Radiative transfer equation for surface and body waves

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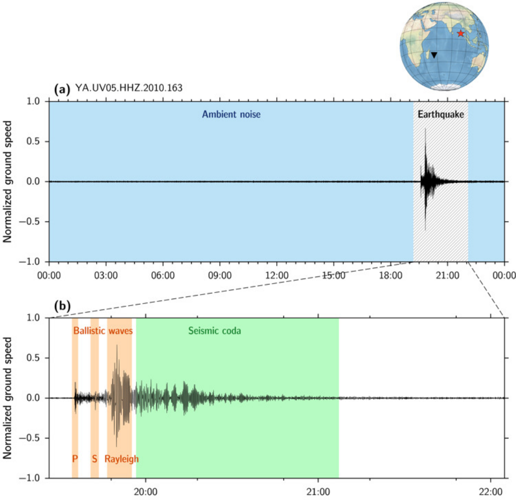
Motivations

• Long-term objective: estimate physical quantities in Earth's interior based on seismic coda waves (multiply scattered waves) or seismic ambient noise, recorded at Earth's surface.



• Short-term objective: derive a radiative transfer equation appropriate for seismology (in a randomly heterogeneous elastic half-space).

Motivations

- Radiative transfer theory describes the propagation of wave energy in random media in the following regime:
  - Medium with weak fluctuations with correlation length $\sim$ wavelength
  - Propagation distance $\gg$ wavelength.
- Radiative transfer in open space (proposed phenomenologically by Chandrasekhar) is rather well established.
  The energy density (specific intensity) $W(t, \boldsymbol{x}, \boldsymbol{\kappa})$ satisfies
  $$\partial_t W + \boldsymbol{v}(\boldsymbol{\kappa}) \cdot \nabla_{\boldsymbol{x}} W = \int \sigma(\boldsymbol{\kappa}, \boldsymbol{\kappa}') W(\boldsymbol{\kappa}') d\boldsymbol{\kappa}' - \Lambda(\boldsymbol{\kappa}) W(\boldsymbol{\kappa})$$
  with the scattering cross section and the extinction rate related by
  $$\int \sigma(\boldsymbol{\kappa}, \boldsymbol{\kappa}') d\boldsymbol{\kappa}' = \Lambda(\boldsymbol{\kappa})$$
  so that
  $$\partial_t W + \boldsymbol{v}(\boldsymbol{\kappa}) \cdot \nabla_{\boldsymbol{x}} W = \int \sigma(\boldsymbol{\kappa}, \boldsymbol{\kappa}') \big[W(\boldsymbol{\kappa}') - W(\boldsymbol{\kappa}) \big] d\boldsymbol{\kappa}'$$
  [Keller, Papanicolaou, Ryzhik, 90's]

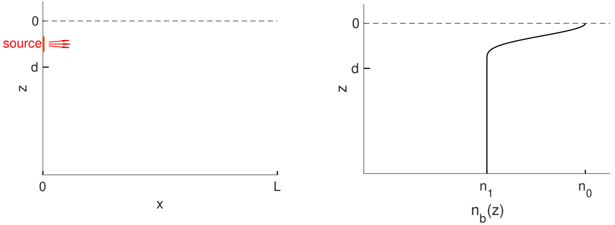
Motivations

- Boundary effects are intensively studied.
  No satisfactory radiative transfer theory for a elastic half-space.
- Idea: Derive a system of radiative transfer equations for coupled "surface" and "body" waves in a scalar approximation [Margerin 19].
- Framework: A two-dimensional scalar model involving a thick waveguide or a half-space with a thin surface layer and random heterogeneities.

A half-space containing a non-scattering thin layer

$$\frac{\mathrm{n}_\mathrm{b}^2(z)}{c_o^2}\partial_t^2 u - \Delta u = \delta(x) f(z;t), \quad (x,z) \in \mathbb{R} \times (0,+\infty),$$

with $u(x,z=0;t)=0$, $\Delta = \partial_x^2 + \partial_z^2$ and n_b non-increasing



$n_0 > n_1$

Take Fourier transform: $\hat{u}(x,z;\omega) = \int_\mathbb{R} u(x,z;t)\exp(i\omega t)dt$

$$\Delta \hat{u} + k^2 \mathrm{n}_\mathrm{b}^2(z)\hat{u} = -\delta(x)\hat{f}(z;\omega), \quad (x,z) \in \mathbb{R} \times (0,+\infty),$$

with $k = \omega / c_o$.

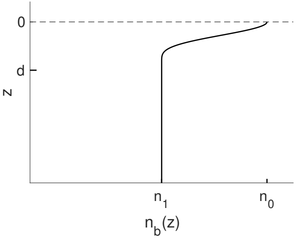
Fix ω. Denote $k = \omega / c_o$.

