Lippman-Schwinger-Lanczos algorithm for inverse scattering problems.

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October 17, 2022

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- Reduced Order Models (ROMs) for forward problems: If e.g. PDE is linear, find a low dimensional matrix that acts like the differential operator.
- Model reduction theory is a large field, but only recently have data driven ROMs been used for *inverse problems*.

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- Recent work LS in time domain (Borcea et. al 2022 archived)

Time domain SISO problem

$$\begin{aligned} u_{tt} + Au &= 0 \text{ in } \Omega \times [0, \infty) & (1) \\ u(t = 0) &= g \text{ in } \Omega & (2) \\ u_t(t = 0) &= 0 \text{ in } \Omega & (3) \end{aligned}$$

where

$$A = A_0 + q \tag{4}$$

- $A_0 \geq 0$ is known background, (for example $A_0 = -\Delta$),
- $q(x) \geq 0$ is our unknown potential
- initial data g is localized (approximate delta) source
- assume homogeneous Neumann boundary conditions on the spatial boundary $\partial\Omega$.

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- The inverse problem is as follows: Given

$$\{F(k\tau)\} \quad \text{for} \quad k = 0, \ldots, 2n-2,$$

reconstruct q.

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- then the $n \times n$ mass matrix $k, l = 0, \ldots, n-1$

$$M_{kl} = \int_\Omega u_k u_l dx \tag{7}$$

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- from the cosine angle sum formula

$$M_{kl} = \frac{1}{2} \left(F((k-l)\tau) + F((k+l)\tau) \right), \tag{9}$$

M can be obtained directly from the data.

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$$v_k = \sum_l u_l U^{-1}_{lk}. \tag{10}$$

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- . The functions $\{v_k\}$ will be orthonormal in the L^2 norm (Gram-Schmidt).

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- This is because we start with a local source, orthogonalize sequentially, reflections overlap with previous times.
- So do all of the above for the known background problem

Time domain SISO problem

- Background exact solution

$$u^0(x,t) = \cos(\sqrt{A_0}t)g(x). \tag{11}$$

and snapshots $\{u_j^0\}$

- mass matrix

$$M_{kl}^0 = \int_\Omega u_k^0 u_l^0 dx, \tag{12}$$

- Cholesky decomposition

$$M^0 = (U^0)^\top U^0,$$

- orthogonalized background snapshots

$$\vec{v}^0 = \vec{u}^0 (U^0)^{-1}. \tag{13}$$

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- Definition of our data generated snapshots

$$\begin{aligned} \vec{\mathbf{u}} &:= \vec{v}^0 U \\ &= \vec{u}^0 (U^0)^{-1} U. \end{aligned} \tag{15}$$

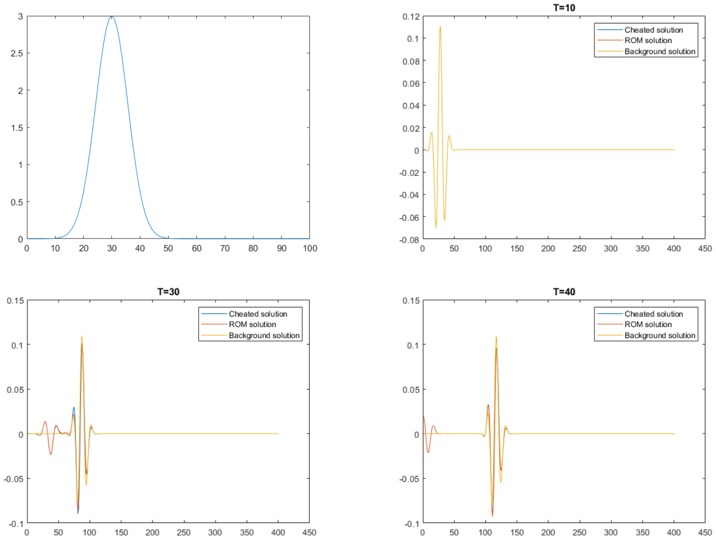

Figure: Data generated internal snapshots

Lippmann-Schwinger-Lanczos equation

- Time domain Lippmann-Schwinger

$$F_0(k\tau) - F(k\tau) = \int_0^{k\tau} \int_\Omega u_0(x, k\tau - t) u(x, t) q(x) dx dt. \qquad (16)$$

