Asymptotics of dissipative eigenvalues and application to scattering

VESSELIN PETKOV

University of Bordeaux

Scattering and Inverse Scattering, Linz, November 2, 2022

Outline

1. Dissipative eigenvalues and application to scattering theory
2. Results
3. Dirichlet-to-Neumann map and trace formula
4. Idea of the proof of Theorem 3
5. The case $0 < \gamma(x) < 1$

Dissipative eigenvalues and application to scattering theory.

Dissipative eigenvalues. Let $K \subset \mathbb{R}^d$, $d \geq 2$, be a bounded non-empty domain and let $\Omega = \mathbb{R}^d \setminus \bar{K}$ be connected. We suppose that the boundary Γ of K is C^∞. Consider the boundary problem

$$\begin{cases} u_{tt} - \Delta_x u + c(x) u_t = 0 \text{ in } \mathbb{R}_t^+ \times \Omega, \\ \partial_\nu u - \gamma(x) u_t - \sigma(x) u = 0 \text{ on } \mathbb{R}_t^+ \times \Gamma, \\ u(0, x) = f_1, \; u_t(0, x) = f_2 \end{cases} \tag{1}$$

with initial data $f = (f_1, f_2)$ in the energy space $\mathcal{H} = H^1(\Omega) \times L^2(\Omega)$ with norm

$$\|f\| = \Big(\int_\Omega (|\nabla_x f_1|^2 + |f_2|^2) dx + \int_\Gamma \sigma(x) |f_1|^2 dS_x \Big)^{1/2}.$$

Here ν is the unit outward normal to Γ pointing into Ω, $\gamma(x) \geq 0$, $\sigma(x) \geq 0$ are C^∞ functions on Γ and $0 \leq c(x) \in C_0^\infty(\mathbb{R}^d)$.

The solution of (1) is given by $V(t)f = e^{tG} f,\ t \geq 0$, where

$V(t)$ is a contraction semi-group in $\mathcal{H}$ whose generator $G = \begin{pmatrix} 0 & 1 \\ \Delta & c \end{pmatrix}$ has a domain $D(G)$ which is the closure in the graph norm $(\|f\|^2 + \|Gf\|^2)^{1/2}$ of functions $(f_1, f_2) \in C^\infty_{(0)}(\mathbb{R}^n) \times C^\infty_{(0)}(\mathbb{R}^n)$ satisfying the boundary condition $\partial_\nu f_1 - \gamma f_2 - \sigma(x) f_1 = 0$ on Γ. The spectrum of G in $\mathrm{Re}\, z < 0$ is formed by isolated eigenvalues with finite multiplicity. For simplicity in the following we assume that $c(x) = 0, \sigma(x) = 0$. Notice that if $Gf = \lambda f$ with $f = (f_1, f_2) \neq 0$ and $\partial_\nu f_1 - \gamma f_2 = 0$ on Γ, we get

$$\begin{cases} (\Delta - \lambda^2) f_1 = 0 \text{ in } \Omega, \\ \partial_\nu f_1 - \lambda \gamma f_1 = 0 \text{ on } \Gamma. \end{cases} \tag{2}$$

Moreover, $u(t, x) = V(t)f = e^{\lambda t} f(x), \mathrm{Re}\, \lambda < 0$, is a solution of (1) with exponentially decreasing global energy. Such solutions are called asymptotically disappearing and they perturb the scattering. On the other hand, a solution $V(t)f$ is called disappearing if there exists $T > 0$ such that $V(t)f \equiv 0$ for $\forall t \geq T$.

I. Let

$$W_- f = \lim_{t \to +\infty} V(t) J U_0(-t) f, \; W_+ f = \lim_{t \to +\infty} V^*(t) J U_0(t) f, \; f \in H_0$$

be the wave operators, where $U_0(t) : H_0 \to H_0$ is the unitary group corresponding to Cauchy problem in $\mathbb{R}^d$, $H_0 = H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ and $J : H_0 \to \mathcal{H}$ is projection. It was proved (Colombini, P., Rauch, (2014)) that if we have a least one eigenvalue λ of G with $\operatorname{Re} \lambda < 0$, then the wave operators $W_\pm$ are not complete, that is $\text{Ran}\, W_- \neq \text{Ran}\, W_+$ and we cannot define the scattering operator S by $S = W_+^{-1} \circ W_-$.
Idea of the proof. Introduce the spaces

$$H_+ = \{f \in \mathcal{H} : \; V(t) f \to 0 \text{ as } t \to +\infty\}, \; H_- = \{f \in \mathcal{H} : \; V^*(t) f \to 0 \text{ as } t \to +\infty\}.$$

First one proves that $\overline{\text{Ran}\, W_\pm} = \mathcal{H} \ominus H_\pm$. The equality $\text{Ran}\, W_- = \text{Ran}\, W_+$ yields $H_+ = H_-$. If f is an eigenfunction with eigenvalue $\lambda, \mathrm{Re}\, \lambda < 0$, clearly $f \in H_+$. Second, we show that $f \in H_- \cap H_+$ implies that $V(t) f$ is disappearing which is impossible. Thus $f \notin H_-$. We may define S by using another evolution operator.

II. For problems associated to unitary groups (the global energy is conserved in time) the associated scattering operator $S(z) : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$ satisfies

$$S^{-1}(z) = S^*(\bar{z}),\ z \in \mathbb{C},$$

if $S(z)$ is invertible at z. This implies that $S(z)$ is invertible for $\operatorname{Im} z > 0$, since $S(z)$ and $S^*(z)$ are analytic for $\operatorname{Im} z < 0$. For dissipative boundary problems the above relation is not true and $S(z_0)$ may have a non trivial kernel for some $z_0, \operatorname{Im} z_0 > 0$. In this case for d odd, Lax and Phillips (1973) proved that $\mathbf{i}z_0$ is an eigenvalue of G.