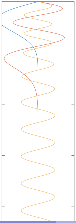
Spectral problem associated to the one-dimensional Helmholtz equation

$$[\partial_z^2 + k^2 \mathrm{n}_\mathrm{b}^2(z)]\phi(z) = \gamma \phi(z), \quad z \in (0, +\infty)$$

with Dirichlet boundary condition at $z = 0$:

- The spectrum is of the form $(-\infty, n_1^2 k^2) \cup \{\beta_{N-1}^2, \dots, \beta_0^2\}$.
- The N modal wavenumbers β_j are in $(n_1 k, n_0 k)$.
- The functions ϕ_j, $j = 0, \dots, N-1$, are the modes corresponding to the discrete spectrum. They decay exponentially in z for $z > d$.
- The functions ϕ_γ, $\gamma \in (-\infty, n_1^2 k^2)$, are the modes corresponding to the continuous spectrum. They are oscillatory and bounded at infinity.





Fix ω. Denote $k = \omega / c_o$.

Spectral problem associated to the one-dimensional Helmholtz equation

$$[\partial_z^2 + k^2 \mathrm{n}_\mathrm{b}^2(z)]\phi(z) = \gamma \phi(z), \quad z \in (0, +\infty)$$

with Dirichlet boundary condition at $z = 0$:

- The set of modes is complete.
  Any function v can be expanded on this complete set:

$$v(z) = \sum_{j=0}^{N-1} v_j \phi_j(z) + \int_{-\infty}^{n_1^2 k^2} v_\gamma \phi_\gamma(z) d\gamma,$$

  with $v_j = \int_0^{+\infty} \phi_j(z) v(z) dz$ and $v_\gamma = \int_0^{+\infty} \phi_\gamma(z) v(z) dz$.

Cf. Magnanini and Santosa 00 (using the Levitan-Levinson transform method).

- The solution of the 2D Helmholtz equation

$$\Delta \hat{u}(x,z) + k^2 \mathrm{n}_{\mathrm{b}}^2(z)\hat{u}(x,z) = -\delta(x)\hat{f}(z)$$

can be expanded as:

$$\hat{u}(x,z) = \sum_{j=0}^{N-1} \hat{u}_j(x)\phi_j(z) + \int_{-\infty}^{n_1^2 k^2} \hat{u}_\gamma(x)\phi_\gamma(z)d\gamma.$$

- Project the 2D Helmholtz equation onto ϕ_j (or ϕ_γ):

$$\int_0^\infty dz\phi_j(z)\big[(\partial_x^2 + \partial_z^2)\hat{u}(x,z) + k^2 \mathrm{n}_{\mathrm{b}}^2(z)\hat{u}(x,z)\big] = -\int_0^\infty dz\phi_j(z)\delta(x)\hat{f}(z)$$

$\hookrightarrow$ The complex mode amplitudes satisfy uncoupled equations:

$$\partial_x^2 \hat{u}_j + \beta_j^2 \hat{u}_j = -\delta(x)\int_0^\infty \hat{f}(z)\phi_j(z)dz, \quad j = 0,\dots,N-1,$$

$$\partial_x^2 \hat{u}_\gamma + \gamma\hat{u}_\gamma = -\delta(x)\int_0^\infty \hat{f}(z)\phi_\gamma(z)dz, \quad \gamma \in (-\infty, n_1^2 k^2).$$

- Therefore we have for $x \in (0, +\infty)$:

$$\hat{u}(x,z) = \sum_{j=0}^{N-1} \frac{a_{j,\mathrm{s}}}{\sqrt{\beta_j}} e^{i\beta_j x} \phi_j(z) + \int_0^{n_1^2 k^2} \frac{a_{\gamma,\mathrm{s}}}{\gamma^{1/4}} e^{i\sqrt{\gamma}x} \phi_\gamma(z) d\gamma$$
$$+ \int_{-\infty}^{0} \frac{a_{\gamma,\mathrm{s}}}{|\gamma|^{1/4}} e^{-\sqrt{|\gamma|}x} \phi_\gamma(z) d\gamma,$$

where $a_{j,\mathrm{s}}$, $a_{\gamma,\mathrm{s}}$ are constant and determined by the source:

$$a_{j,\mathrm{s}} = \frac{\sqrt{\beta_j}}{2} \int_0^\infty \phi_j(z) \hat{f}(z) dz, \quad j = 0, \dots, N-1,$$
$$a_{\gamma,\mathrm{s}} = \frac{|\gamma|^{1/4}}{2} \int_0^\infty \phi_\gamma(z) \hat{f}(z) dz, \quad \gamma \in (-\infty, n_1^2 k^2).$$

- The modes for $0 \le j \le N-1$ are guided ("surface modes"), the modes for $\gamma \in (0, n_1^2 k^2)$ are radiating ("body modes"), the modes for $\gamma \in (-\infty, 0)$ are evanescent.

