Diffraction Tomography for Generalized Incident Fields

joint work with Otmar Scherzer and Clemens Kirisits

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Workshop 2: "Inverse Problems on Small Scales"

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Inverse scattering problem

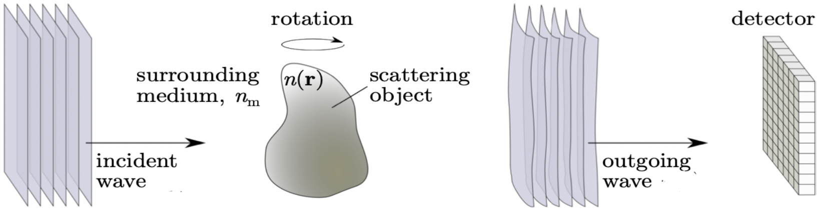

Figure: Conceptual setup, tomographic data acquisition. [1]

[1] from P. Müller, M. Schürrmann, and J. Guck. "The Theory of Diffraction Tomography", 2016

Inverse scattering problem

- $n : \mathbb{R}^3 \to \mathbb{C}$ refractive index, $n(\mathbf{r}) = 1$ outside of $\mathcal{B}_{r_s}(\mathbf{0})$
- $k_0 = \omega / c_0$ wave number
- Consider the system

$$\Delta u^{\text{inc}} + k_0^2 u^{\text{inc}} = 0$$
$$\Delta u^{\text{tot}} + k_0^2 n^2 u^{\text{tot}} = 0$$
$$u^{\text{tot}} = u^{\text{sca}} + u^{\text{inc}}$$

- $f(\mathbf{r}) := k_0^2[n(\mathbf{r})^2 - 1]$ scattering potential
- $\{\mathbf{r} \in \mathbb{R}^3 \mid r_3 = \pm r_M\}$ detector plane

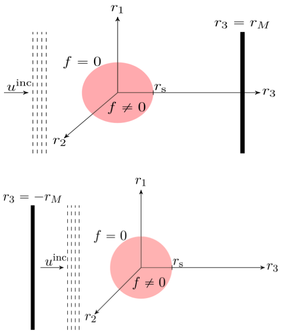

Figure: Reflection and transmission imaging [2]

- Measurement data $\mathcal{M}(r_1, r_2, \pm r_M) := [u^{\text{tot}} - u^{\text{inc}}](r_1, r_2, \pm r_M)$ include information about amplitudes and phase.

Reconstruct the scattering potential $f(\mathbf{r})$ from the received information $\mathcal{M}(r_1, r_2, \pm r_M)$.

[2] from C. Kirisits, M. Quellmalz, M. Ritsch-Marte, O. Scherzer, E. Setterqvist, and G. Steidl. "Fourier reconstruction for diffraction tomography of an object rotated into arbitrary orientations", 2021.

Diffraction tomography provides under Born approximation an analytical solution to the inverse scattering problem.

It is valid if...

- ...n differs only weakly from the homogeneous background (small contrast)
- ...scattering is sufficiently weak such that multiple scattering can be neglected

and highly efficient with respect to computational time!

We combine the system and obtain

$$\left.\begin{array}{c}\Delta u^{\text{inc}} + k_0^2 u^{\text{inc}} = 0\\ \Delta u^{\text{tot}} + k_0^2 n^2 u^{\text{tot}} = 0\\ u^{\text{tot}} = u^{\text{sca}} + u^{\text{inc}}\end{array}\right\} \quad \longrightarrow \quad \Delta u^{\text{sca}} + k_0^2 u^{\text{sca}} = -f(u^{\text{inc}} + u^{\text{sca}}).$$

Assuming $u^{\text{sca}} \ll u^{\text{inc}}$ we obtain the Born approximation u, that satisfies

$$\Delta u + k_0^2 u = -f u^{\text{inc}}$$

and a convolution with the fundamental solution $\mathbf{x} \to G(\mathbf{x}) = \frac{e^{ik_0 \|\mathbf{x}\|}}{4\pi \|\mathbf{x}\|}$, $\mathbf{x} \neq 0$ gives

$$u(\mathbf{r}) = \int_{\mathbb{R}^3} G(\mathbf{r} - \mathbf{r}') f(\mathbf{r}') u^{\text{inc}}(\mathbf{r}') d\mathbf{r}', \quad \mathbf{r} \in \mathbb{R}^3$$

The incident field

In conventional diffraction tomography u^{inc} is assumed to be a monochromatic plane wave

$$u^{\text{inc}}(\mathbf{r}) = e^{ik_0 \mathbf{s}_0 \cdot \mathbf{r}}$$

arriving from the direction $\mathbf{s}_0 \in \mathbb{S}^2$.