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- Use data generated internal solution (interpolated in time)

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Spectral domain SISO problem

- Find u such that

$$\begin{aligned} -u'' + q(x)u + \lambda u &= 0 \quad \text{for} \quad x \quad \text{on} \quad (0,1) \\ -u'(0) &= 1 \\ u(1) &= 0 \end{aligned}$$

- Define the transfer function $F(\lambda) := u(0;\lambda)$.
- Consider the inverse problem: Given $\{F(\lambda), F'(\lambda) : \lambda = b_1, \ldots b_m\}$, find $q(x)$

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- Given $2m$ spectral data values to reconstruct $q(x)$
- Can do a modified version of what follows for other forms of spectral data
- We will construct a ROM that matches this data exactly

Spectral domain SISO

- Consider exact solutions to above $u_1, \ldots, u_m$ corresponding to spectral points $\lambda = b_1, \ldots b_m$. and the subspace

$$G = \text{span}\{u_1, \ldots, u_m\}$$

- Although we do not know these solutions, we can obtain the Galerkin system (ROM) from the data
- Given by the mass and stiffness matrices

$$M_{ij} = \int_0^1 u_i u_j$$

and

$$S_{ij} = \int_0^1 u_i' u_j' + \int_0^1 q u_i u_j.$$

They are given by the formulas

$$M_{ij} = \frac{F(\lambda_i) - F(\lambda_j)}{\lambda_j - \lambda_i}, \quad M_{ii} = -\frac{dF}{d\lambda}(\lambda_i). \tag{18}$$

and

$$S_{ij} = \frac{F(\lambda_j)\lambda_j - F(\lambda_i)\lambda_i}{\lambda_j - \lambda_i}, \quad S_{ii} = \frac{d(\lambda F)}{d\lambda}(\lambda_i). \tag{19}$$

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- These Krylov subspaces *are the same as* those generated by time snapshots corresponding to the ROM!

Spectral domain SISO

- That is, if $d \in \mathbb{R}^m$ satisfies the Galerkin problem

$$Sd(t) + Md(t)_{tt} = 0, \qquad d(0) = b, \qquad d_{t\,t=0} = 0,$$

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- Then $d(\tau i)$ satisfy the second order finite-difference scheme

$$d[\tau(i+1)] = (2I - \tau A)d[\tau i] - d[\tau(i-1)], i = i, \ldots, m-1,$$

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- span$\{d(\tau i)\}$ are the *same* as the above Krylov subspaces w/ powers of A.

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- So the entries of this orthogonalized reduced order model (which can be obtained from the data) are the entries of the stiffness matrix

$$\hat{S}_{ij} = \int \hat{u}_i' \hat{u}_j' + \int_0^1 q \hat{u}_i \hat{u}_j$$

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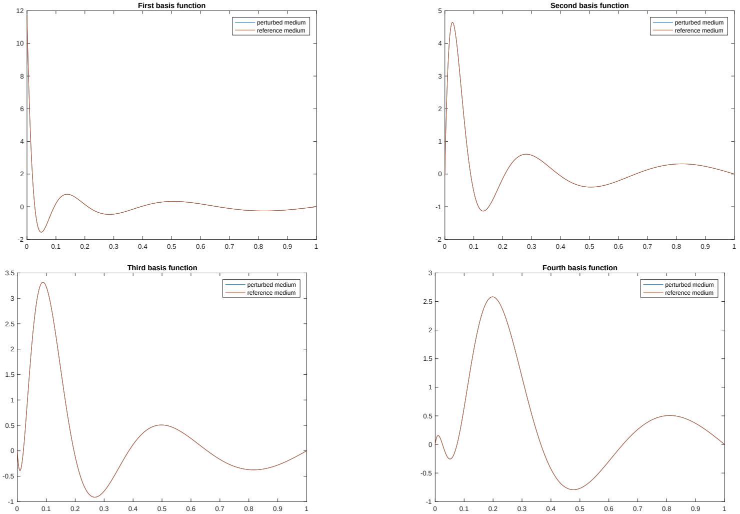
- correspond to orthogonalized projected time snapshots, which *depend only very weakly on the coefficient* .

