Acoustically-Modulated Electromagnetic Inverse Source Problems

John C. Schotland

Department of Mathematics
Yale University

john.schotland@yale.edu

Joint work with Wei Li, Yang Yang and Yimin Zhong

Backstory

This talk is about a problem I encountered many years go, when I was a medical student in neurosurgery, rotating on the epilepsy service. At the time, I was just becoming interested in inverse problems and discovered that they were everywhere.

The problem is also connected to my recent work in multiwave (hybrid) inverse problems such as acousto-optic imaging.

References

- Li, W., Schotland, J., Yang, Y. and Zhong, Y. Acousto-Electric Inverse Source Problem. SIAM Journal Imaging Science **14**, 1601–1616 (2021)

- Li, W., Schotland, J., Yang, Y. and Zhong, Y. Inverse Source Problem for Acoustically Modulated Electromagnetic Waves. (ArXiv)

- Schotland, J. Acousto-optic Imaging of Random Media. Prog. Opt. **65**, 347–380 (2020)

Epilepsy

Epilepsy is a common neurological disorder that is characterized by recurrent seizures. The prevalence of epilepsy is approximately 1% in the US population. It is among the most treatable of neurological diseases.

Diagnosis is based on patient history and the results of tests, including electroencephalography (EEG) and imaging studies.

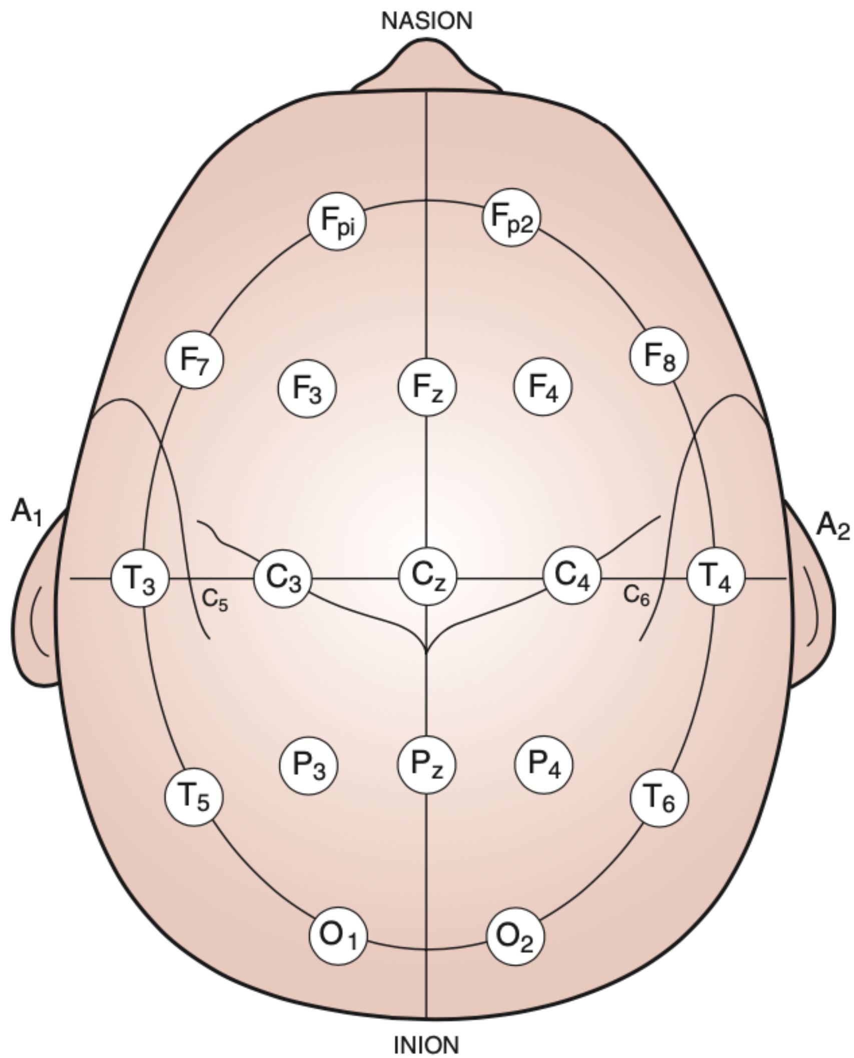
Medications achieve complete remission of seizures in 80% of patients. Of the remaining 20%, about half are candidates for various surgical procedures. Surgery substantially reduces the occurrence of seizures in 70% of these cases.