III. Following Lax and Phillips, for d odd, we can decompose the energy space

$$\mathcal{H} = D_-^{\rho} \oplus \mathcal{K} \oplus D_+^{\rho},$$

where $D_{\pm}^{\rho} = U_0(\pm\rho)D_{\pm},\ \rho > 0$ are the Lax-Phillips spaces ($U_0(t)$ is the unitary group related to Cauchy problem for the wave equation). A function g is called outgoing (resp. incoming) if its component in D_-^{ρ}(resp. D_+^{ρ}) vanishes. An eigenfunction f of G is always incoming and $V(t)f$ is incoming for all $t \geq 0$. It is easy to see that if we have one disappearing solution, then the space

$$H_T = \{f \in \mathcal{H} :\ V(t)f \equiv 0,\ t \geq T\}$$

has infinite dimension. On the other hand, Majda (1975) established that if K and $\gamma(x)$ are real analytic, then in the case $\gamma(x) \neq 1,\ \forall x \in \Gamma$, there are no disappearing solutions.

2.Results

Throughout the talk we consider two cases:

$$(\mathbf{A}):\ 0<\gamma(x)<1, \forall x\in\Gamma,\ (\mathbf{B}):\ \gamma(x)>1,\ \forall x\in\Gamma.$$

Let $\sigma_p(G)$ denote the set of eigenvalues of G.

Proposition 1 (P. (2016), (2021))

Let $K=B_3=\{x\in\mathbb{R}^3:\ \|x\|\leq 1\}$ and suppose that $\gamma\equiv const$. Then

(1) $\gamma\equiv 1$. *There are no eigenvalues of G in $\mathbb{C}$,*

(2) $\gamma>1$. *All eigenvalues of G are real, we have an infinite number of eigenvalues of G and*

$$\sigma_p(G)\subset(-\infty,-\frac{1}{\gamma-1}],$$

(3) $0<\gamma<1$. *All eigenvalues of G are not real, we have an infinite number eigenvalues of G and*

$$\sigma_p(G)\subset\{\lambda\in\mathbb{C}:\ |\operatorname{Re}\lambda|<(1-\gamma)|\operatorname{Im}\lambda|^2,\ \operatorname{Re}\lambda<0\}.$$

Eigenvalues free regions

Theorem 1 (P. (2016))

In the case (A) *for every* ϵ, $0 < \epsilon \ll 1$, *the eigenvalues of* G *lie in the region*

$$\Lambda_\epsilon = \{\lambda \in \mathbb{C} : \ |\operatorname{Re}\lambda| \leq C_\epsilon(|\operatorname{Im}\lambda|^{\frac{1}{2}+\epsilon} + 1), \ \operatorname{Re}\lambda < 0\}.$$

In the case (B) *for every* ϵ, $0 < \epsilon \ll 1$, *and every* $M \in \mathbb{N}$ *the eigenvalues of* G *lie in the region* $\Lambda_\epsilon \cup \mathcal{R}_M$, *where*

$$\mathcal{R}_M = \{|\operatorname{Im}\lambda| \leq C_M(1 + |\operatorname{Re}\lambda|)^{-M}, \ \operatorname{Re}\lambda < 0\}.$$

For strictly convex obstacles K we improve the above result in the case (B).

Theorem 2 (P. (2016))

Assume K *strictly convex. In the case* (B) *for every* $M \in \mathbb{N}$ *the eigenvalues of* G *lie in the region* $\mathcal{R}_M \cup \{|\lambda| < R, \operatorname{Re}\lambda < 0\}$.

By applying the results of Vodev (2017) for the Dirichlet-to-Neumann map, it is possible to improve the above result replacing the region Λ_ϵ by a strip

$$\mathcal{M} = \{\lambda \in \mathbb{C} : \ -R_0 \leq \mathrm{Re}\, \lambda < 0\}, \ R_0 > 0.$$

Thus for strictly convex obstacles the eigenvalue free regions correspond to the case of a ball.

Previous results have been proved by Majda (1976). He proved that in the case (A) the eigenvalues lie in

$$E_1 = \{\lambda \in \mathbb{C} : \ |\,\mathrm{Re}\, \lambda| \leq C_1(|\,\mathrm{Im}\, \lambda|^{3/4} + 1), \ \mathrm{Re}\, \lambda < 0\},$$

while in the case (B) he showed that the eigenvalues lie in $E_1 \cup E_2$, where

$$E_2 = \{\lambda \in \mathbb{C} : \ |\,\mathrm{Im}\, \lambda| \leq C_2(|\,\mathrm{Re}\, \lambda|^{1/2} + 1), \mathrm{Re}\, \lambda < 0\}.$$

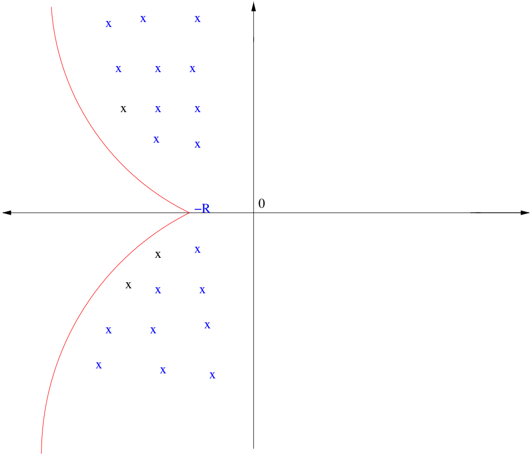

Figure 1: Eigenvalues for $0 < \gamma(x) < 1$

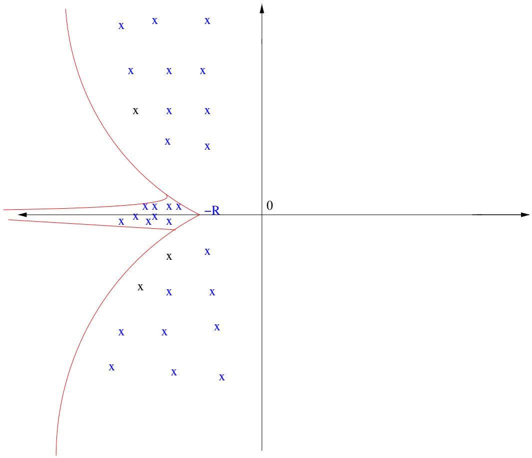

Figure 2: Eigenvalues for $\gamma(x) > 1$

Weyl asymptotic for the eigenvalues in the case (B)

