$$\mathbb{S} := \mathscr{W}_{\mathrm{e}}^*,\ \Gamma(X) = \beta\gamma(X), \mathbb{D} = g^{\mu\nu}\Gamma(e_\mu)\nabla^S_{e_\nu}.$$

   Weyl equation $\mathbb{D}\phi = 0$.

2. Conserved current

$$(\phi_1|\phi_2)_{\mathbb{D}} = i\int_S \bar{\phi_1}\cdot\Gamma(\nu)\phi_2\, d\mathrm{vol}_h.$$

   is independent on S spacelike (h induced metric, ν normal).

3. Tetrads. With suitable choice of a Newman Penrose tetrad, the Weyl equation reads :

$$i\partial_t\Psi = H\Psi.$$

   where $\Psi = (\psi_0, \psi_1)$ are the components of the spinor in an associated spin frame.

Asymptotic velocity

Theorem (Nicolas-H '03)

There exists a selfadjoint operator $v \in B(\mathscr{H})$, called the past asymptotic velocity *such that:*

$$\chi(v) = \mathrm{s-}\lim_{t \to -\infty} \mathrm{e}^{-itH} \chi\left(\frac{r_*}{t}\right) \mathrm{e}^{itH}, \quad \forall \chi \in C_{\mathrm{c}}^{\infty}(\mathbb{R}).$$

The spectrum of v is $\mathrm{sp}(v) = \{-1, 1\}$.

We set

$$\pi_{\mathscr{H}_-} := \mathbf{1}_{\{1\}}(v), \quad \pi_{\mathscr{I}_-} := \mathbf{1}_{\{-1\}}(v).$$

Asymptotic completeness

Proposition

1. *For* $\phi \in \mathrm{Sol}_{\mathrm{sc}}(\mathrm{M_I})$, *the trace* $\mathrm{T}_{\mathscr{H}_-}\phi = \phi_{|\mathscr{H}_-} \in C^\infty(\mathscr{H}_-;\mathbb{C}^2)$ *is well defined and uniquely extends to a bounded operator* $\mathrm{T}_{\mathscr{H}_-} : \mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I}) \to \mathrm{L}^2(\mathscr{H}_-)$.
2. *For* $\phi \in \mathrm{Sol}_{\mathrm{sc}}(\mathrm{M_I})$ *the trace* $\mathrm{T}_{\mathscr{I}_-}\phi := \hat{\phi}_{|\mathscr{I}_-}, \quad \hat{\phi} = r\phi \in \mathrm{Sol}_{\mathrm{sc}}(\hat{\mathbb{D}})$ *is well defined and uniquely extends to a bounded operator* $\mathrm{T}_{\mathscr{I}_-} : \mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I}) \to \mathrm{L}^2(\mathscr{I}_-)$.

Theorem (H-Nicolas '03)

The map $\mathrm{T}_{\mathrm{M_I}} = \mathrm{T}_{\mathscr{H}_-} \oplus \mathrm{T}_{\mathscr{I}_-}$ *from* $\mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I})$ *to* $\mathrm{L}^2(\mathscr{H}_-) \oplus \mathrm{L}^2(\mathscr{I}_-)$ *is unitary.*

IV. Elements of the proof

Construction of the Unruh state

- We restrict our discussion to block I.
- In $\mathrm{M_I}$ we construct the Unruh state on $\mathrm{Ker}_{L^2} \not{D}$ by

$$C^+ = \mathrm{P}_{\mathscr{H}_-}\, f(i^{-1}\mathscr{L}_{\mathscr{H}}) + \mathrm{P}_{\mathscr{I}_-}\, \mathbf{1}_{\mathbb{R}^+}(i^{-1}\mathscr{L}_{\mathscr{I}}),$$

  where $\mathrm{P}_{\mathscr{H}_-}$ and $\mathrm{P}_{\mathscr{I}_-}$ project to solutions that go to $\mathscr{H}_-$ and $\mathscr{I}_-$. $\mathscr{L}_{\mathscr{H}}$ and $\mathscr{L}_{\mathscr{I}}$ are Lie derivatives of spinors along the vector fields $v_{\mathscr{H}}$ and $v_{\mathscr{I}}$.
- Idea : estimate WF of $C^+\phi$ in terms of wavefront set on $\mathscr{H}_-$, $\mathscr{I}_-$ using reconstruction formulae :

$$\phi(x) = -\int_S \mathbb{G}(x,y)\Gamma(g^{-1}\nu)(y)\phi(y)i_l^*(d\mathrm{vol}_g)(y).$$