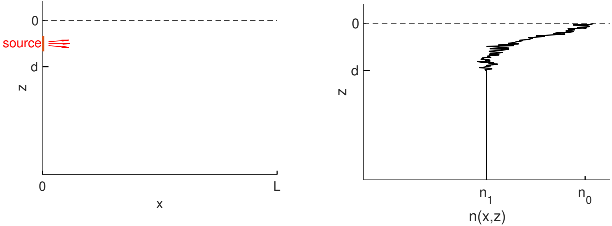
A half-space containing a scattering thin layer

$$\frac{n^2(x,z)}{c_o^2}\partial_t^2 u - \Delta u = \delta(x) f(z;t), \quad (x,z) \in \mathbb{R} \times (0,+\infty).$$

We consider that the thin layer is scattering

$$n^2(x,z) = \mathrm{n}_\mathrm{b}^2(z) + \varepsilon \nu(x,z),$$

where ν is a zero-mean, bounded, random process, stationary in x, with integrable in x covariance, compactly supported in z:



- We still have

$$\hat{u}(x,z) = \sum_{j=0}^{N-1} \hat{u}_j(x)\phi_j(z) + \int_{-\infty}^{n_1^2 k^2} \hat{u}_\gamma(x)\phi_\gamma(z)d\gamma,$$

but the $\hat{u}_j(x)$, $\hat{u}_\gamma(x)$ satisfy coupled equations (for $x > 0$):

$$\partial_x^2 \hat{u}_j + \beta_j^2 \hat{u}_j = -\varepsilon k^2 \sum_{l=0}^{N-1} C_{j,l}(x)\hat{u}_l - \varepsilon k^2 \int_{-\infty}^{n_1^2 k^2} C_{j,\gamma'}(x)\hat{u}_{\gamma'} d\gamma',$$

for $j = 0, \ldots, N-1$,

$$\partial_x^2 \hat{u}_\gamma + \gamma \hat{u}_\gamma = -\varepsilon k^2 \sum_{l=0}^{N-1} C_{\gamma,l}(x)\hat{u}_l - \varepsilon k^2 \int_{-\infty}^{n_1^2 k^2} C_{\gamma,\gamma'}(x)\hat{u}_{\gamma'} d\gamma',$$

for $\gamma \in (-\infty, n_1^2 k^2)$, where

$$C_{j,l}(x) = \int_0^{+\infty} \phi_j(z)\phi_l(z)\nu(x,z)dz,$$

and similarly for $C_{j,\gamma'}(x)$, $C_{\gamma,l}(x)$, $C_{\gamma,\gamma'}(x)$.

- We introduce the forward-going and backward-going mode amplitudes

$$a_j(x),\ b_j(x),\ j=0,\dots,N-1$$

which are defined such that

$$\hat{u}_j(x) = \frac{1}{\sqrt{\beta_j}}\Big(a_j(x)e^{i\beta_j x} + b_j(x)e^{-i\beta_j x}\Big),$$

$$\partial_x \hat{u}_j(z) = i\sqrt{\beta_j}\Big(a_j(x)e^{i\beta_j x} - b_j(x)e^{-i\beta_j x}\Big),$$

The mode amplitudes satisfy

$$\partial_x a_j(x) = \frac{i\varepsilon k^2}{2}\sum_{l'=0}^{N-1}\frac{C_{j,l'}(x)}{\sqrt{\beta_{l'}\beta_j}}\Big[a_{l'}(x)e^{i(\beta_{l'}-\beta_j)x} + b_{l'}(x)e^{i(-\beta_{l'}-\beta_j)x}\Big]$$
$$+ \frac{i\varepsilon k^2}{2}\int_0^{n_1^2k^2}\frac{C_{j,\gamma'}(x)}{\sqrt[4]{\gamma'}\sqrt{\beta_j}}\Big[a_{\gamma'}(x)e^{i(\sqrt{\gamma'}-\beta_j)x} + b_{\gamma'}(x)e^{i(-\sqrt{\gamma'}-\beta_j)x}\Big]d\gamma'$$
$$+ \frac{i\varepsilon k^2}{2}\int_{-\infty}^{0}\frac{C_{j,\gamma'}(x)}{\sqrt{\beta_j}}\hat{u}_{\gamma'}(x)e^{-i\beta_j x}d\gamma'$$

Idem for $b_j(x)$, $a_\gamma(x)$, $b_\gamma(x)$.