Introduce the set
$\Lambda := \{\lambda \in \mathbb{C} : |\mathrm{Im}\,\lambda| \leq C_2(1+|\mathrm{Re}\,\lambda|)^{-2},\ \mathrm{Re}\,\lambda \leq -C_0 \leq -1\}$, $\frac{2C_2}{C_0} \leq 1$, containing $\mathcal{R}_M$, $\forall M \geq 2$ modulo a compact set. Given $\lambda \in \sigma_p(G)$, define the algebraic multiplicity of $\lambda \in \sigma_p(G)$ by

$$\mathrm{mult}\,(\lambda) := \mathrm{tr}\,\frac{1}{2\pi \mathbf{i}} \int_{|z-\lambda|=\epsilon} (z-G)^{-1} dz$$

with $0 < \epsilon \ll 1$ sufficiently small.

Theorem 3 (P. (2021))

Assume $\gamma(x) > 1$ for all $x \in \Gamma$. Then the counting function of the eigenvalues of G in Λ taken with their multiplicities has the asymptotic

$$\sharp\{\lambda_j \in \sigma_p(G) \cap \Lambda : |\lambda_j| \leq r,\ r \geq C_\gamma\}$$
$$= \frac{\omega_{d-1}}{(2\pi)^{d-1}} \Big(\int_\Gamma (\gamma^2(x) - 1)^{(d-1)/2} dS_x \Big) r^{d-1} + \mathcal{O}_\gamma(r^{d-2}),\ r \to \infty, \qquad (3)$$

ω_{d-1} being the volume of the unit ball $\{x \in \mathbb{R}^{d-1} : |x| \leq 1\}$.

Remark 1

For strictly convex obstacles we obtain the asymptotic of all eigenvalues. The constant C_γ depend on γ. When $\max_{x\in\Gamma}\gamma(x) \searrow 1$, one has $C_\gamma \to +\infty$. This is justified by the proof of Theorem 3 and by the example for the ball B_3.

We obtain an analog of Theorem 3 for the Maxwell system with dissipative boundary conditions

$$\begin{cases} \partial_t E = \mathrm{curl}\, H, \qquad \partial_t H = -\mathrm{curl}\, E \quad \text{in} \quad \mathbb{R}_t^+ \times \Omega, \\ \nu \wedge E - \gamma(x)(\nu \wedge (\nu \wedge H)) = 0 \quad \text{on} \quad \mathbb{R}_t^+ \times \Gamma, \\ E(0,x) = E_0(x), \qquad H(0,x) = H_0(x),\ x \in \Omega \end{cases} \tag{4}$$

with initial data $F_0 = (E_0, H_0) \in \mathcal{H} = L^2(\Omega;\mathbb{C}^3) \times L^2(\Omega;\mathbb{C}^3)$. The solution is described by a contraction semigroup

$$(E,H)(t) = V(t)f = e^{tG_b}F_0,\ t \geq 0.$$

Here the generator G_b is the operator

$$G = \begin{pmatrix} 0 & \mathrm{curl} \\ -\mathrm{curl} & 0 \end{pmatrix}$$

with domain $D(G_b) \subset \mathcal{H}$.

Theorem 4 (P. (2022))

Assume $\gamma(x) > 1, \forall x \in \Gamma$ *or* $0 < \gamma(x) < 1, \forall x \in \Gamma$. *Set* $\gamma_0(x) = \max\{\gamma(x), \frac{1}{\gamma(x)}\}$. *Then the counting function of the eigenvalues of* G_b *in* Λ *counted with their multiplicities for* $r \to \infty$ *has the asymptotic*

$$\sharp\{\lambda_j \in \sigma_p(G) \cap \Lambda : \ |\lambda_j| \leq r, \ r \geq C_{\gamma_0}\} = \frac{1}{4\pi}\Big(\int_\Gamma (\gamma_0^2(x) - 1)dS_x\Big) r^2 + \mathcal{O}_{\gamma_0}(r). \quad (5)$$

The proof is more complicated since it is impossible to reduce the analysis to that of the wave equation. It was proved by Colombini, P., Rauch (2016) that if $\gamma \equiv 1$ and $K = B_3$ there are no eigenvalues of G_b. The partial case when K is the ball B_3 and $\gamma = const \neq 1$ has been studied by Colombini-P. (2018).

For $\operatorname{Re} \lambda < 0$ introduce the exterior Dirichlet-to-Neumann map

$$\mathcal{N}(\lambda) : H^s(\Gamma) \ni f \longrightarrow \partial_\nu u|_\Gamma \in H^{s-1}(\Gamma),$$

where u is the solution of the problem

$$\begin{cases} (-\Delta + \lambda^2)u = 0 \text{ in } \Omega, \\ u = f \text{ on } \Gamma, \\ u : (\mathbf{i}\lambda) - \text{outgoing}. \end{cases} \tag{6}$$

A function $u(x)$ is $(\mathbf{i}\lambda)$-outgoing if there exists $R > \rho_0$ and $g \in L^2_{comp}(\mathbb{R}^d)$ such that

$$u(x) = (-\Delta_0 + \lambda^2)^{-1} g, \; |x| \geq R,$$

where $R_0(\lambda) = (-\Delta_0 + \lambda^2)^{-1}$ is the outgoing resolvent of the free Laplacian $-\Delta_0$ in $\mathbb{R}^d$ which is analytic in $\mathbb{C}$ for d odd and on the logarithmic covering of $\mathbb{C} \setminus \{0\}$ for d even. For $d = 3$ this resolvent has kernel $\frac{e^{\lambda|x-y|}}{4\pi|x-y|}$.