EEG

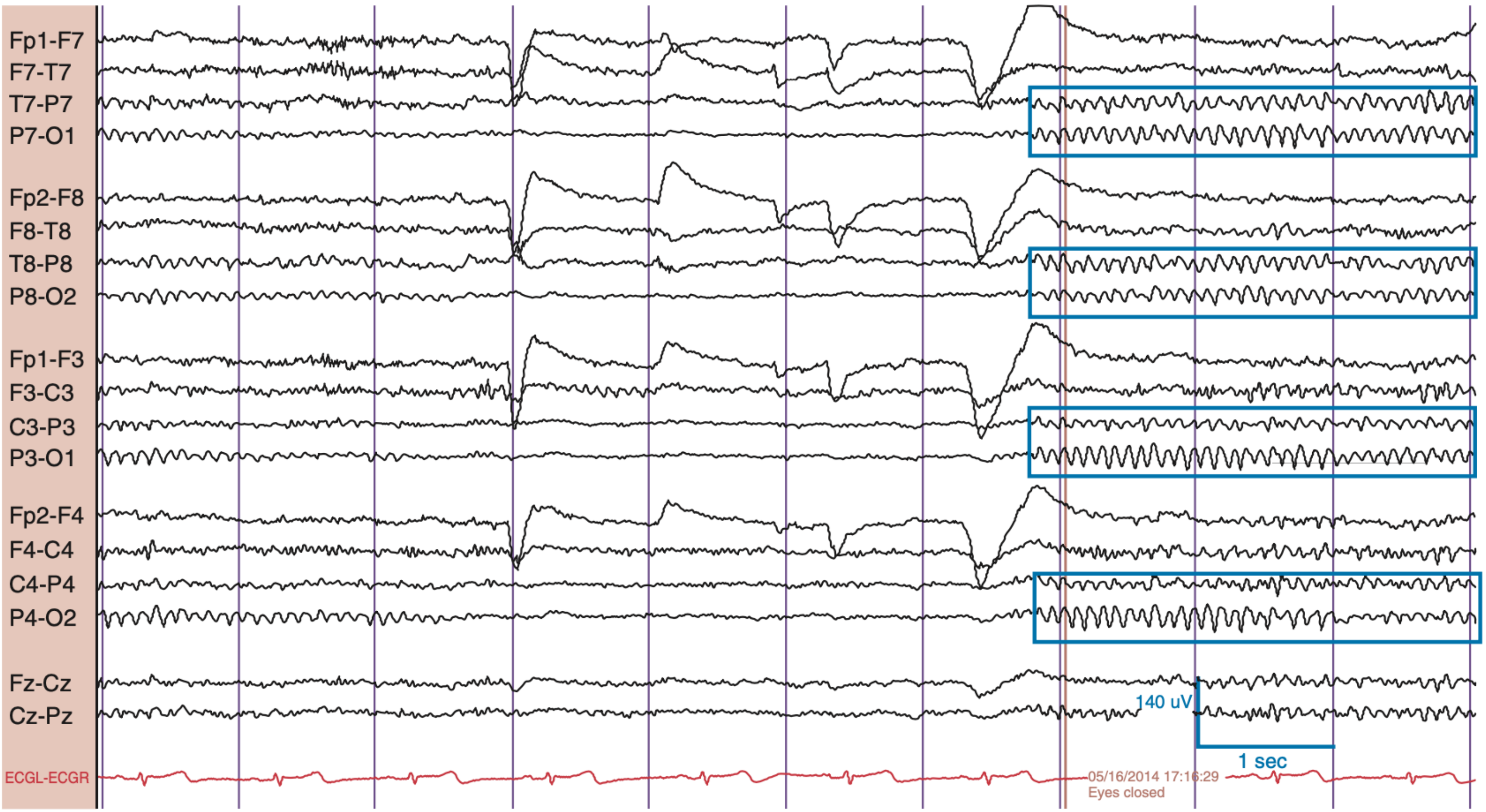
The EEG records electrical activity originating in the brain. In a typical clinical setting, the electrical signal is recorded from electrodes that are placed either on the scalp or surgically implanted in the brain.

The objective is to localize the current source that produces the measured signal.