- Consider large propagation distances

$$a_j^\varepsilon(x) = a_j\Big(\frac{x}{\varepsilon^2}\Big), \quad b_j^\varepsilon(x) = b_j\Big(\frac{x}{\varepsilon^2}\Big), \quad j = 0, \ldots, N-1$$

The normalized mode amplitudes satisfy

$$\begin{aligned}
\partial_x a_j^\varepsilon(x) = & \frac{ik^2}{2\varepsilon} \sum_{l'=0}^{N-1} \frac{C_{j,l'}\big(\frac{x}{\varepsilon^2}\big)}{\sqrt{\beta_{l'}\beta_j}} \Big[a_{l'}^\varepsilon(x) e^{i(\beta_{l'}-\beta_j)\frac{x}{\varepsilon^2}} + b_{l'}^\varepsilon(x) e^{i(-\beta_{l'}-\beta_j)\frac{x}{\varepsilon^2}} \Big] \\
& + \frac{ik^2}{2\varepsilon} \int_0^{n_1^2 k^2} \frac{C_{j,\gamma'}\big(\frac{x}{\varepsilon^2}\big)}{\sqrt[4]{\gamma'}\sqrt{\beta_j}} \Big[a_{\gamma'}^\varepsilon(x) e^{i(\sqrt{\gamma'}-\beta_j)\frac{x}{\varepsilon^2}} + b_{\gamma'}^\varepsilon(x) e^{i(-\sqrt{\gamma'}-\beta_j)\frac{x}{\varepsilon^2}} \Big] d\gamma' \\
& + \frac{ik^2}{2\varepsilon} \int_{-\infty}^{0} \frac{C_{j,\gamma'}\big(\frac{x}{\varepsilon^2}\big)}{\sqrt{\beta_j}} \hat{u}_{\gamma'}^\varepsilon(x) e^{-i\beta_j \frac{x}{\varepsilon^2}} d\gamma'
\end{aligned}$$

Idem for $b_j^\varepsilon(x)$, $a_\gamma^\varepsilon(x)$, $b_\gamma^\varepsilon(x)$.
$\hookrightarrow$ Application of diffusion-approximation theory gives the statistics of the mode amplitudes in the limit $\varepsilon \to 0$.

The coherent field

- Quantity of interest: mean normal derivative of the field at the surface.
- The coherent (mean) field is of the form

$$\mathbb{E}\big[\partial_z \hat{u}(\frac{x}{\varepsilon^2}, 0)\big] = \sum_{j=0}^{N-1} \frac{a_{j,\mathrm{coh}}(x)}{\sqrt{\beta_j}} e^{i\beta_j \frac{x}{\varepsilon^2}} \partial_z \phi_j(0)$$

The mean mode amplitudes decay exponentially

$$a_{j,\mathrm{coh}}(x) = a_{j,\mathrm{s}} \exp\Big(-\frac{x}{\ell_{j,\mathrm{sca}}} + i\kappa_{j,\mathrm{sca}} x\Big)$$

The mode-dependent scattering mean free path $\ell_{j,\mathrm{sca}}$ can be expressed in terms of the two-point statistics of the fluctuations of the random medium.

The Wigner transform

• Quantity of interest: Wigner transform of the normal derivative of the field at the surface

$$
\begin{aligned}
W^{\rm s}(x,\kappa;t,\omega) &= \lim_{\varepsilon\to 0} \frac{1}{2\pi} \iint d\omega' dx' \exp(-i\omega' t - i\kappa x') \\
&\quad \times \mathbb{E}\Big[\partial_z \hat{u}\big(\frac{x}{\varepsilon^2}+\frac{x'}{2},0;\omega+\frac{\varepsilon^2}{2}\omega'\big)\partial_z \overline{\hat{u}}\big(\frac{x}{\varepsilon^2}-\frac{x'}{2},0;\omega-\frac{\varepsilon^2}{2}\omega'\big)\Big] \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \iint dt' dx' \exp(i\omega t' - i\kappa x') \\
&\quad \times \mathbb{E}\Big[\partial_z u\big(\frac{x}{\varepsilon^2}+\frac{x'}{2},0;\frac{t}{\varepsilon^2}+\frac{t'}{2}\big)\partial_z u\big(\frac{x}{\varepsilon^2}-\frac{x'}{2},0;\frac{t}{\varepsilon^2}-\frac{t'}{2}\big)\Big].
\end{aligned}
$$

Mathematically: local Fourier transform of the field covariance.
Physically: energy density at time t/ε^2 and frequency ω that arrives at x/ε^2 with the angle determined by the longitudinal wavenumber κ.