The operator $\mathcal{N}(\lambda)$ can be expressed by the cut-off resolvent $\chi(-\Delta_D + \lambda^2)^{-1}\chi, \chi \in C_0^\infty(\mathbb{R}^d)$ of the Dirichlet Laplacian Δ_D, hence $\mathcal{N}(\lambda)$ is analytic in $\{\lambda : \mathrm{Re}\,\lambda < 0\}$. The boundary condition for an eigenfunction g becomes

$$\mathcal{C}(\lambda)f := \mathcal{N}(\lambda)f - \lambda\gamma f = 0,\ f = g|_\Gamma.$$

The operator $\mathcal{N}(\lambda)^{-1}:\ H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is compact and analytic in $\{z : \mathrm{Re}\,\lambda < 0\}$ since there are no resonances of the Neumann problem in $\{z : \mathrm{Re}\,z < 0\}$. We write

$$\mathcal{C}(\lambda) = (Id - \lambda\gamma\mathcal{N}(\lambda)^{-1})\mathcal{N}(\lambda).$$

and by the analytic Fredholm theorem one deduces that $\mathcal{C}(\lambda)^{-1}$ is meromorphic in $\{\lambda : \mathrm{Re}\,\lambda < 0\}$.

Trace formula

Proposition 2 (P. (2016))

Let $\alpha \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < 0\}$ be a closed positively oriented curve without self intersections. Assume that $\mathcal{C}(\lambda)^{-1}$ has no poles on α . Then

$$\operatorname{tr}_{\mathcal{H}} \frac{1}{2\pi \mathbf{i}} \int_\alpha (\lambda - G)^{-1} d\lambda = \operatorname{tr}_{H^{1/2}(\Gamma)} \frac{1}{2\pi \mathbf{i}} \int_\alpha \mathcal{C}(\lambda)^{-1} \frac{\partial \mathcal{C}}{\partial \lambda}(\lambda) d\lambda. \tag{7}$$

Since G has only point spectrum in $\operatorname{Re} \lambda < 0$, the left hand term in (7) is equal to the number of the eigenvalues of G in the domain ω bounded by α counted with their algebraic multiplicities. Setting $\tilde{\mathcal{C}}(\lambda) = \frac{\mathcal{N}(\lambda)}{\lambda} - \gamma$, we write the right hand side of (7) as

$$\operatorname{tr}_{H^{1/2}(\Gamma)} \frac{1}{2\pi \mathbf{i}} \int_\alpha \tilde{\mathcal{C}}(\lambda)^{-1} \frac{\partial \tilde{\mathcal{C}}}{\partial \lambda}(\lambda) d\lambda. \tag{8}$$

Set $\lambda = -\frac{1}{\tilde{h}}$, $0 < \mathrm{Re}\,\tilde{h} \ll 1$ and consider the problem

$$\begin{cases} (-\tilde{h}^2\Delta + 1)u = 0 \text{ in } \Omega, \\ -\tilde{h}\partial_\nu u - \gamma u = 0 \text{ on } \Gamma, \\ u - \text{outgoing}. \end{cases} \tag{9}$$

We introduce the operator $C(\tilde{h}) := -\tilde{h}\mathcal{N}(-\tilde{h}^{-1}) - \gamma$ and using (8), the trace formula (7) becomes

$$\mathrm{tr}\, \frac{1}{2\pi \mathbf{i}} \int_\alpha (\lambda - G)^{-1} d\lambda = \mathrm{tr} \frac{1}{2\pi \mathbf{i}} \int_{\tilde{\alpha}} C(\tilde{h})^{-1} \dot{C}(\tilde{h}) d\tilde{h}, \tag{10}$$

where $\dot{C}$ denote the derivative with respect to $\tilde{h}$ and $\tilde{\alpha}$ is the curve $\tilde{\alpha} = \{z \in \mathbb{C} : z = -\frac{1}{w},\ w \in \alpha\}$.

Recall that $\Lambda = \{\lambda \in \mathbb{C} : \ |\operatorname{Im}\lambda| \leq C_2(|\operatorname{Re}\lambda| + 1)^{-2}, \ \operatorname{Re}\lambda \leq -C_0 \leq -1\}$. For $\lambda \in \Lambda$ one has $|\operatorname{Im}\lambda| \leq 1$ and this implies $\tilde{h} \in L$, where

$$L := \{\tilde{h} \in \mathbb{C} : \ |\operatorname{Im}\tilde{h}| \leq C_1|\tilde{h}|^4, \ |\tilde{h}| \leq C_0^{-1}, \ \operatorname{Re}\tilde{h} > 0\}. \tag{11}$$

Write the points in L as $\tilde{h} = h(1 + \mathbf{i}\eta)$ with $0 < h \leq h_0 \leq C_0^{-1}$, $\eta \in \mathbb{R}$, $|\eta| \leq h^2$. Therefore the problem (9) becomes

$$\begin{cases} (-h^2\Delta - z)u = 0 \text{ in } \Omega, \\ -(1 + \mathbf{i}\eta)h\partial_\nu u - \gamma u = 0 \text{ on } \Gamma, \\ u - \text{outgoing} \end{cases} \tag{12}$$

with $z = -\frac{1}{(1+\mathbf{i}\eta)^2} = -1 + s(\eta)$, $|s(\eta)| \leq (2 + h^2)h^2 \leq 3h^2$.

Semi-classical parametrix

Given $f \in H^s(\Gamma)$, consider the problem

$$\begin{cases} (-h^2\Delta - z)u = 0 \; in \; \Omega, \\ u = f \text{ on } \Gamma, u - \mathrm{outgoing}. \end{cases} \tag{13}$$

Consider the problem (13) for $z = -1 + s(\eta)$, $|s(\eta)| \leq 3h^2$ and define the semi-classical exterior Dirichlet-to-Neumann map

$$\mathcal{N}_{ext}(z,h) : H_h^s(\Gamma) \ni f \longrightarrow h\partial_\nu u|_\Gamma \in H_h^{s-1}(\Gamma).$$

Here $H_h^s(\Gamma)$ is the semi-classical Sobolev space with norm $\|\langle hD\rangle^s u\|_{L^2(\Gamma)}$.