Scalp electrodes

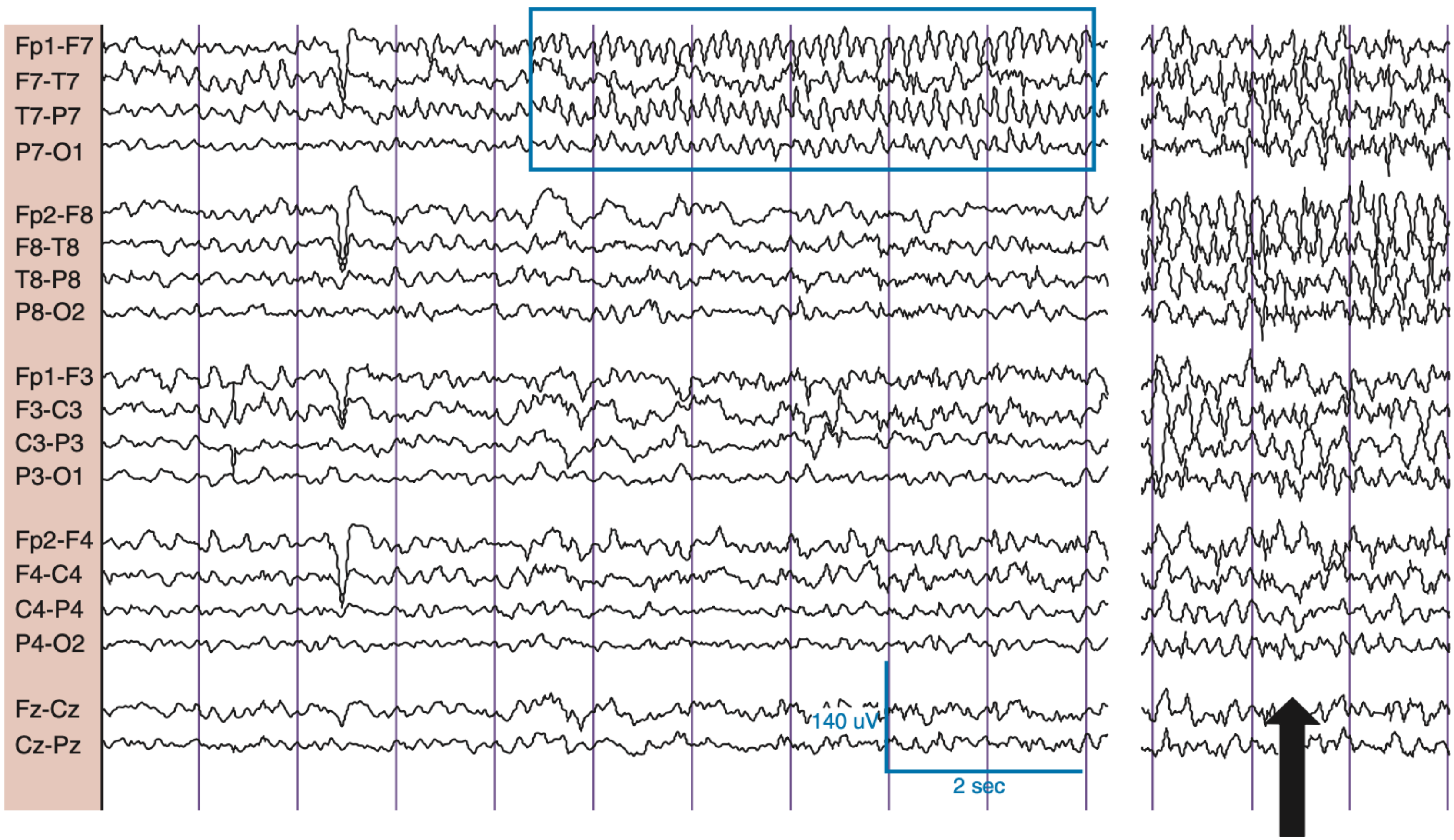


Normal EEG

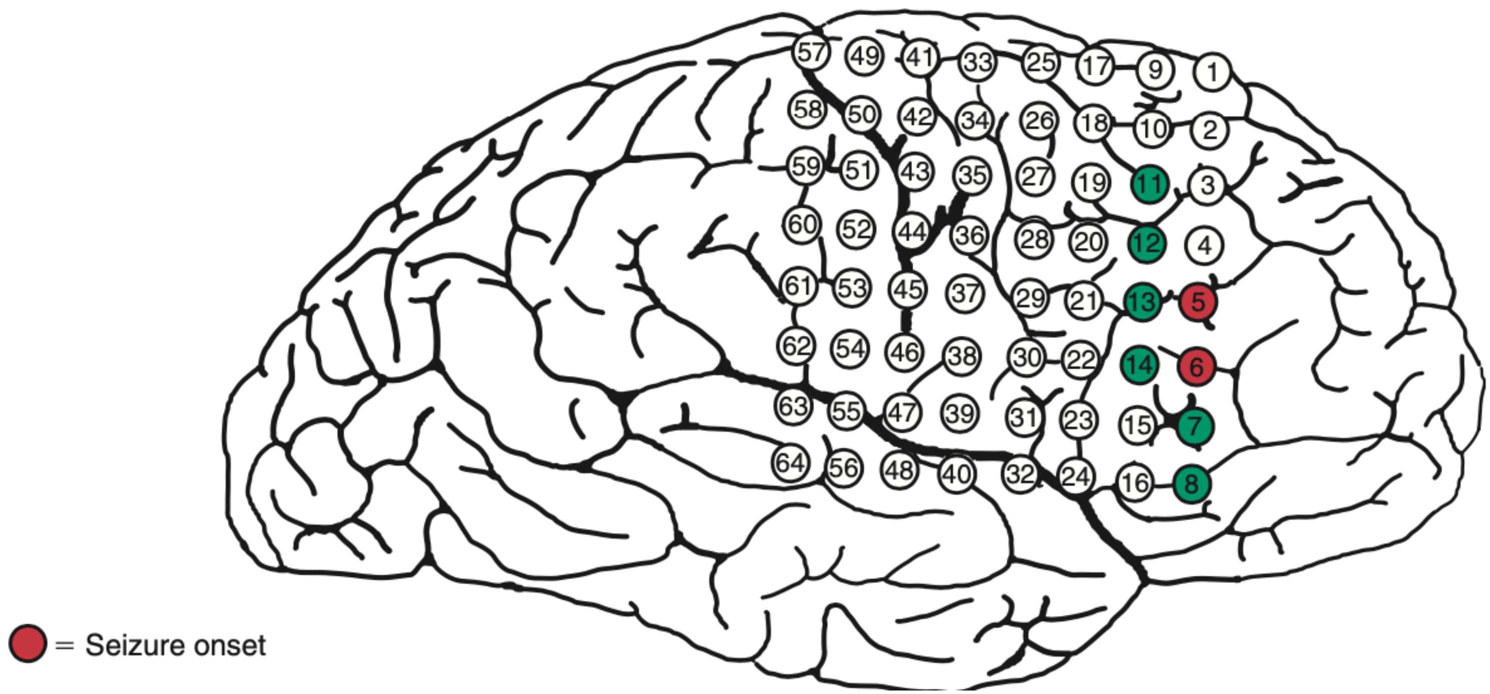


Fp1-F7
F7-T7
T7-P7
P7-O1
Fp2-F8
F8-T8
T8-P8
P8-O2
Fp1-F3
F3-C3
C3-P3
P3-O1
Fp2-F4
F4-C4
C4-P4
P4-O2
Fz-Cz
Cz-Pz
ECGL-ECGR
140 uV
1 sec
05/16/2014 17:16:29
Eyes closed

Temporal lobe seizure



Cortical EEG



Surgical patients usually require intracranial EEG for seizure localization

Frontal lobe seizure

Challenges of epilepsy surgery

Surgical treatment of epilepsy requires precise localization of the seizure focus.

The goal of surgery is maximize the volume of resection while minimizing postoperative neurological deficits.

The most common cause of surgical failure is incomplete resection of the focus.

The problem of localizing the seizure focus can be viewed an inverse source problem.

Forward problem

Consider the flow of current from a source in a bounded domain. The total current has contributions from the source and volume:

$$\mathbf{J} = \mathbf{J}_0 + \sigma \mathbf{E} \ ,$$

where $\mathbf{J}_0$ is the source current, σ is the conductivity and $\mathbf{E}$ is the electric field. Under static conditions, the conservation of charge takes the form $\nabla \cdot \mathbf{J} = 0$. In addition, $\mathbf{E} = -\nabla u$, where u is the potential. The potential then obeys the equation

$$\begin{aligned} \nabla \cdot \sigma(\mathbf{x}) \nabla u &= \nabla \cdot \mathbf{J}_0 \quad \text{in} \quad \Omega \ , \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \partial\Omega \ , \end{aligned}$$

where the Neumann boundary condition prevents the outward flow of current through $\partial\Omega$.

Inverse problem

$$\begin{aligned} \nabla \cdot \sigma(\mathbf{x}) \nabla u &= S \quad \text{in} \quad \Omega \ , \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \partial\Omega \ , \end{aligned}$$

Assuming σ is know, the **inverse source problem** is to determine S from boundary measurements of u. This problem is underdetermined and does not have a unique solution.

Uniqueness can be restored if a priori information about the source is known. For instance, if the source consists of a single current dipole (or even a fixed number of dipoles), then its position and strength can be uniquely determined.

Breaking the rules

Measurements of the potential are generally taken outside the conducting medium. The availability of internal measurements fundamentally alters the inverse problem.

The source can be determined directly from the governing PDE.

Unfortunately, the necessary measurements of the potential cannot generally be acquired in practice.

Acousto-electric source imaging

To overcome the problem of nonuniqueness requires a new physical idea.

Replace inverse boundary value problem by inverse problem with internal data.

Internal data obtained from control of boundary measurements by an acoustic wave.

Inverse problem

Acousto-electric imaging utilizes two interacting fields. The electric current density is spatially modulated by an acoustic wave, while measurements of the potential are recorded.

The inverse problem consists of two steps.

1. Recover an internal functional of the source from boundary measurements. The internal functional is defined at every point of the medium and serves as a proxy for measurements of the potential within the medium.

2. Reconstruct the source from the internal functional.

Outline

- Acousto-electric effect

  — mechanics

  — electrical conduction

- Internal functional

- Stability and inversion

Mechanics

Consider a system of charge carriers in a conducting fluid. If a small-amplitude acoustic wave is incident on the system, the particles will oscillate about their local equilibrium positions. It is then possible to treat the motion of each particle as independent, neglecting hydrodynamic interactions.