Weak dependence of orthogonalized bases on q



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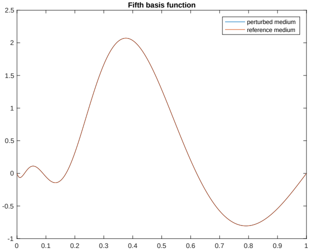

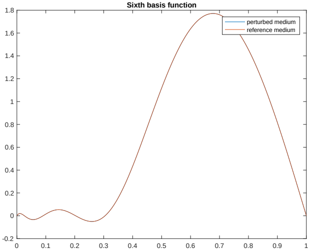

A one-dimensional example: Inversion

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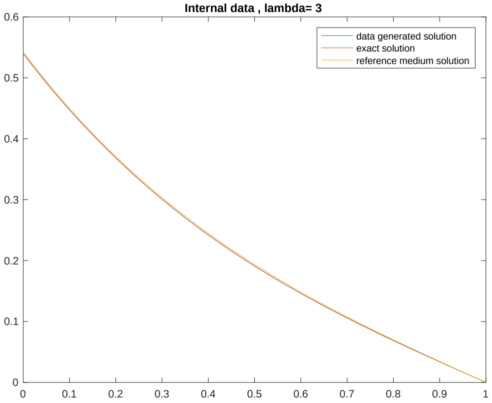
- From the reference medium, we have a highly accurate approximation to the orthogonalized basis.
- By solving the Galerkin system, we get the coefficients
- This yields boundary data generated *internal solutions*

Internal solution

Internal solution for arbitrarily chosen spectral value $\lambda = 3$ generated from data.



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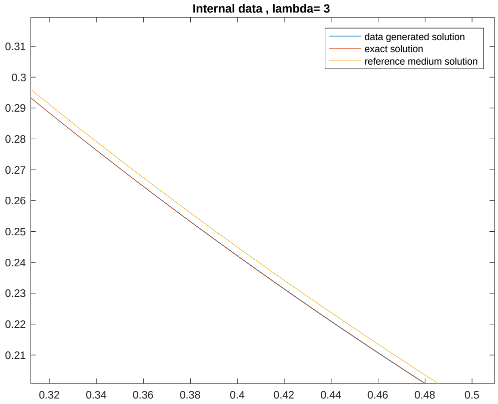

Spectral domain MIMO

- For higher dimensional problems, we can use multiple k sources/receivers:

$$\begin{aligned} -\Delta u_i^r + q(x)u_i^r + b_i u_i^r &= 0 \quad \text{in} \quad \Omega \\ \frac{\partial u_i^r}{\partial \nu} &= g_r \quad \text{on } \partial\Omega \end{aligned} \tag{20}$$

"source" (Neumann data) g_r and spectral value b_i

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- Now spectral data is in the form of a $k \times k$ block

$$F_{rl}^i := F_{rl}(b_i) = \int_{\partial\Omega} u_i^r g_l$$

and

$$DF_{rl}^i := \frac{dF_{rl}}{d\lambda}(\lambda)|_{\lambda=b_i}$$

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- is again obtained directly from boundary data :

$$M_{irjl} = \frac{F^j_{lr} - F^i_{lr}}{b_i - b_j}, \tag{21}$$

$$M_{iril} = -DF^i_{lr}, \tag{22}$$

$$S_{irjl} = \frac{b_j F^j_{lr} - b_i F^i_{lr}}{b_j - b_i}, \tag{23}$$

and

$$S_{iril} = (\lambda F_{rl})'(b_i). \tag{24}$$

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- A natural way that uses internal solutions - in the Lippmann-Schwinger equation
- Adds versatility , computationally simple

Spectral domain Lippmann-Schwinger Lanczos approach

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- With data generated ROM u with its data generated internal solution.