G. Vodev (2015) constructed for domains with arbitrary geometry a semi-classsical parametrix for the interior problem (13) (Ω replaced by K) as a Fourier integral operator with complex phase $\varphi(x, \xi'; z)$ in a small neighborhood of the boundary Γ. Close to the boundary introduce geodesic normal coordinates $x = (x', x_d)$ in a neighborhood of a point $x_0 \in \Gamma$ with $x_d = 0$ on Γ(we take $x_d = \mathrm{dist}(x, \Gamma)$). Set $\xi = (\xi', \xi_d)$. We say that $a(x, \xi'; h) \in S^k_\delta(\Gamma)$, $\delta \geq 0$, $k \in \mathbb{R}$, if

$$\sup_{x \in \mathbb{R}^d, \xi' \in \mathbb{R}^{d-1}} |\partial^\alpha_{x'} \partial^\beta_{\xi'} a(x, \xi'; h)| \leq C_{\alpha,\beta} h^{-\delta(|\alpha|+|\beta|)} \langle \xi' \rangle^{k-|\beta|}, \ \forall \alpha, \forall \beta,$$

where $\langle \xi' \rangle = (1 + |\xi'|^2)^{1/2}$. For $a \in S^k_\delta(\Gamma)$, consider the semi-classical $h-$

pseudo-differential operator

$$\Big(Op_h(a)f\Big)(x) = (2\pi h)^{-d+1} \iint e^{i\langle x'-y', \xi' \rangle/h} a(x, \xi'; h) f(y') dy' d\xi'.$$

and the classical one

$$\Big(Op_h(a)f\Big)(x) = (2\pi)^{-d+1} \iint e^{i\langle x'-y', \xi' \rangle} a(x, h\xi'; h) f(y') dy' d\xi'.$$

We have a calculus for the $h-$pseudo-differential operators with symbols in S^k_δ if $0 < \delta < 1/2$. In local coordinates the semi-classical symbol of $-h^2\Delta$ becomes $\xi_d^2 + r(x, \xi') + hq(x)\xi_d$ and $r(x', 0, \xi') = r_0(x', \xi')$ is the principal symbol of the Laplace-Beltrami operator $-h^2\Delta|_\Gamma$ on the boundary Γ. Let $\rho(x', \xi', z) = \sqrt{z - r_0(x', \xi')} \in C^\infty(T^*(\Gamma))$, $\mathrm{Im}\,\rho > 0$ be the root of the equation $\rho^2 + r_0(x', \xi') - z = 0$. It is easy to see that $\rho \in S^1_0$.

Proposition 3 (Vodev, (2015))

For $z = -1 + s(\eta)$, $|s(\eta)| \leq h^2$ and $0 < h \leq h_0$ there exists a parametrix $T_N(z, h)$ of (13) such that

$$\| -\mathbf{i}\mathcal{N}_{int}(z, h)f - T_N(z, h)f\|_{H_h^m(\Gamma)} \leq C_{m,N} h^{-s_d + N} \|f\|_{L^2(\Gamma)}, \ \forall m \in \mathbb{N}, \ \forall N \in \mathbb{N}, \quad (14)$$

where $C_{m,N} > 0$ is independent of h, z

Here $T_N(z, h)$ is a pseudo-differential operator with principal symbol $-\mathbf{i}\sqrt{z - r_0} = \sqrt{-z + r_0}$.

The above result was proved by Vodev for the interior Dirichlet-to-Neumann map $\mathcal{N}_{int}(z,h)$, while for the exterior one $\mathcal{N}_{ext}(z,h)$ the same result was established by P. (2016) by using the local construction of parametrix of Vodev.
For Maxwell system introduce the spaces $\mathcal{H}^t_s(\Gamma) = \{f \in H^s(\Gamma; \mathbb{C}^3) : \langle \nu(x), f(x)\rangle = 0\}$ and consider the boundary problem

$$\begin{cases} \mathrm{curl}\, E = -\lambda H, \; x \in \Omega, \\ \mathrm{curl}\, H = \lambda E, \; x \in \Omega, \\ \nu \wedge E = f \text{ for } x \in \Gamma, \\ (E, H) : \mathbf{i}\lambda - \text{outgoing}. \end{cases} \tag{15}$$

Introduce the operator $\mathcal{N}_b(\lambda) : \mathcal{H}^t_1 \ni f \longrightarrow \nu \wedge H|_\Gamma \in \mathcal{H}^t_0$, (E, H) being the solution of (15). This operator is well defined and it plays the role of the Dirichlet-to-Neumann $\mathcal{N}(\lambda)$. By $\mathcal{N}_b(\lambda)$, we write the boundary condition in (4) as follows

$$C(\lambda)g := \mathcal{N}_b(\lambda)(\nu \wedge g) - \frac{1}{\gamma(x)} g = 0, \; g = E|_\Gamma \in \mathcal{H}^t_1(\Gamma).$$

and set $\mathcal{P}(\lambda)g = \mathcal{N}_b(\lambda)(\nu \wedge g)$.

Write $\tilde{h} = h(1 + \mathbf{i}t),\ 0 < h \leq h_0 \leq C_0^{-1},\ t \in \mathbb{R}$. Then for $\tilde{h} \in L$, we have $|t| \leq h^2$ for $\tilde{h} \in L$ and the problem (15) becomes

$$\begin{cases} -\mathbf{i}h \,\mathrm{curl}\, E = zH,\ x \in \Omega, \\ -\mathbf{i}h \,\mathrm{curl}\, H = -zE,\ x \in \Omega, \\ \nu \wedge E = f,\ x \in \Gamma, \\ (E, H) - \mathrm{outgoing} \end{cases}$$

with $-\mathbf{i}z = h\lambda,\ z = -\mathbf{i}(1 + \mathbf{i}t)^{-1}$. A semi-classical parametrix for the above problem has been constructed by Vodev (2021) and the principal symbol of the approximation of $\mathcal{P}(-1/h)$ has the form $m = (\sqrt{1 + r_0})Id - \frac{1}{\sqrt{1+r_0}}\mathcal{B}$, where Id is the identity (3×3) matrix and $\mathcal{B} \in S_0^2$ is a symmetric (3×3) matrix. Setting $\gamma_0(x) = \frac{1}{\gamma(x)}$, the eigenvalues of $m - \gamma_0(x)Id$ are

$$\sqrt{1 + r_0} - \gamma_0, \sqrt{1 + r_0} - \gamma_0, (1 + r_0)^{-1/2} - \gamma_0.$$

The leads to many difficulties in the semi-classical asymptotics of matrix valued p. d. operators ($(1 + r_0)^{-1/2} - \gamma_0 \in S_0^0$ and this eigenvalue does not go to $+\infty$ as $|\xi'| \to \infty$.)