For this case there are reconstruction algorithms of back-propagation type:

- A. C. Kak and M. Slaney. “Principles of Computerized Tomographic Imaging”, 2001.
- A. Devaney. “A filtered backpropagation algorithm for diffraction tomography”, 1982.
- C. Kirisits, M. Quellmalz, M. Ritsch-Marte, O. Scherzer, E. Setterqvist, and G. Steidl. “Fourier reconstruction for diffraction tomography of an object rotated into arbitrary orientations”, 2021.
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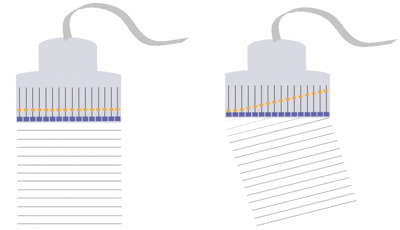
What if the plane wave assumption is not compatible with the emitting device used in practice?

Novel ultrasound devices allow to control the shape of the incident wave:



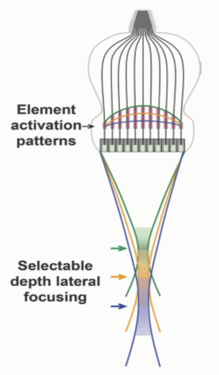


Figure: beam focusing[3]

- Limited width of the plane wave
- Focused imaging

➔ Make diffraction tomography applicable to these setups.

[3] from https://radiologykey.com/ultrasound-12/

The generalized incident field

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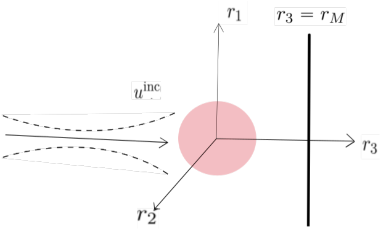
We extend the incident wave to be a superposition of plane waves interacting with different propagation directions:

$$u^{\text{inc}}(\mathbf{r}) = \int_{\mathbb{S}^2} a(\mathbf{s}) e^{ik_0 \mathbf{s}\cdot\mathbf{r}} d\mathbf{s},$$

where $a \in L^2(\mathbb{S}^2)$.



Then, the resulting scattered wave reads as

$$u(\mathbf{r}) = \int_{\mathbb{S}^2} a(\mathbf{s}) \int_{\mathbb{R}^3} \frac{e^{ik_0(\|\mathbf{r}-\mathbf{r}'\|+\mathbf{r}'\cdot\mathbf{s})}}{4\pi\|\mathbf{r}-\mathbf{r}'\|} f(\mathbf{r}') d\mathbf{r}' d\mathbf{s}, \qquad \mathbf{r} \in \mathbb{R}^3$$

and is evaluated at the measurement plane.

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We define

- $\overline{\mathbf{k}} := (k_1, k_2) \in \mathbb{R}^2$
- $\mathbf{k}^{\pm} = \mathbf{k}^{\pm}(\overline{\mathbf{k}}) := (\overline{\mathbf{k}}, \pm\kappa)^{\intercal}$
- $\kappa = \kappa(\overline{\mathbf{k}}) := \begin{cases} \sqrt{k_0^2 - |\overline{\mathbf{k}}|^2}, & |\overline{\mathbf{k}}| \leq k_0 \\ i\sqrt{|\overline{\mathbf{k}}|^2 - k_0^2}, & \overline{\mathbf{k}} > k_0 \end{cases}$
- $G_1^{\kappa}(r_M) = \sqrt{\frac{\pi}{2}}\frac{i}{\kappa}e^{i\kappa|r_M|}$

and perform the two-dimensional Fourier transform, such that

$$\mathcal{F}_{1,2}u(\overline{\mathbf{k}}, \pm r_M) = G_1^{\kappa}(r_M)\int_{\mathbb{S}^2} a(\mathbf{s})\mathcal{F}f(\mathbf{k}^{\pm} - k_0\mathbf{s})ds, \quad \overline{\mathbf{k}} \in \mathbb{R}^2,\ |\overline{\mathbf{k}}| < k_0$$

gives the relation between the recorded scattered wave and the scattering potential.[4]

[4] follows from Theorem 3.1 in C. Kirisits, M. Quellmalz, M. Ritsch-Marte, O. Scherzer, E. Setterqvist, and G. Steidl. "Fourier reconstruction for diffraction tomography of an object rotated into arbitrary orientations", 2021.