Spectral domain Lippman-Schwinger Lanczos approach

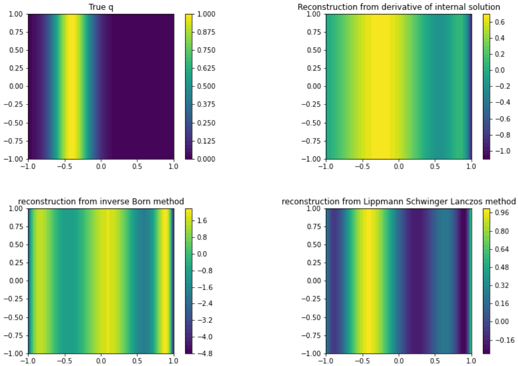

Figure: Lippmann Schwinger Lanczos: Reconstruction of 1-d medium. Two sources total; one on each side, and four spectral values.

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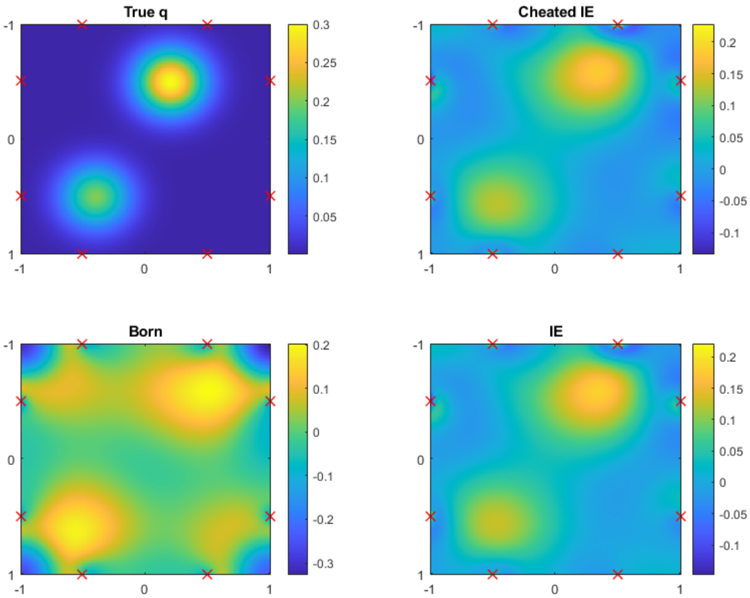

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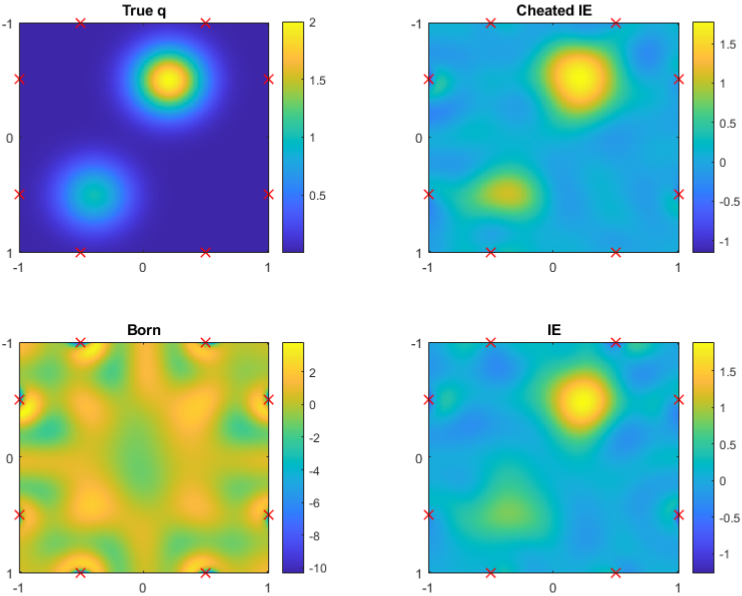