4. Idea of the proof of Theorem 3

Take N large enough and consider the semi-classical parametrix $-\mathbf{i}T(z,h) = -\mathbf{i}T_N(z,h)$ for $-\mathcal{N}_{ext}(z,h)$ with $z = -1 + s(\eta)$, $|s(\eta)| \leq h^2$. By Green formula one deduces that $\mathcal{N}_{ext}(-1,h)$ is a self-adjoint operator. Introduce the self-adjoint operator

$$P(h) := -\mathbf{i}T(-1,h) - \gamma(x'),\ 0 < h \leq h_0.$$

The semiclassical principal symbol of $P(h)$ is $p_1(x',\xi') = -\mathbf{i}\sqrt{-1-r_0} - \gamma(x') = \sqrt{1+r_0} - \gamma(x')$. Since $\min_{x'} \gamma(x') > 1$, this symbol vanishes when

$$r_0(x',\xi') = \gamma^2(x') - 1 > 0.$$

We will treat $P(h)$ as a classical pseudo-differential operator with symbol

$$\sqrt{1+h^2 r_0(x',\xi')} - \gamma(x') + hP_0(h),\ P_0(h) \in S_0^0.$$

We apply the approach of Sjöstrand-Vodev (1997) concerning the asymptotic of Rayleigh resonances of elasticity system close to the real axis. Let

$$\mu_1(h) \leq \mu_2(h) \leq ... \leq \mu_m(h) \leq ...$$

be the eigenvalues of $P(h)$ counted with their multiplicities. The points $0 < h_k \leq h_0$, where $\mu_k(h_k) = 0$ correspond to points for which $P(h)$ is not invertible. For large fixed k_0, depending on h_0, the eigenvalues $\mu_k(h_0)$ are positive, whenever $k > k_0$. Thus if for $r^{-1} < h_0$ we have $\mu_k(r^{-1}) < 0,\ k > k_0$, one obtains $\mu_k(h_k) = 0$ for some $r^{-1} < h_k < h_0$. However, a more precise analysis of the behaviour of $\mu_k(h)$ and the connexion of h_k to eigenvalues $\lambda_j \in \Lambda \cap \sigma_p(G)$ is necessary. Consequently, the problem is reduced to a Weyl asymptotic of the counting function of the negative eigenvalues of $P(r^{-1}),\ r > (h_0(\gamma))^{-1} = C_\gamma$ given by the well known formula

$$\frac{r^{d-1}}{(2\pi)^{d-1}} \iint_{r_0(x',\xi') \leq \gamma^2(x')-1} dx' d\xi' + \mathcal{O}_\gamma(r^{d-2}),\ r > C_\gamma \tag{16}$$

since $\sqrt{1 + r_0(x', \xi')} - \gamma(x') \leq 0 \iff r_0(x', \xi') \leq \gamma^2(x') - 1.$

Main steps

1. Examine $\frac{dP(h)}{dh}$ and $\frac{d\mu_k(h)}{dh}$ and prove that the zero h_k of $\mu_k(h)$ for $k > k_0$ is unique.
2. Study the continuation $P(\tilde{h})$ for $\tilde{h} = h(1 + \mathbf{i}\eta)$, $|\eta| \leq h^2$ and show that on suitable curves $\gamma_{p,k} \in \mathbb{C}$ one has

$$\|P^{-1}(\tilde{h})\|_{H^s(\Gamma) \to H^{s+1}(\Gamma)} \leq \frac{C_s}{h^p}, \ \tilde{h} \in \gamma_{p,k}, \ \eta \neq 0. \tag{17}$$

3. Establish a trace formula

$$\mathrm{tr}_{H^{1/2}(\Gamma)} \frac{1}{2\pi \mathbf{i}} \int_{\gamma_{k,p}} P^{-1}(\tilde{h}) \frac{dP(\tilde{h})}{d\tilde{h}} d\tilde{h}$$

and count the number of h_k in a domain bounded by $\gamma_{k,p}$.

4. Show that the trace formulas for $C(\tilde{h})$ and $P(\tilde{h})$ over $\gamma_{k,p}$ differ by a negligible term $\mathcal{O}_m(h^m)$, $\forall m \in \mathbb{N}$. Thus we obtain a bijection map $\ell : (0, h_0] \ni h_k \to \ell(h_k) = \lambda_k \in \Lambda$ between the set of points $h_k \in (0, h_0]$ and the eigenvalues $\lambda_k \in \Lambda \cap \sigma_p(G)$.