- The Wigner transform $W^{\rm s}$ is of the form:

$$W^{\rm s}(x,\kappa;t,\omega) = \sum_{j=0}^{N(\omega)-1} \frac{\partial_z \phi_j(0;\omega)^2}{\beta_j(\omega)} W_j(x;t,\omega)\delta(\kappa-\beta_j(\omega)),$$

where the $W_j(x;t,\omega)$'s satisfy

$$\partial_x W_j + \frac{1}{v_j(\omega)}\partial_t W_j = \sum_{l=0,l\neq j}^{N(\omega)-1} \Gamma_{jl}(\omega) W_l - \Lambda_j(\omega) W_j .$$

- β_j is the phase velocity of the j-th surface mode,
- v_j is the group velocity of the j-th surface mode ($v_j(\omega) = 1/\partial_\omega \beta_j(\omega)$),
- Γ_{jl} is the scattering coefficient (power coming from the l-th surface mode to the j-th surface mode),
- Λ_j is the extinction coefficient, that takes into account leakage towards the body modes and scattering to other surface modes.

The scattering and extinction coefficients depend on the two-point statistics of the fluctuations of the random medium.

- The $W_j(x;t,\omega)$'s satisfy

$$\partial_x W_j + \frac{1}{v_j(\omega)}\partial_t W_j = \sum_{l=0,l\neq j}^{N(\omega)-1} \Gamma_{jl}(\omega) W_l - \Lambda_j(\omega) W_j,$$

with $v_j(\omega) = 1/\partial_\omega \beta_j(\omega)$,

$$\Gamma_{jl}(\omega) = \frac{k^4(\omega)}{2\beta_j\beta_l(\omega)} \int_0^\infty \mathcal{R}_{jl}(x;\omega) \cos\big((\beta_l(\omega) - \beta_j(\omega))x\big)\, dx,$$

$$\Lambda_j(\omega) = \sum_{l=0,l\neq j}^{N(\omega)-1} \Gamma_{jl}(\omega)$$

$$+ \int_0^{n_1^2 k^2(\omega)} \frac{k^4(\omega)}{2\sqrt{\gamma}\beta_j(\omega)} \int_0^\infty \mathcal{R}_{j\gamma}(x;\omega) \cos\big((\sqrt{\gamma} - \beta_j(\omega))x\big)\, dx d\gamma,$$

$$\mathcal{R}_{jl}(x;\omega) = \int_0^\infty \int_0^\infty \phi_j\phi_l(z;\omega)\mathbb{E}[\nu(0,z)\nu(x,z')]\phi_j\phi_l(z';\omega) dz dz',$$

$$\mathcal{R}_{j\gamma}(x;\omega) = \int_0^\infty \int_0^\infty \phi_j\phi_\gamma(z;\omega)\mathbb{E}[\nu(0,z)\nu(x,z')]\phi_j\phi_\gamma(z';\omega) dz dz'.$$

$\rightarrow$ Energy is lost to the body modes ($\Lambda_j(\omega) > \sum_{l=0,l\neq j}^{N(\omega)-1} \Gamma_{jl}(\omega)$).

An anomalous quasi-equilibrium

The mean mode powers

$$P_j(x;\omega) = \int_{-\infty}^{\infty} W_j(x;t,\omega)dt$$

satisfy

$$\partial_x P_j = \sum_{l=0,l\neq j}^{N-1} \Gamma_{jl} P_l - \Lambda_j P_j, \quad P_j(x=0;\omega) = P_{j,o}(\omega)$$

The solution $\boldsymbol{P}(x;\omega) = (P_j(x;\omega))_{j=1}^N$ is of the form

$$\boldsymbol{P}(x;\omega) = \exp\big(-\mathbf{M}(\omega)x\big)\boldsymbol{P}_o(\omega), \qquad M_{jl}(\omega) = \Lambda_j(\omega)\delta_{jl} - \Gamma_{jl}(\omega)(1-\delta_{jl})$$

Analysis of the matrix $\mathbf{M}$: it is diagonalizable, positive, with a simple minimal eigenvalue $\lambda_1 > 0$ with associated eigenvector $\boldsymbol{V}_1$. For large x:

$$\boldsymbol{P}(x;\omega) \simeq \exp\big(-\lambda_1(\omega)x\big)\big(\boldsymbol{P}_o(\omega)\cdot\boldsymbol{V}_1(\omega)\big)\boldsymbol{V}_1(\omega)$$

If $\Lambda_j \equiv \Lambda$, then $\boldsymbol{V}_1$ is uniform: equipartition of energy.
Otherwise $\boldsymbol{V}_1$ is not uniform !

An anomalous quasi-equilibrium

The mean mode power distribution

$$\frac{P_j(x;\omega)}{\sum_{l=0}^{N(\omega)-1} P_l(x;\omega)} \stackrel{x\to+\infty}{\longrightarrow} \frac{V_{1,j}(\omega)}{\sum_{l=0}^{N(\omega)-1} V_{1,l}(\omega)}$$

reaches a quasi-equilibrium state $\boldsymbol{V}_1(\omega)$ that is *not* the equipartitioned distribution state.