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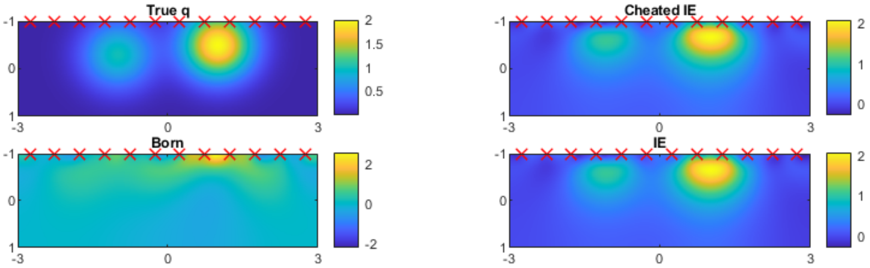


Figure: Experiment 3: True medium (top left) and its reconstructions using 'Cheated IE' (top right), Born linearization (bottom left) and our approach (bottom right)

symmetric data: Lippman-Schwinger Lanczos approach

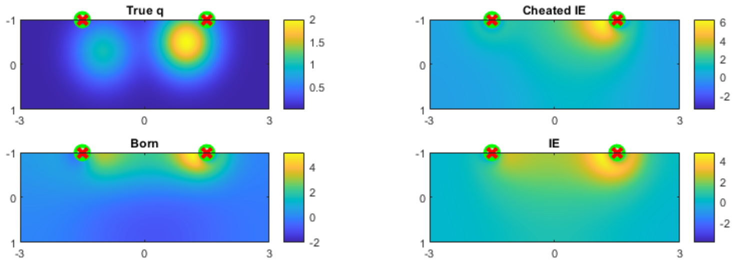
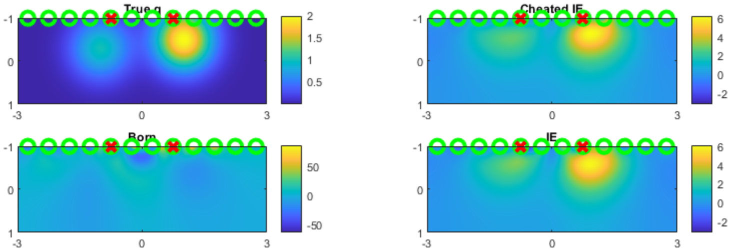


Figure: Experiment 1: True medium (top left) and its reconstructions using 'Cheated IE' (top right), Born linearization (bottom left) and our approach (bottom right)

Non-symmetric data: Lippman-Schwinger Lanczos approach



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Time domain MIMO problem

$$u_{tt} + Au = 0 \text{ in } \Omega \times [0, \infty) \quad (27)$$
$$u(t = 0) = g \text{ in } \Omega \quad (28)$$
$$u_t(t = 0) = 0 \text{ in } \Omega \quad (29)$$

operator $A = A_0 + q$. Source/receivers modeled by $\{g_j\}$, data

$$F^{ji}(k\tau) = \int_\Omega g_j(x) \cos(\sqrt{A} k\tau) g_i(x) dx, \quad (30)$$

receiver j from source i at time $k\tau$.

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- data generated internal solutions directly

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Time domain multistatic 2.5 D

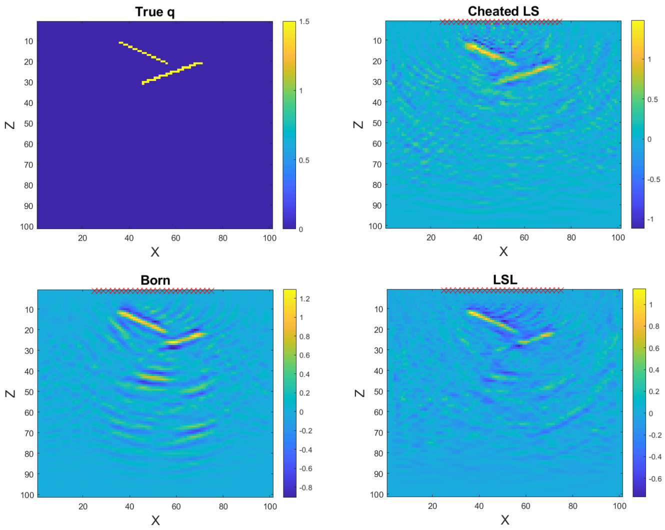

Figure: 2-D varying medium in. 3-D

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- Reconstructions can be improved with iteration (recent work Borcea et. al).