Idea for the Step 1

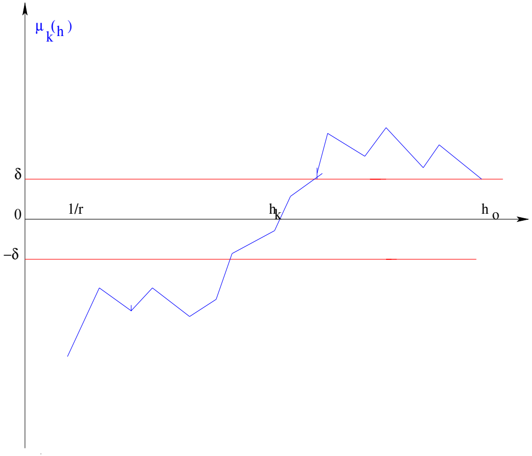

Figure 3: eigenvalue $\mu_k(h)$ for $1/r < h \leq h_o,\ k > k_0$

Set $\min_{x\in\Gamma}\gamma(x) = c_0 > 1$, $\max_{x\in\Gamma}\gamma(x) = c_1 \geq c_0$ and choose a constant $C = \frac{1}{c_1^2}$. We denote by $(.,.)$ the scalar product in $L^2(\Gamma)$ and for two self adjoint operators L_1, L_2 the inequality $L_1 \geq L_2$ means $(L_1 u, u) \geq (L_2 u, u)$, $\forall u \in L^2(\Gamma)$.

Proposition 4

Let $\langle h\Delta\rangle = (1 - h^2\Delta_\Gamma)^{1/2}$ and let $\epsilon = \frac{C}{2}(c_0 - 1)^2 < 1$. Then for h sufficiently small we have

$$h\frac{\partial P(h)}{\partial h} + CP(h)\langle h\Delta\rangle^{-1}P(h) \geq \epsilon\langle h\Delta\rangle. \tag{18}$$

Remark 2

The values of ϵ depends on $(c_0 - 1)^2$ and $\epsilon \searrow 0$ when $c_0 \searrow 1$. Also $0 < h \leq \frac{\epsilon}{C_3}$ so h_0 and $h_k \in]0, h_0]$ must have order $o(\epsilon)$. Hence we need to take $r \geq \frac{1}{o(\epsilon)}$ in (16).

The principal symbol of the operator of the left hand part of (18) has the form

$$(1+C-\epsilon)\sqrt{1+h^2r_0}+\epsilon\sqrt{1+h^2r_0}+(C\gamma^2-1)(1+h^2r_0)^{-1/2}-2C\gamma$$

$$=(1+C-\epsilon)\sqrt{1+h^2r_0}+C\gamma^2-2C\gamma-1$$

$$+(C\gamma^2-1)((1+h^2r_0)^{-1/2}-1)+\epsilon\sqrt{1+h^2r_0}.$$

We have $(1+C-\epsilon)(Op(\sqrt{1+h^2r_0})u,u)\geq(1+C-\epsilon)\|u\|^2$ and

$$\Big((C(\gamma-1)^2-\epsilon)u,u\Big)\geq(C(c_0-1)^2-\epsilon)\|u\|^2=\epsilon\|u\|^2.$$

On the other hand, $C\gamma^2-1\leq 0$ and the symbol in red is non-negative and has order 0. By applying the sharp semi-classical Gärding inequality this term is greater that $-C_2h\|u\|^2$ and it can be absorbed for small h by $\epsilon\|u\|^2$. The same is true for lower order term $(hP_0(h)u,u)$ which is bounded by $C_1h\|u\|^2$ and we take $0<h_0\leq(C_1+C_2)^{-1}\epsilon$.

Let h_1 be small and let $\mu_k(h_1)$ have multiplicity m. For h close to h_1 one has exactly m eigenvalues and we denote by $F(h)$ the space spanned by them. We can find a small interval (α, β) around $\mu_k(h_1)$, independent on h, containing the eigenvalues spanning $F(h)$. Given $h_2 > h_1$ close to h_1, consider a normalised eigenfunction $e(h_2)$ with eigenvalue $\mu_k(h_2)$. Denote by dot the derivative with respect to h. Let $\pi(h) = E_{(\alpha,\beta)}$ be the spectral projection of $P(h)$, hence $F(h) = \pi(h)L^2(\Gamma)$. Then $(\pi(h) - I)\pi(h) = 0$ yields $\pi(h)\dot{\pi}(h)\pi(h) = 0$. Construct a smooth extension $e(h) = \pi(h)e(h_2) \in F(h),\ h \in [h_1, h_2]$ of $e(h_2)$ with $\|e(h)\| = 1,\ \dot{e}(h) \in F(h)^{\perp}$. Obviously, $e(h_1)$ will be normalised eigenfunction with eigenvalue $\mu_k(h_1)$. One obtains

$$h\dot{P}(h) = h^2\Delta\langle hD\rangle^{-1} + hL_0 = P(h) + \gamma(x) - \langle hD\rangle^{-1} + hL_1$$

with zero order operators L_0, L_1 and this implies $|(\dot{P}(h)e(h), e(h))| \leq C_0 h^{-1},\ h \in [h_1, h_2]$. Therefore

$$|\mu_k(h_2) - \mu_k(h_1)| = \Big|\int_{h_1}^{h_2} \frac{d}{dh}(P(h)e(h), e(h))dh\Big| \leq C_0 \int_{h_1}^{h_2} h^{-1}dh \leq \frac{C_0}{h_1}(h_2 - h_1).$$

Assuming $\mu_k(h) \in [-\delta, \delta]$ for $h \in [h_1, h_2]$, we deduce that $\mu_k(h)$ is locally Lipschitz function in h and its almost defined derivative satisfies $|h\frac{\partial \mu_k(h)}{\partial h}| \leq C_0$. To estimate $h\frac{\partial \mu_k(h)}{\partial h}$ from below, we exploit Proposition 4. For $0 < h \leq h_0$ we have

$$h\frac{\partial \mu_k(h)}{\partial h} = (h\dot{P}(h)e(h), e(h))$$

$$\geq \epsilon(\langle hD\rangle e(h), e(h)) - C(\langle hD\rangle^{-1}P(h)e(h), P(h)e(h)) \geq \epsilon - C\delta^2 \geq \frac{\epsilon}{2},$$

choosing $C\delta^2 \leq \epsilon/2 = \frac{C(c_0-1)^2}{4}$ which yields $0 < \delta \leq \frac{c_0-1}{2}$. Consequently, for $h \in [h_1, h_2]$ one has

$$\mu_k(h_2) - \mu_k(h_1) \geq \frac{\epsilon}{2}\int_{h_1}^{h_2} h^{-1}dh \geq \frac{\epsilon}{2h_2}(h_2 - h_1)$$

and we obtain $\frac{\epsilon}{2} \leq h\frac{d\mu_k(h)}{dh} \leq C_0$.