A half-space containing a scattering thin layer: summary

- The Wigner transform $W^{\rm s}$ is of the form:

$$W^{\rm s}(x,\kappa;t,\omega) = \sum_{j=0}^{N(\omega)-1} \frac{\partial_z \phi_j(0;\omega)^2}{\beta_j(\omega)} W_j(x;t,\omega)\delta(\kappa-\beta_j(\omega)),$$

where the $W_j(x;t,\omega)$'s satisfy

$$\partial_x W_j + \frac{1}{v_j(\omega)}\partial_t W_j = \sum_{l=0,l\neq j}^{N(\omega)-1} \Gamma_{jl}(\omega) W_l - \Lambda_j(\omega) W_j.$$

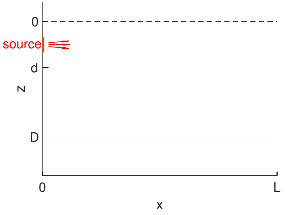
$\to$ Energy is lost to the body modes ($\Lambda_j(\omega) > \sum_{l=0,l\neq j}^{N(\omega)-1} \Gamma_{jl}(\omega)$).
$\to$ No effective coupling from body to surface modes.
$\to$ No non-zero equilibrium distribution.
$\to$ An anomalous quasi-equilibrium distribution (given by the first eigenvector of the matrix $\Lambda_j(\omega)\delta_{jl} - \Gamma_{jl}(\omega)(1-\delta_{jl})$).

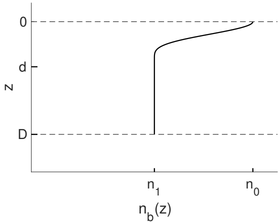
How to get a model with all types of coupling ?

A thick layer containing a thin non-scattering layer

$$\frac{\mathrm{n}_\mathrm{b}^2(z)}{c_o^2}\partial_t^2 u - \Delta u = \delta(x) f(z;t), \quad (x,z) \in \mathbb{R} \times (0,D),$$

with $u(x,z=0;t)=0$, $\partial_z u(x,z=D;t)=0$, and $\mathrm{n}_\mathrm{b}(z)$ of the form





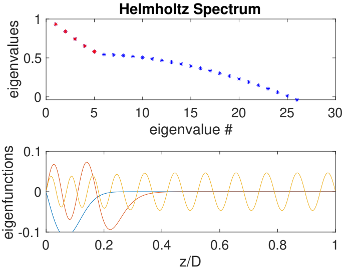
Take Fourier transform: $\hat{u}(x,z;\omega) = \int_{\mathbb{R}} u(x,z;t)\exp(i\omega t)dt$

$$\Delta\hat{u} + k^2 \mathrm{n}_\mathrm{b}^2(z)\hat{u} = -\delta(x)\hat{f}(z;\omega), \quad (x,z) \in \mathbb{R} \times (0,D),$$

with $k = \omega / c_o$.

Spectral problem associated to the one-dimensional Helmholtz equation $[\partial_z^2 + k^2 \mathrm{n}_\mathrm{b}^2(z)]\phi(z) = \gamma\phi(z)$ in $(0, D)$ with Dirichet boundary condition at $z = 0$ and Dirichlet or Neumann boundary condition at $z = D$:

- The spectrum is discrete. The eigenvalues are of the form $\gamma_{j,D}$ with $\cdots < \gamma_{j+1,D} < \gamma_{j,D} < \cdots < \gamma_{0,D} < n_0^2 k^2$.
- We denote N_D such that $\gamma_{N_D,D} \leq n_1^2 k^2 < \gamma_{N_D-1,D}$, we denote M_D such that $\gamma_{M_D,D} \leq 0 < \gamma_{M_D-1,D}$.
- For $j \leq M_D$ we write $\beta_{j,D}(\omega) = \sqrt{\gamma_{j,D}}$.
- The eigenfunctions are exponentially decaying in (d, D) for $j < N_D$ and oscillatory for $j \geq N_D$.



Here $d = 0.2D$, $N_D = 5,\ M_D = 26$.

$\phi_{j,D}(z)$ for $j = 1,\ 4,\ 20$.

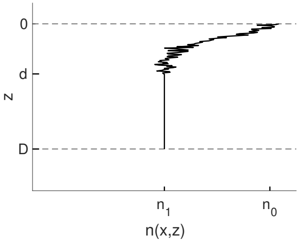
- The set of eigenfunctions is complete.

A thick layer containing a thin scattering layer

$$\frac{n^2(x,z)}{c_o^2}\partial_t^2 u - \Delta u = \delta(x) f(z;t), \quad (x,z) \in \mathbb{R} \times (0,D),$$

with $u(x,z=0;t)=0$, $\partial_z u(x,z=D;t)=0$, and $n(x,z)$ of the form

$$n^2(x,z) = \mathrm{n}_{\mathrm{b}}^2(z) + \varepsilon \nu(x,z)$$



• We can write a RTE for the truncated problem with a fixed D as $\varepsilon \to 0$.
In this truncated problem there are only discrete modes.