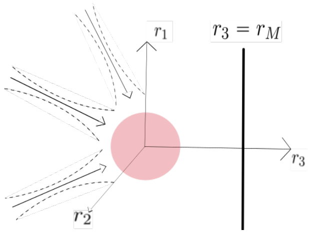
Measurement setup

For 3D tomographic reconstruction:

- rotate the direction of incidence by $\mathcal{R} \in SO(3)$
- $u^{\text{inc}}(\mathbf{r}, \mathcal{R}) = \int_{\mathbb{S}^2} a(\mathbf{s}) e^{ik_0 \mathcal{R}^\intercal \mathbf{s} \cdot \mathbf{r}} d\mathbf{s}$
- resulting scattered wave $u(\mathbf{r}, \mathcal{R})$ satisfies

$$(\Delta + k_0^2) u(\mathbf{r}, \mathcal{R}) = -f(\mathbf{r}) u^{\text{inc}}(\mathbf{r}, \mathcal{R})$$

- measurements given by $\mathcal{M}(\bar{\mathbf{k}}, \pm r_M, \mathcal{R}) = \mathcal{F}_{1,2} u(\bar{\mathbf{k}}, \pm r_M, \mathcal{R})$



We propose the following setups:

- Rotation of incidence direction and fixed measurement plane
- Rotation of the object

The measurements are related to the scattering potential via

$$\mathcal{M}(\bar{\mathbf{k}}, \pm r_M, \mathcal{R}) = G_1^\kappa(r_M) \int_{\mathbb{S}^2} a(\mathcal{R}^{\intercal^{-1}} \mathbf{s}) \mathcal{F} f(\mathbf{k}^\pm - k_0 \mathbf{s}) d\mathbf{s},$$

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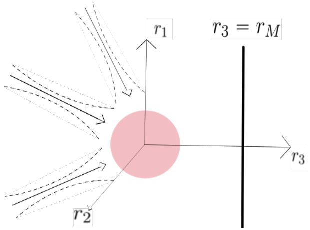

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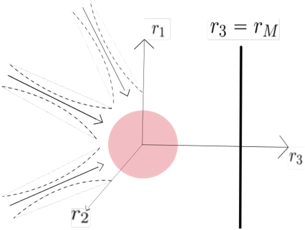

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for all $\bar{\mathbf{k}} \in \mathbb{R}^2$ and all $\mathcal{R} \in SO(3)$.

We rewrite the equation to

$$\mathcal{M}(\mathcal{R}) = G_1^{\kappa}(r_M) \int_{\mathbb{S}^2} a(\mathcal{R}^{\intercal\,-1}\mathbf{s})g(\mathbf{s})d\mathbf{s},$$

where $g(\mathbf{s}) = \mathcal{F}f(\mathbf{k}^{\pm} - k_0\mathbf{s})$ for fixed $\overline{\mathbf{k}} \in \mathbb{R}^2$.

- $g \in L^2(\mathbb{S}^2)$ has a expansion in spherical Fourier series

$$g(s) = \sum_{n\geq 0} \sum_{k=-n}^{n} g_n^k Y_n^k(s),$$

where the basis elements Y_n^k are spherical harmonics (Basis of $L^2(\mathbb{S}^2)$).

- The rotated function $a \in L^2(\mathbb{S}^2)$ has the expansion

$$a(\mathcal{R}^{\intercal\,-1}\mathbf{s}) = \sum_{n\geq 0} \sum_{k=-n}^{n} a_n^k Y_n^k(\mathcal{R}^{\intercal\,-1}\mathbf{s}) = \sum_{n\geq 0} \sum_{k=-n}^{n} a_n^k \sum_{j=-n}^{n} D_n^{j,k}(\mathcal{R}^{\intercal}) Y_n^j(\mathbf{s}),$$

where $a_n^k = \langle a, Y_n^k \rangle_{L^2(\mathbb{S}^2)}$ and $D_n^{j,k}$ are Wigner D-matrices. [5]

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$$\mathcal{M}(\mathcal{R}) = G_1^\kappa(r_M) \int_{\mathbb{S}^2} a(\mathcal{R}^{\intercal\,-1}\mathbf{s})g(\mathbf{s})ds, \quad \text{for all } \mathcal{R} \in SO(3)$$