The equation of motion of a single particle is of the form

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p \ .$$

Here $\mathbf{v}$ is the velocity of the particle, ρ is its mass density, and p is the pressure in the fluid. We consider a standing time-harmonic plane wave of frequency ω with

$$p = A \cos(\omega t) \cos(\mathbf{k} \cdot \mathbf{x} + \varphi) \ ,$$

where A is the amplitude of the wave, $\mathbf{k}$ is the wave vector and φ is the phase. For simplicity, we have assumed that the speed of sound c_s is constant with $|\mathbf{k}| = \omega / c_s$.

The oscillatory solution to the equation of motion is given by

$$\mathbf{v} = \frac{A}{\rho\omega} \sin(\omega t) \sin(\mathbf{k}\cdot\mathbf{x} + \varphi)\mathbf{k} \ .$$

Thus apart from a transient, the particle moves with the fluid.

In the presence of an applied field, the charge carriers move and generate a current. The current density is of the form $\sum_i q_i \mathbf{v}_i \delta(\mathbf{x} - \mathbf{R}_i(t))$, where $\mathbf{R}_i$ is the position of the ith charge carrier, $\mathbf{v}_i$ is the velocity and q_i is the charge. Since each particle is independent, it follows that the modulated current $\mathbf{J}_\delta$ is given by

$$\mathbf{J}_\delta(\mathbf{x}) = \mathbf{J}_0(\mathbf{x}) \left[1 + \delta \cos(\mathbf{k}\cdot\mathbf{x} + \varphi)\right] \ ,$$

where $\mathbf{J}_0$ is the current in the absence of the acoustic wave and $\delta = A/(\rho c_s^2) \ll 1$ is a small parameter.

The conductivity σ_δ of the medium is proportional to the density of conducting particles and is given by

$$\sigma_\delta(\mathbf{x}) = \sigma(\mathbf{x})\,[1 + \delta\beta \cos(\mathbf{k}\cdot\mathbf{x} + \varphi)] \ ,$$

where σ is the unmodulated conductivity and β is the zero-frequency elasto-electric constant. We conclude that the acoustic wave leads to spatial modulation of the current and the conductivity.

Electric potential

The total current

$$\mathbf{J} = \mathbf{J}_\delta + \sigma_\delta \mathbf{E} \ ,$$

has contributions from the source and the volume. Since $\nabla \cdot \mathbf{J} = 0$ and $\mathbf{E} = -\nabla u$, the potential obeys

$$\begin{aligned} \nabla \cdot \sigma_\delta(\mathbf{x}) \nabla u_\delta &= \nabla \cdot \mathbf{J}_\delta \quad \text{in} \quad \Omega \ , \\ \frac{\partial u_\delta}{\partial n} &= 0 \quad \text{on} \quad \partial\Omega \ . \end{aligned}$$

Next we turn to the derivation of an internal functional from boundary measurements of the potential.

Internal functional

We consider the following auxiliary boundary value problem

$$\nabla \cdot \sigma(\mathbf{x}) \nabla v_j = 0 \qquad \text{in} \quad \Omega\,,$$
$$\frac{\partial v_j}{\partial n} = g_j \qquad \text{on} \quad \partial\Omega\,,$$

where g_j are prescribed boundary sources. Since the unmodulated conductivity σ is known, the solutions v_j in principle can be computed. The identity

$$\Sigma_\delta^{(j)} = \int_\Omega \Big[(\sigma_\delta - \sigma) \nabla u_\delta \cdot \nabla v_j + v_j \nabla \cdot \mathbf{J}_\delta \Big]\, dx \ ,$$

follows from an integration by parts and use of the boundary conditions. Here the surface term $\Sigma_\delta^{(j)}$ is defined by

$$\Sigma_\delta^{(j)} = \int_{\partial\Omega} u_\delta \sigma g_j dx \ .$$

Evidently, $\Sigma_\delta^{(j)}$ can be determined from boundary measurement of u_δ.

We now introduce the asymptotic expansions for u_δ and $\Sigma_\delta^{(j)}$ as

$$u_\delta = u_0 + \delta u_1 + \cdots ,$$

$$\Sigma_\delta^{(j)} = \Sigma_0^{(j)} + \delta \Sigma_1^{(j)} + \cdots .$$

At $O(\delta)$ we have

$$\Sigma_1^{(j)} = \int_\Omega (\beta\sigma\nabla u_0 - \mathbf{J}_0) \cdot \nabla v_j \cos(\mathbf{k}\cdot\mathbf{x} + \varphi)\, dx.$$

Since $\Sigma_\delta^{(j)}$ is determined by boundary measurements, $\Sigma_1^{(j)}$ is known. By varying $\mathbf{k}$ and φ and inverting a Fourier transform, we can recover the internal functional

$$H_j = (\beta\sigma\nabla u_0 - \mathbf{J}_0) \cdot \nabla v_j$$

at every point in Ω.

Inverse problem

The inverse problem now consists of recovering the source $\mathbf{J}_0$ from the internal functional H_j. We emphasize that this is an unusual inverse problem, since the data H_j is known everywhere in Ω.

Suppose there are multiple boundary sources. We then solve the system of linear equations

$$H_j = (\beta\sigma\nabla u_0 - \mathbf{J}_0) \cdot \nabla v_j$$

for the vector field

$$\mathbf{A} = \beta\sigma\nabla u_0 - \mathbf{J}_0 \ .$$

Here we assume that (∇v_j) are linearly independent. It can be seen that this condition holds if the boundary sources g_j are appropriately chosen. After determining u_0 by solving the boundary value problem,

$$\begin{aligned} \nabla \cdot \sigma(\mathbf{x})\nabla u_0 &= \nabla \cdot \mathbf{J}_0 \quad \text{in} \quad \Omega \ , \\ \frac{\partial u_0}{\partial n} &= 0 \quad \text{on} \quad \partial\Omega \ , \end{aligned}$$