• Then we consider $D \to +\infty$.

The Wigner transform

- In the regime $\varepsilon \to 0$, the Wigner transform W^{s} is of the form:

$$W^{\mathrm{s}}_D(x,\kappa;t,\omega) = \sum_{j=0}^{M_D(\omega)-1} \frac{\partial_z \phi_{j,D}(0;\omega)^2}{\beta_{j,D}(\omega)} W_{j,D}(x;t,\omega)\delta\big(\kappa - \beta_{j,D}(\omega)\big),$$

where the $W_{j,D}(x;t,\omega)$'s satisfy

$$\partial_x W_{j,D} + \frac{1}{v_{j,D}(\omega)}\partial_t W_{j,D} = \sum_{l=0,l\neq j}^{M_D(\omega)-1} \Gamma_{jl,D}(\omega) W_{l,D} - \Lambda_{j,D}(\omega) W_{j,D}.$$

$\to$ Here $\Lambda_{j,D}(\omega) = \sum_{l=0,l\neq j}^{M_D(\omega)-1} \Gamma_{jl,D}(\omega)$ so no power is lost:

$$\partial_x W_{j,D} + \frac{1}{v_{j,D}(\omega)}\partial_t W_{j,D} = \sum_{l=0,l\neq j}^{M_D(\omega)-1} \Gamma_{jl,D}(\omega)\big(W_{l,D} - W_{j,D}\big).$$

The scattering coefficients $\Gamma_{jl,D}$ depend on the two-point statistics of the fluctuations of the random medium.

- When $kD \gg 1$ [Codington-Levinson 84],

$$W_D^{\rm s}(x,\kappa;t,\omega) \simeq \sum_{j=0}^{N(\omega)-1} \frac{\partial_z \phi_j(0;\omega)^2}{\beta_j(\omega)} W_j(x;t,\omega)\delta\big(\kappa-\beta_j(\omega)\big) + \frac{\partial_z \tilde{\phi}_\xi(0;\omega)^2}{k\xi} \tilde{\mathcal{N}}(\xi)\tilde{W}_\xi(x;t,\omega)\big|_{\xi=\kappa/k},$$

and the Wigner transforms $(W_j,\tilde{W}_\xi)$ satisfy the coupled RTEs:

$$\partial_x W_j + \frac{1}{v_j}\partial_t W_j = \sum_{l=0,l\neq j}^{N-1} \Gamma_{jl} W_l + \int_0^\infty \tilde{\Gamma}_{j\xi'} \tilde{W}_{\xi'} \tilde{\mathcal{N}}(\xi')d\xi' - \Lambda_j W_j,$$

$$\partial_x \tilde{W}_\xi + \frac{1}{v_\xi}\partial_t \tilde{W}_\xi = \frac{1}{kD}\sum_{l=0}^{N-1} \tilde{\Gamma}_{\xi l} W_l + \frac{1}{kD}\int_0^\infty \tilde{\Gamma}_{\xi\xi'} \tilde{W}_{\xi'} \tilde{\mathcal{N}}(\xi')d\xi' - \frac{1}{kD}\tilde{\Lambda}_\xi \tilde{W}_\xi,$$

for $j=0,\dots,N-1$ and $\xi \in (0,n_1)$, where the group velocities are

$$v_j(\omega) = \frac{1}{\partial_\omega \beta_j(\omega)}, \qquad v_\xi = \frac{c_o \xi}{n_1^2},$$

and the density of states is $\tilde{\mathcal{N}}(\xi) = \frac{\xi}{\pi\sqrt{n_1^2-\xi^2}} \mathbf{1}_{(0,n_1)}(\xi)$.

- The mean mode powers,

$$P_j(x;\omega) = \int_{-\infty}^{\infty} W_j(x;t,\omega)dt, \qquad \tilde{P}_\xi(x;\omega) = \int_{-\infty}^{\infty} \tilde{W}_\xi(x;t,\omega)dt,$$

satisfy

$$\partial_x P_j = \sum_{l=0, l\neq j}^{N-1} \Gamma_{jl} P_l + \int_0^\infty \tilde{\Gamma}_{j\xi'} \tilde{P}_{\xi'} \tilde{\mathcal{N}}(\xi') d\xi' - \Lambda_j P_j,$$

$$\partial_x \tilde{P}_\xi = \frac{1}{kD} \sum_{l=0}^{N-1} \tilde{\Gamma}_{\xi l} P_l + \frac{1}{kD} \int_0^\infty \tilde{\Gamma}_{\xi\xi'} \tilde{P}_{\xi'} \tilde{\mathcal{N}}(\xi') d\xi' - \frac{1}{kD} \tilde{\Lambda}_\xi \tilde{P}_\xi.$$

- The total power

$$\mathcal{P}(x;\omega) = \sum_{j=0}^{N(\omega)-1} P_j(x;\omega) + kD \int_0^\infty \tilde{P}_\xi(x;\omega) \tilde{\mathcal{N}}(\xi) d\xi$$

is a conserved quantity, that is, $\partial_x \mathcal{P} = 0$.