Idea for the Step 2

We fix $\alpha_0 = \frac{\epsilon}{2}$ and $h_0 > 0$. Let $p > d$ be fixed and let

$$I_{k,p} = \{h \in]0, h_0] : |h - h_k| \leq \frac{h^{p+1}}{\alpha_0}\}.$$

Then for $h \in]0, h_0] \setminus I_{k,p}$ one has $|\mu_k(h)| \geq h^p$. Thus for $h \in]0, h_0] \setminus \Big(\bigcup_{k \geq k_0} I_{k,p}\Big)$ one obtains

$$\|P^{-1}(h)\|_{L^2 \to L^2} = \mathcal{O}(h^{-p}). \tag{19}$$

On the other hand, $\sum_{k \geq k_0} |I_{k,p}| = \mathcal{O}(h^{p+2-d})$. We construct disjoint intervals $J_{k,p}$ so that the estimate (19) holds for $h \in]0, h_0] \setminus \Big(\bigcup_{k \geq k_0} J_{k,p}\Big)$ with $|J_{k,p}| = \mathcal{O}(h^{p+2-d})$.

We choose a curve $\gamma_{k,p} \subset \mathbb{C}$ bounded by four segments $\mathrm{Re}\, \tilde{h} \in \partial J_{k,p}$, $\mathrm{Im}\, \tilde{h} = \pm(\mathrm{Re}\, \tilde{h})^{p+1}$, $|\mathrm{Im}\, \tilde{h}| \leq (\mathrm{Re}\, \tilde{h})^{p+1}$ and extend the estimate (19) for $\tilde{h} \in \gamma_{k,p}$. To do this, by using Prop. 4, one proves

$$\|P^{-1}(\tilde{h})\|_{\mathcal{L}(H^{-1/2}, H^{1/2})} \leq C_3 \frac{\mathrm{Re}\, \tilde{h}}{|\mathrm{Im}\, \tilde{h}|},\ \mathrm{Re}\, \tilde{h} > 0,\ \mathrm{Im}\, \tilde{h} \neq 0,$$

$$\|P^{-1}(\tilde{h})\|_{\mathcal{L}(H^{-1/2}, H^{1/2})} \leq \mathcal{O}((\mathrm{Re}\, \tilde{h})^{-p}),\ \tilde{h} \in \gamma_{k,p}. \tag{20}$$

5. The case $0 < \gamma(x) < 1$

The case $0 < \gamma(x) < 1$ for the wave equation is significally different from $\gamma(x) > 1$ and we are going to study the eigenvalues close to the imaginary axis. The first difficulty is that G has continuous spectrum on $\mathbf{i}\mathbb{R}$. To avoid this, for cut-off resolvent $\chi(G-\lambda)^{-1}\chi$ we prove the following

Proposition 5 (P. (2022))

Let d be odd. Assume that $\chi(-\Delta_D - \lambda^2)^{-1}\chi$ is analytic for $\operatorname{Re}\lambda < a$, $a > 0$. Let $\beta \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < a\}$ be a closed positively oriented curve without self intersections. Then if $\mathcal{C}(\lambda)^{-1}$ has no poles on α we have

$$\operatorname{tr}_{\mathcal{H}} \frac{1}{2\pi\mathbf{i}} \int_\beta \chi(\lambda - G)^{-1}\chi d\lambda = \operatorname{tr}_{H^{1/2}(\Gamma)} \frac{1}{2\pi\mathbf{i}} \int_\beta \mathcal{C}(\lambda)^{-1} \frac{\partial \mathcal{C}}{\partial \lambda}(\lambda) d\lambda. \tag{21}$$

Notice that the assumption that the cut-off resolvent $\chi(-\Delta_D - \lambda^2)^{-1}\chi$ is analytic implies that $\chi(\lambda - G)^{-1}\chi$ and $\mathcal{C}(\lambda)^{-1}$ are meromorphic in $\operatorname{Re}\lambda < a$ and they have the same singularities. Moreover, $\mathcal{C}(\lambda)^{-1}$ is analytic for $0 \leq \operatorname{Re}\lambda < a$.

Set $\frac{\mathbf{i}\sqrt{z}}{h} = \lambda$. We must contstruct a semi-classical paramterix of the problem (13) for $z = 1 + \mathbf{i}s(\eta)$, $|s(\eta)| \leq h^2$ with $h = \frac{1}{|\operatorname{Im}\lambda|}$. For strictly convex obstacles such parametrix has been constructed by Sjöstrand (2014). The principal symbol of this parametrix is $\sqrt{1 - r_0}$. Restricted to the elliptic region $r_0(x', \xi') \geq 1 + \delta$, $0 < \delta \ll 1$, the operator $Op_h(\sqrt{1 - r_0}) - \gamma(x')$ is invertible. The same is true in the diffractive region $|r_0(x', \xi') - 1| \leq \delta$. choosing δ sufficiently small. Thus in the trace formula (21) we can reduce the analysis to the hyperbolic region $r_0(x', \xi') \leq 1 - \delta$. In this region we have zeros determined by the equation $\sqrt{1 - r_0(x', \xi')} - \gamma(x') = 0$. In a work in progress we expect to prove that in the case $0 < \gamma(x) < 1$, $\forall x \in \Gamma$, for strictly convex obstacles we have the asymptotics

$$\sharp\{\lambda_j \in \sigma_p(G),\ -R_0 < \operatorname{Re}\lambda_j < 0,\ |\lambda_j| \leq r,\ r \geq C_\gamma\}$$
$$= \frac{\omega_{d-1}}{(2\pi)^{d-1}} \Big(\int_\Gamma (1 - \gamma^2(x))^{(d-1)/2} dS_x \Big) r^{d-1} + \mathcal{O}_\gamma(r^{d-2}),\ r \to \infty.$$

THANK YOU FOR ATTENTION !