we can find the source from the inversion formula

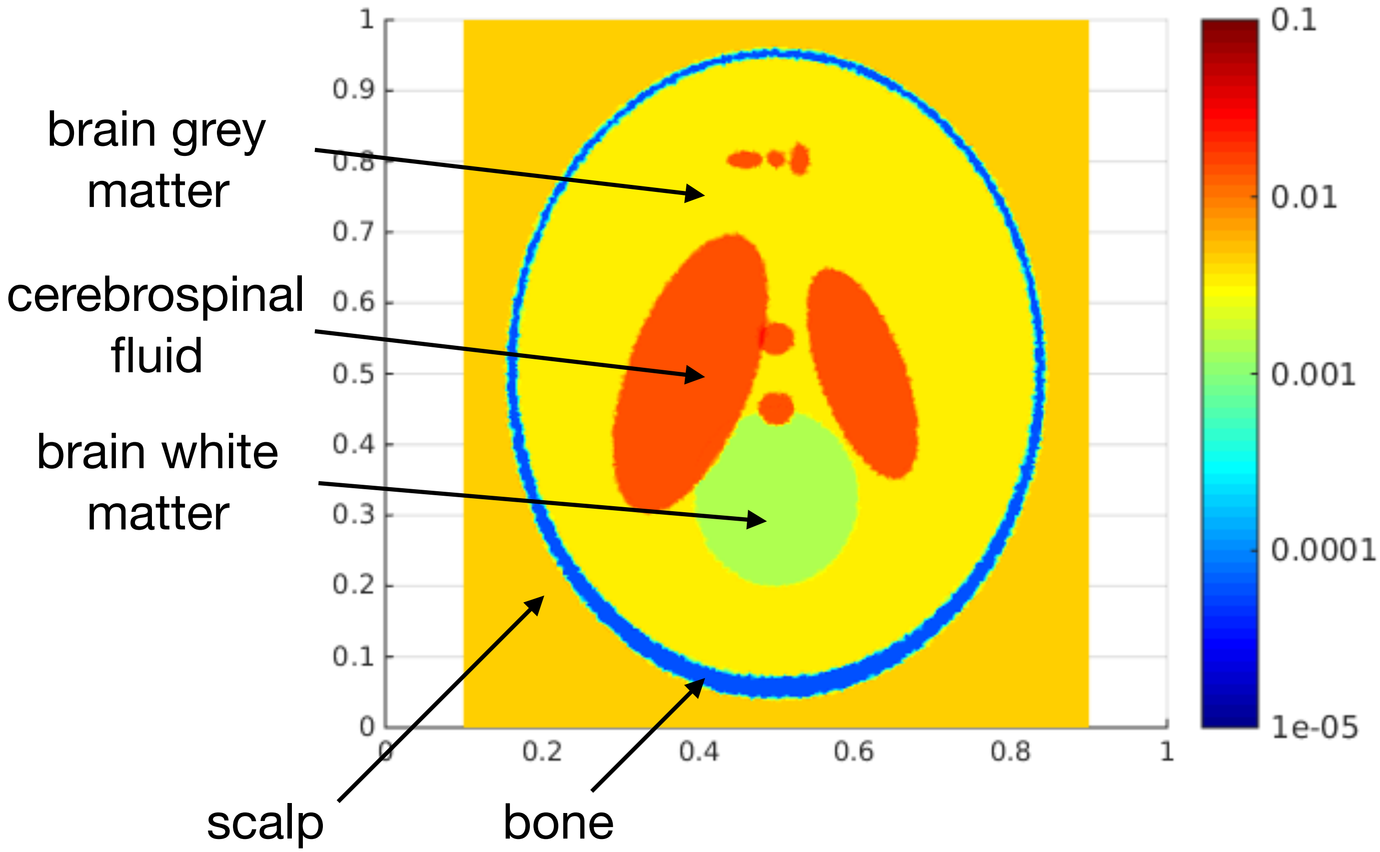
$$\mathbf{J}_0 = \beta\sigma\nabla u_0 - \mathbf{A}.$$

The reconstruction procedure uniquely recovers $\mathbf{J}_0$ with Lipschitz stability.

Theorem (Li, S, Yang, Zhong). Let $\mathbf{J}_0$ and $\tilde{\mathbf{J}}_0$ be currents reconstructed from the corresponding internal functionals H_j and $\tilde{H}_j$. Then

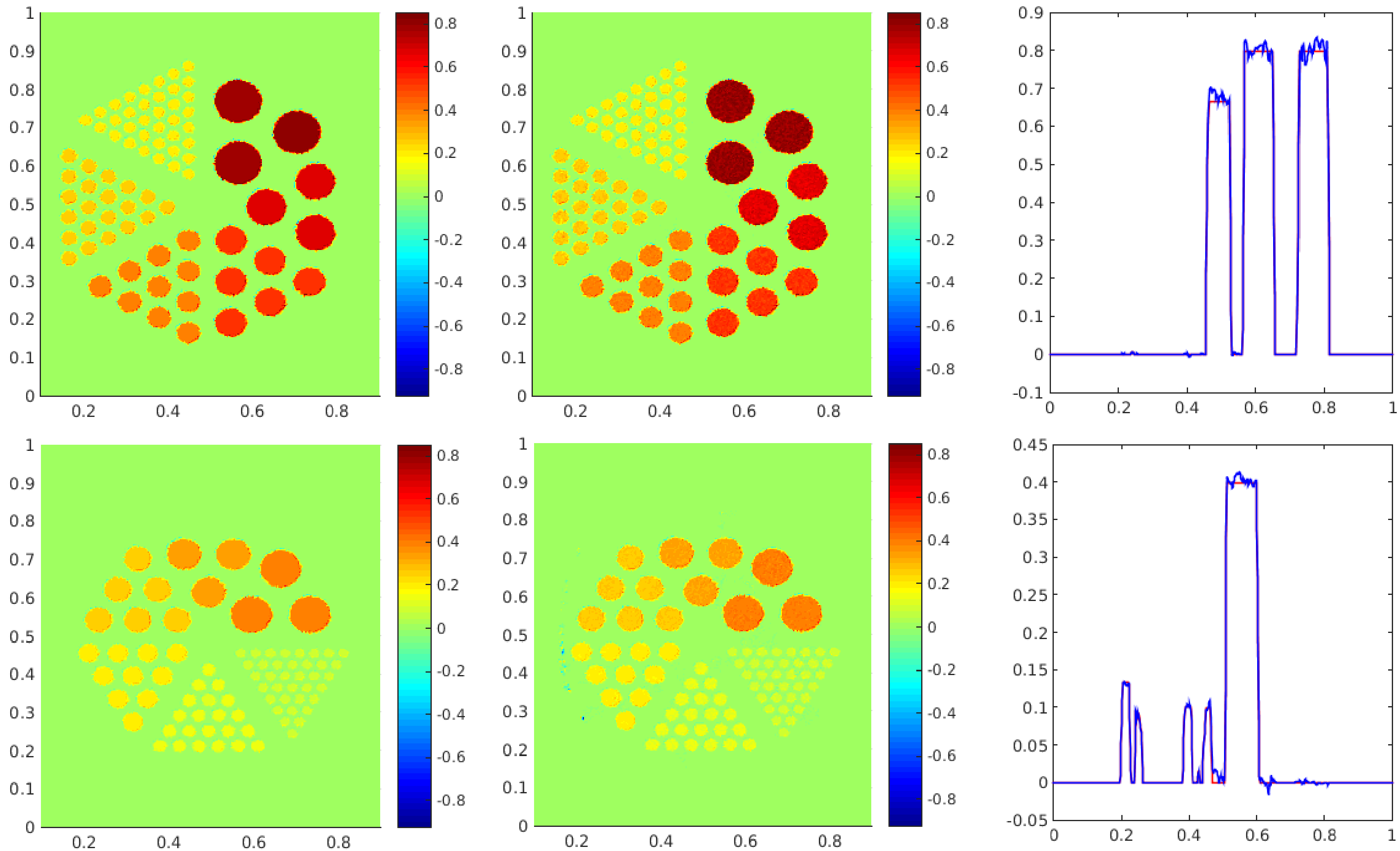
$$\|\mathbf{J}_0 - \tilde{\mathbf{J}}_0\|_{(L^2(\Omega))^n} \le C \sum_j \|H_j - \tilde{H}_j\|_{L^2(\Omega)} \; .$$