Short-range propagation

• The parameters Γ and $\tilde{\Gamma}$ are of order $k^2\sigma^2\ell_c$ where σ and ℓ_c are the standard deviation and the correlation length (in x) of the random fluctuations $\nu(x,z)$.

Assume a source that generates surface waves.

For propagation distances of the order of $1/(k^2\sigma^2\ell_c)$, the RTE for the surface modes can be reduced to

$$\partial_x W_j + \frac{1}{v_j}\partial_t W_j = \sum_{l=0, l\neq j}^{N-1} \Gamma_{jl} W_l - \Lambda_j W_j,$$

which is the RTE determined in the half-space case.

• For x of the order of $1/(k^2\sigma^2\ell_c)$,

$$\big(P_j(x)\big)_{j=0}^{N-1} \simeq \exp(-\mathbf{M}x)\big(P_j(0)\big)_{j=0}^{N-1},$$

where $\mathbf{M}$ is the positive matrix with entries

$$M_{jl} = \Lambda_j \delta_{jl} - \Gamma_{jl}(1-\delta_{jl}).$$

$\hookrightarrow$ the total mass carried by the surface modes decays.
$\hookrightarrow$ the distribution of the mean surface mode powers adopts a quasi-equilibrium state.

• The decay is in fact a transfer of power from the surface modes to the body modes, expressed by

$$\partial_x \tilde{P}_\xi = \frac{1}{kD} \sum_{l=0}^{N-1} \tilde{\Gamma}_{\xi l} P_l$$

$\hookrightarrow$ the mean body mode powers are determined by the leakage of the surface modes.

Medium-range propagation

• For propagation distances $x \sim D/(k\sigma^2\ell_c)$ the mean surface mode powers are in a quasi-equilibrium state that is determined by the mean body mode power distribution:

$$\left(P_j(x)\right)_{j=0}^{N-1} = \mathbf{M}^{-1}\left(\int_0^\infty \tilde{\Gamma}_{j\xi'}\tilde{P}_{\xi'}(x)\tilde{\mathcal{N}}(\xi')d\xi'\right)_{j=0}^{N-1}.$$

• The mean body mode powers $\tilde{P}_\xi$ satisfy an effective RTE at the scale $D/(k\sigma^2\ell_c)$:

$$\partial_x \tilde{P}_\xi = \frac{1}{kD}\int_0^\infty \sigma(\xi,\xi')\tilde{P}_{\xi'}d\xi' - \frac{1}{kD}\tilde{\Lambda}_\xi\tilde{P}_\xi,$$

$$\sigma(\xi,\xi') = \sum_{l,l'=0}^{N-1}\tilde{\Gamma}_{\xi l}(\mathbf{M}^{-1})_{ll'}\tilde{\Gamma}_{l'\xi'}\tilde{\mathcal{N}}(\xi') + \tilde{\Gamma}_{\xi\xi'}\tilde{\mathcal{N}}(\xi').$$

First term: conversion from a body mode ξ' to a body mode ξ mediated by surface modes $\xi' \mapsto l' \mapsto l \mapsto \xi$

Second term: direct conversion from a body mode ξ' to a body mode ξ.

Long-range propagation

- As $x \gg D/(k\sigma^2\ell_c)$, $P_j(x)$ and $\tilde{P}_\xi(x)$ converge to

$$\mathcal{P}_\infty(\omega) = \frac{\mathcal{P}(0;\omega)}{kD\int_0^\infty \tilde{\mathcal{N}}(\xi)d\xi} = \frac{\pi\mathcal{P}(0;\omega)}{n_1 kD}.$$

$\hookrightarrow$ Equipartition.
$\hookrightarrow$ Most of the power is carried by body modes (the fraction of power carried by the surface modes is of the order of d/D).
But: the surface modes play an important role in the scattering process.

Conclusions and perspectives

- Two-dimensional scalar toy model for seismic coda: a randomly heterogeneous half-space or thick waveguide that has a thin layer beneath the surface that supports a finite number of guided (surface) modes.
- An original RTE that involves:
  - coupling between all types of modes,
  - a nontrivial coupling mechanism between body modes mediated by surface modes.
  - a slowly evolving metastable surface mode distribution (which is not the equipartitioned distribution) which ultimately leads to energy equipartition between all modes.
- Next steps:
  - analyze the associated inverse problem: the background index of refraction can be robustly determined from the Wigner transform or related albedo operator.
  - address the three-dimensional, elastic system.