Here $\mathbb{G}$ is the causal propagator, $TS = \mathrm{Ker}\,\nu$, l transverse to S, $\nu\cdot l = 1$. By scattering theory this kind of formulae can be extended to L^2 solutions.

A Key Proposition

Proposition

Let (M, g) be an oriented and time oriented Lorentzian manifold of dimension n, and let $S \subset M$ be a null hypersurface equipped with a smooth density dm. For $u \in \mathscr{E}'(S)$ we define $\delta_S \otimes u \in \mathscr{E}'(M)$ by:

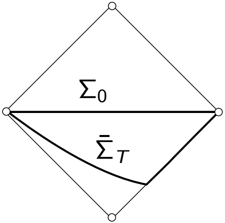
$$\int_M (\delta_S \otimes u)\varphi \, d\mathrm{vol}_g := \int_S u\varphi \, dm, \quad \varphi \in C_{\mathrm{c}}^\infty(M).$$

Let also X be a vector field on M, tangent to S, null, future directed on S and suppose $G \in \mathscr{D}'(M \times M)$ satisfies $\mathrm{WF}(G)' \subset \{(q, q') \,:\, q \sim q'\}$. Then for any $u \in \mathscr{E}'(M)$ one has the implication:

$$\begin{aligned} &\mathrm{WF}(u) \subset \{(y, \eta) \in T^*S \setminus o \,:\, \pm\eta\cdot X(y) \geq 0\} \\ &\quad \Rightarrow \mathrm{WF}(G(\delta_S \otimes u)) \cap \pi^{-1}(M \setminus S) \subset \mathcal{N}^{\pm}. \end{aligned}$$

Refinement of a strategy initiated by Moretti.

Choices of surfaces



Null geodesics that do not reach $\mathscr{H}_-$ nor $\mathscr{I}_-$ are still problematic.

- However, we can use special form $f(i^{-1}\mathscr{L}_{\mathscr{H}})$ and $\mathbf{1}_{\mathbb{R}+}(i^{-1}\mathscr{L}_{\mathscr{I}})$ to control wavefront set in region where $v_{\mathscr{H}}$ and $v_{\mathscr{H}}$ are time-like.
- If $|a| \ll 1$, then *all bad null geodesics reach the time-like regions*, so we can use propagation of singularities.

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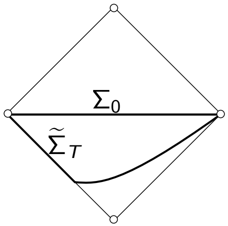

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- If $\underline{|a| \ll 1}$, then *all bad null geodesics reach the time-like regions*, so we can use propagation of singularities.

A bit of bibliography

- **Hadamard states from scattering data** (asymptotically Minkowski/asymptotically de Sitter/asymptotically static)

  [Hollands '00], [Moretti '06-'08], [Dappiaggi–Pinamonti–Moretti '09], [Dappiaggi–Siemssen '13], [Benini–Dappiaggi–Murro '14], [Gérard–Wrochna '16–'17], [Vasy–Wrochna '18]

- **Non-rotating (eternal) black holes** (Schwarzschild, Schwarzschild-de Sitter, etc.)

  [Dappiaggi–Moretti–Pinamonti '11] , [Sanders '15], [Gérard '20], [Hollands–Wald–Zahn '20].

- **Non-existence theorems on rotating black holes** (Kerr, Kerr-de Sitter) [Kay–Wald '92], [Pinamonti–Sanders–Verch '18]

- **De Sitter Kerr (bosons)** [Klein '22]

The Unruh state is the starting point for Hawking effect:

- **massless Dirac on Kerr** [H '09] (cf. (De Sitter) Schwarzschild [Bachelot '99], [Drouot '17])

Thank you for your attention!