Numerical reconstructions



The conductivity σ

Model Reconstruction Cross section



$$g_1(\mathbf{x}) = (1,0) \cdot \hat{\mathbf{n}}(\mathbf{x}) \text{ and } g_2(\mathbf{x}) = (0,1) \cdot \hat{\mathbf{n}}(\mathbf{x})$$

Is it practical?

- Acousto-electric effect has been measured in tissue, but $\beta \ll 1$.

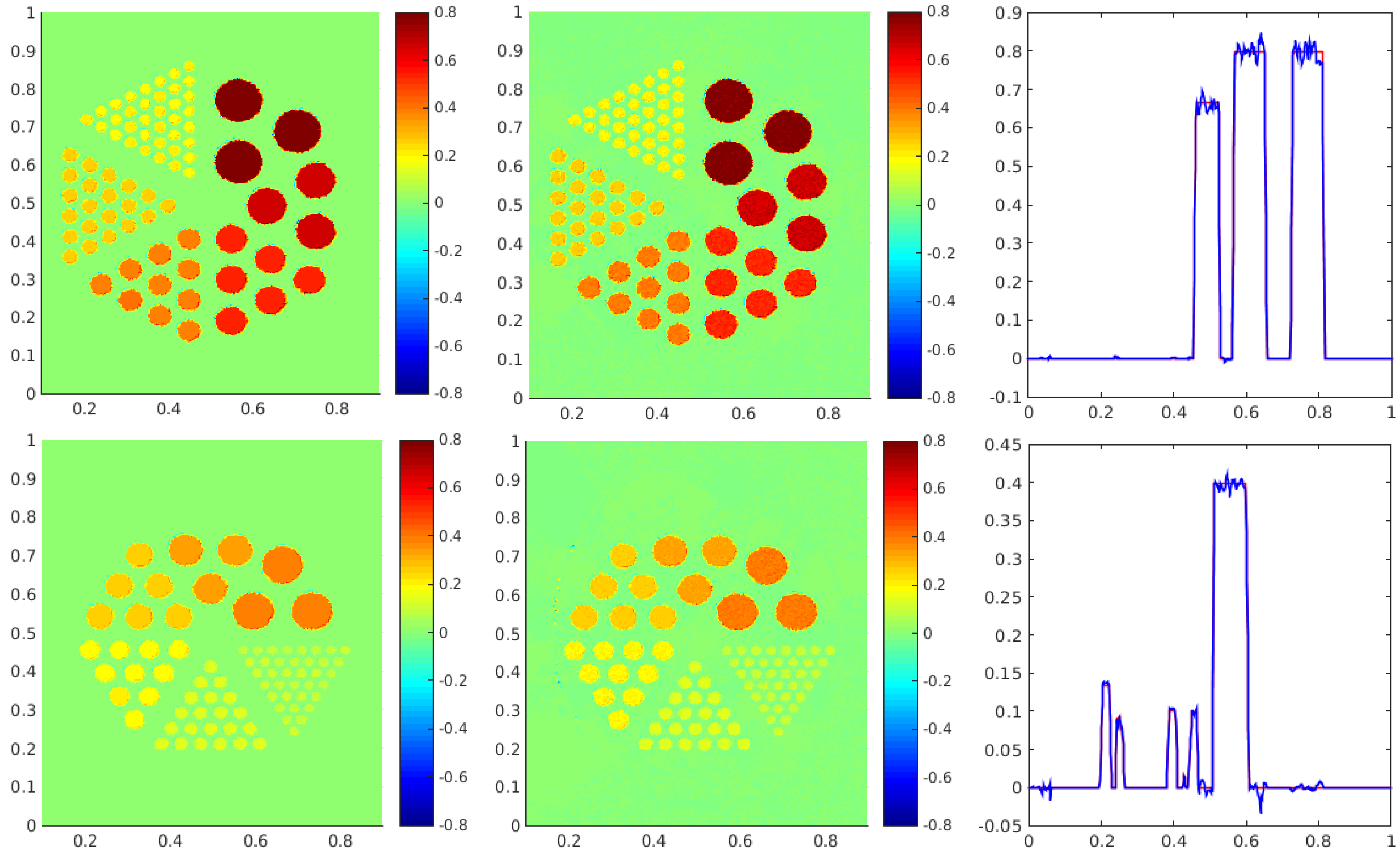
- Conductivity is unknown

  — ultrasound-modulated EIT and its variants

- Band-limited approximation to internal functional

- Partial data

Partial data



Measurements and boundary sources on bottom face are not used

Maxwell equations

The preceding theory holds for static sources and fields. More generally, the inverse source problem for electrodynamics can be considered. This contains as a special case the problem of magnetoencephalography.