Using

- the basis representations of $a, g \in L^2(\mathbb{S}^2)$ and
- calculating the rotational Fourier coefficients $\mathcal{M}_n^{j,k} = \langle \mathcal{M}, D_n^{j,k} \rangle_{L^2(SO(3))}$,
- the orthogonality properties

$$\langle Y_n^k, Y_{n'}^{k'} \rangle_{L^2(\mathbb{S}^2)} = \delta_{n,n'}\delta_{k,k'}$$

$$\langle D_n^{j,k}, D_{n'}^{j',k'} \rangle_{L^2(SO(3))} = \frac{8\pi^2}{2n+1}\delta_{n,n'}\delta_{j,j'}\delta_{k,k'}$$

gives the spherical Fourier coefficients

$$g_n^j = \frac{2n+1}{8\pi^2 G_1^\kappa(r_M)} \frac{\mathcal{M}_n^{j,k}}{a_n^k}, \quad \text{for all } n \in \mathbb{N}_0,\ j,k = -n \dots, n.$$

Inversion

Let $\mathcal{M} \in L^2(SO(3))$ and $a \in L^2(\mathbb{S}^2)$ be given. Find $g \in L^2(\mathbb{S}^2)$ from

$$\mathcal{M}(\mathcal{R}) = G_1^{\kappa}(r_M) \int_{\mathbb{S}^2} a(\mathcal{R}^{\intercal^{-1}}\mathbf{s}) g(\mathbf{s}) ds,$$

for all $\mathcal{R} \in SO(3)$.

Calculate

1. $a_n^k = \langle a, Y_n^k \rangle_{L^2(\mathbb{S}^2)}$, for all $n \in \mathbb{N}_0$, $k = -n \dots, n$,
2. $\mathcal{M}_n^{j,k} = \langle \mathcal{M}, D_n^{j,k} \rangle_{L^2(SO(3))}$, for all $n \in \mathbb{N}_0$, $j, k = -n \dots, n$,
3. $g_n^j = \frac{2n+1}{8\pi^2 G_1^{\kappa}(r_M)} \frac{\mathcal{M}_n^{j,k}}{a_n^k}$, for all $n \in \mathbb{N}_0$, $j, k = -n \dots, n$,

then we reconstruct

$$g(\mathbf{s}) = \sum_{n \geq 0} \sum_{j=-n}^{n} g_n^j Y_n^j(\mathbf{s}).$$

Finally, we obtain the relation

$$\sum_{n\geq 0}\sum_{j=-n}^{n}\frac{2n+1}{8\pi^2 G_1^{\kappa}(r_M)}\frac{\mathcal{M}_n^{j,k}(\bar{\mathbf{k}})}{a_n^k}Y_n^j(\mathbf{s})=\mathcal{F}f(\mathbf{k}^{\pm}-k_0\mathbf{s}),$$

for all $\bar{\mathbf{k}}\in\mathbb{R}^2$, $|\bar{\mathbf{k}}|<k_0$, $\mathbf{s}\in\mathbb{S}^2$ and $n\in\mathbb{N}_0,\ j,k=-n\dots,n$.

- scattering potential f can be obtained via the inverse Fourier transformation
- measurements provide access to f on a volume

$$\left\{(\mathbf{k}^{\pm}-k_0\mathbf{s}):\ \bar{\mathbf{k}}\in\mathbb{R}^2,|\bar{\mathbf{k}}|<k_0,\ \mathbf{s}\in\mathbb{S}^2\right\}$$

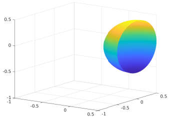
in k-space.

Frequency coverage

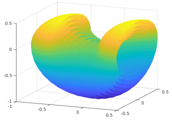
3D frequency coverage for reflection imaging according to a parametrization $\mathbf{s}(\theta, \varphi) \in \mathbb{S}^2$ reads as

$$\left\{ \begin{pmatrix} k_1 - k_0 \sin\theta \cos\varphi \\ k_2 - k_0 \sin\theta \sin\varphi \\ -\kappa(\overline{\mathbf{k}}) - k_0 \cos\theta \end{pmatrix} : \ \overline{\mathbf{k}} \in \mathbb{R}^2, |\overline{\mathbf{k}}| < k_0, \ \theta \in [0,\pi], \ \varphi \in [0, 2\pi) \right\}$$

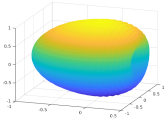
and we illustrate for $k_0 = 0.5$:



$\theta = 0$, $\varphi = 0$



$\theta = [0, \pi]$, $\varphi = 0$



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Looking forward

- Currently, we work on implementation and numerical tests
- Limited view: Reconstructing f from a few rotation angles, i.e. if the available measurement data are limited.

Thank you!