The time-harmonic Maxwell equations in a bounded domain $\Omega \subset \mathbb{R}^3$ are of the form

$$\begin{aligned} i\omega\varepsilon\,\mathbf{E} + \nabla \times \mathbf{H} &= \mathbf{J} + \sigma\mathbf{E} \qquad \text{in} \quad \Omega, \\ -i\omega\mathbf{H} + \nabla \times \mathbf{E} &= 0 \qquad \text{in} \quad \Omega\ . \end{aligned}$$

We also impose the impedance boundary condition

$$\mathbf{H} \times \hat{\mathbf{n}} - \lambda(\hat{\mathbf{n}} \times \mathbf{E}) \times \hat{\mathbf{n}} = 0 \qquad \text{on} \quad \partial\Omega\ .$$

The inverse source problem is to reconstruct the source $\mathbf{J}$ from boundary measurements, assuming that the coefficients ε and σ are known. A typical measurement is the tangential electric field on the boundary:

$$\mathbf{g} = (\hat{\mathbf{n}} \times \mathbf{E}) \times \hat{\mathbf{n}}|_{\partial\Omega}.$$

This problem is underdetermined and does not have a unique solution.

Acoustic modulation

We now examine the effect of acoustic modulation. As usual, the current density $\mathbf{J}_\delta$ is modulated according to

$$\mathbf{J}_\delta = \mathbf{J}(1 + \delta \cos(\mathbf{k} \cdot \mathbf{x} + \varphi)).$$

Likewise, the conductivity σ_δ and permittivity ε_δ are also modulated,

$$\begin{aligned} \varepsilon_\delta &= \varepsilon(1 + \delta\gamma_\varepsilon \cos(\mathbf{k} \cdot \mathbf{x} + \varphi)), \\ \sigma_\delta &= \sigma(1 + \delta\gamma_\sigma \cos(\mathbf{k} \cdot \mathbf{x} + \varphi)). \end{aligned}$$

It follows that the modulated electric and magnetic fields $\mathbf{E}_\delta$ and $\mathbf{H}_\delta$ satisfy the modified Maxwell equations

$$\begin{aligned} i\omega\varepsilon_\delta \mathbf{E}_\delta + \nabla \times \mathbf{H}_\delta = \mathbf{J}_\delta + \sigma_\delta \mathbf{E}_\delta \qquad & \text{in} \quad \Omega, \\ -i\omega \mathbf{H}_\delta + \nabla \times \mathbf{E}_\delta = 0 \qquad & \text{in} \quad \Omega, \end{aligned}$$

together with the boundary condition

$$\mathbf{H}_\delta \times \hat{\mathbf{n}} - \lambda(\hat{\mathbf{n}} \times \mathbf{E}_\delta) \times \hat{\mathbf{n}} = 0 \qquad \text{on} \quad \partial\Omega.$$

The corresponding boundary measurement becomes

$$\mathbf{g}_\delta := (\hat{\mathbf{n}} \times \mathbf{E}_\delta) \times \hat{\mathbf{n}}|_{\partial\Omega}.$$

Internal functional

We consider the auxiliary fields $\mathbf{F}$ and $\mathbf{G}$ which obey the Maxwell equations

$$i\omega\varepsilon\,\mathbf{F}^* + \nabla\times\mathbf{G}^* = \sigma\mathbf{F}^* \quad \text{in} \quad \Omega\ ,$$

$$-i\omega\mu\mathbf{G}^* + \nabla\times\mathbf{F}^* = 0 \quad \text{in} \quad \Omega\ ,$$

We find that the internal functional is of the form

$$\mathbf{Q} = i\omega\gamma_J\mathbf{J} + \left(\omega^2\varepsilon\gamma_\varepsilon + i\omega\sigma\gamma_\sigma\right)\mathbf{E}\ ,$$

which is known at every point in Ω.

The electric field $\mathbf{E}$ obeys

$$\nabla\times\nabla\times\mathbf{E} - k^2\mathbf{E} = \frac{\mathbf{Q}}{\gamma_J}\ ,$$

where $k^2 = \omega^2\varepsilon\left(1-\frac{\gamma_\varepsilon}{\gamma_J}\right) + i\omega\sigma\left(1-\frac{\gamma_\sigma}{\gamma_J}\right)$. Solving for $\mathbf{E}$ leads to the formula for the current

$$\mathbf{J} = \frac{1}{i\omega\gamma_J}\left[\mathbf{Q} - \left(\omega^2\varepsilon\gamma_\varepsilon + i\omega\sigma\gamma_\sigma\right)\mathbf{E}\right].$$

Stability

Theorem (Li, S, Yang, Zhong).

1. If $\gamma_\varepsilon = \gamma_J$, $\gamma_\sigma \neq \gamma_J$, and $\Omega \subseteq \operatorname{supp}\sigma$, then the source $\mathbf{J}$ is uniquely determined. If in addition $|\mathrm{Re}(k^2)|$ is strictly bounded away from zero, then we have the following stability estimates. If $\gamma_\sigma/\gamma_J < 1$,

$$\|\mathbf{J} - \tilde{\mathbf{J}}\|_{(L^2(\Omega))^3} \leq C\|\mathbf{Q} - \tilde{\mathbf{Q}}\|_{(L^2(\Omega))^3},$$

and if $\gamma_\sigma/\gamma_J > 1$,

$$\|\mathbf{J} - \tilde{\mathbf{J}}\|_{(L^2(\Omega))^3} \leq C(\|\mathbf{Q} - \tilde{\mathbf{Q}}\|_{(L^2(\Omega))^3} + \|\mathbf{g} - \tilde{\mathbf{g}}\|_{(L^2(\partial\Omega))^3}.$$

2. If $\gamma_\varepsilon \neq \gamma_J$, then the source $\mathbf{J}$ is uniquely determined.

3. If $\gamma_\varepsilon/\gamma_J > 1$, we have the following stability estimate

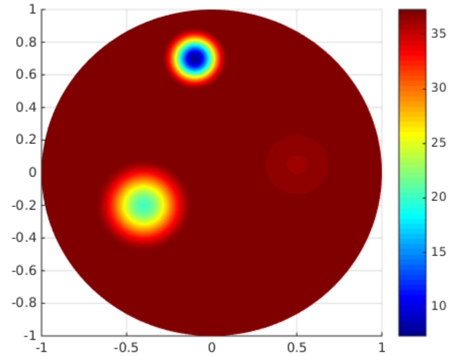
$$\|\mathbf{J} - \tilde{\mathbf{J}}\|_{(L^2(\Omega))^3} \leq C\|\mathbf{Q} - \tilde{\mathbf{Q}}\|_{(L^2(\Omega))^3},$$

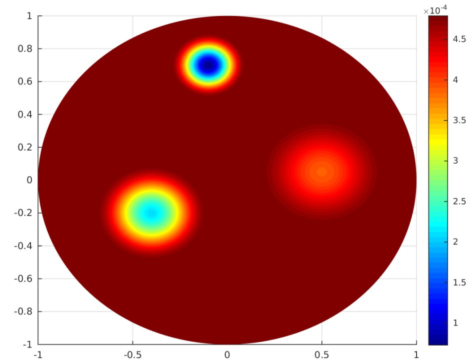
and if $\gamma_\varepsilon/\gamma_J < 1$,

$$\|\mathbf{J} - \tilde{\mathbf{J}}\|_{(L^2(\Omega))^3} \leq C(\|\mathbf{Q} - \tilde{\mathbf{Q}}\|_{(L^2(\Omega))^3} + \|\mathbf{g} - \tilde{\mathbf{g}}\|_{(L^2(\partial\Omega))^3}.$$

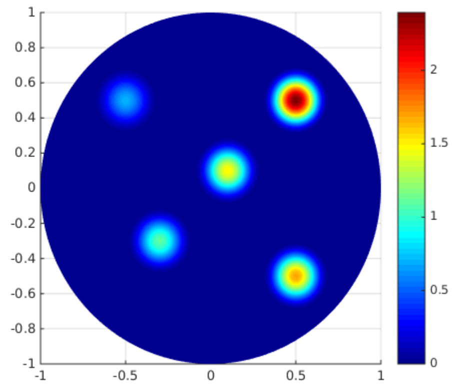
Numerical simulations

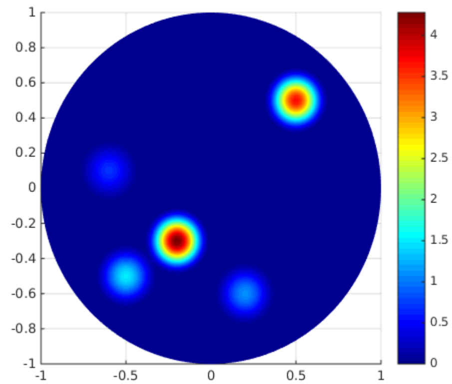
ϵ, σ



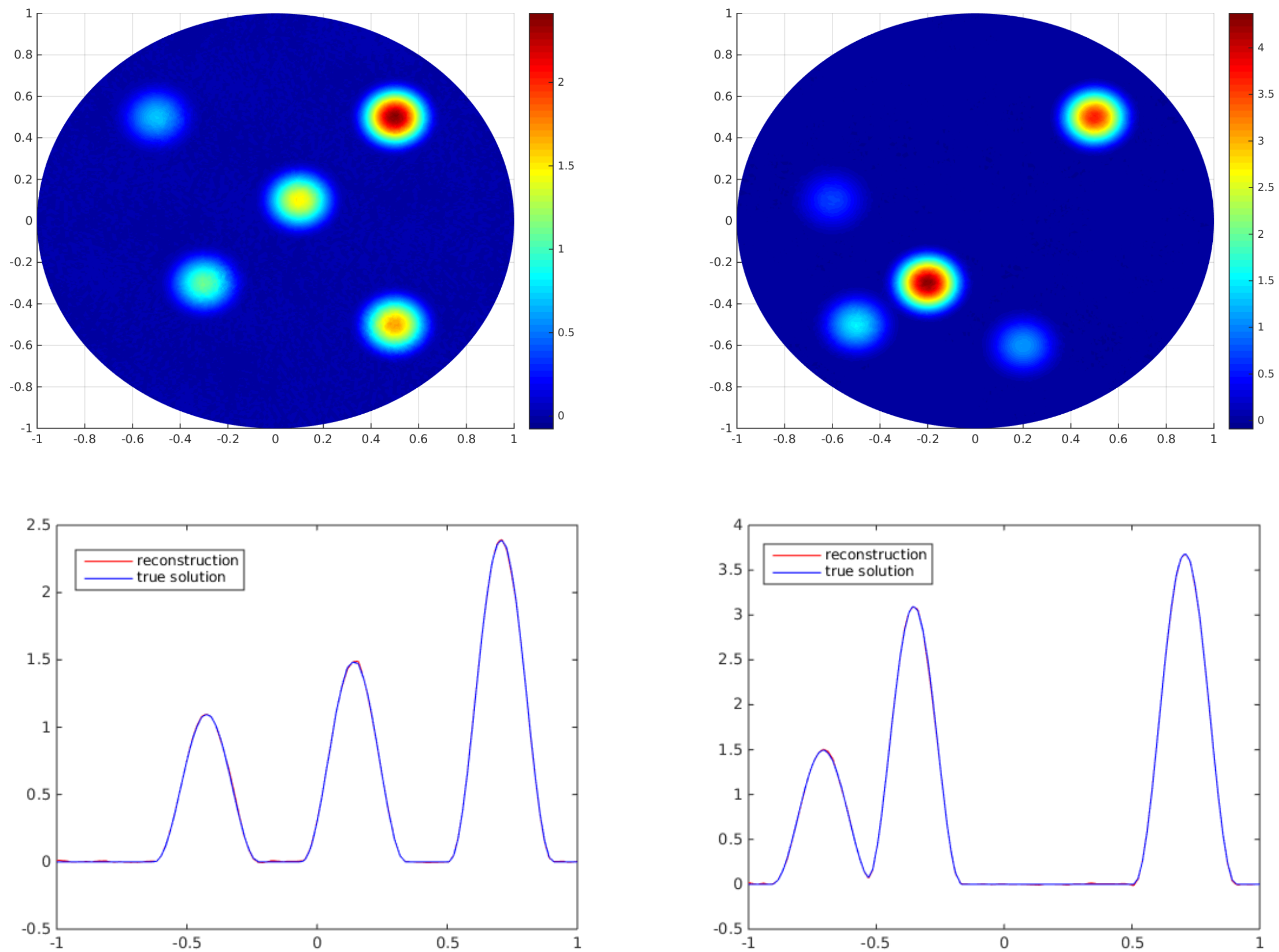


$\mathbf{J}$





Reconstructions



Conclusions

Model of acoustically-modulated electrical source problem.

Boundary measurements of the electric potential in the presence of acoustic modulation determines an internal functional of the source.

The source current can be reconstructed from the internal functional.

The reconstruction is unique with Lipschitz stability.

Numerical implementations of the proposed method with both full and partial boundary measurements.

Results extent to the full Maxwell system (including MEG).

THANK YOU!