Sampling the X-ray transform on simple surfaces

François Monard

Dept. of Mathematics, University of California Santa Cruz

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W2 "Inverse Problems at Small Scales"
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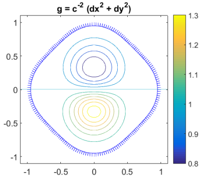
Joint with Plamen Stefanov (Purdue)

Outline

1. Introduction
2. The X-ray transform on CCD's
   - Coordinate systems
   - Canonical relation, microlocal range
3. Sampling issues
   - Sharp sampling rates
   - Predicting aliasing artifacts
   - Non 'box-based' considerations

The geodesic X-ray transform

(M, g), ∂M strictly convex.
$\partial_+ SM$: "inward" boundary ('fan-beam').
Geodesics: $\gamma_{x,v}(t)$.



Problem: to recover $f \in L^2(M)$ from its Geodesic X-ray transform:

$$I_0 f(x, v) = \int_0^{\tau(x,v)} f(\gamma_{x,v}(t))\, dt, \qquad (x, v) \in \partial_+ SM = \mathbb{S}^1 \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

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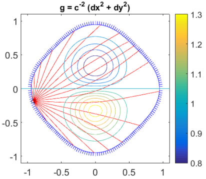

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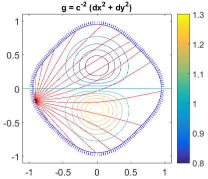

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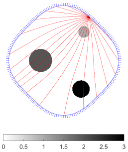

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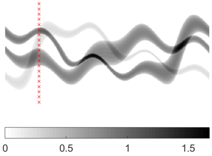

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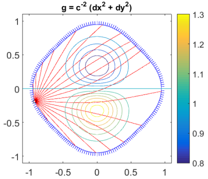

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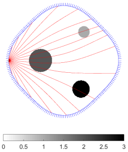

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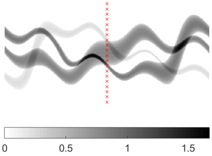

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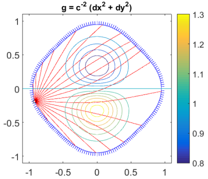

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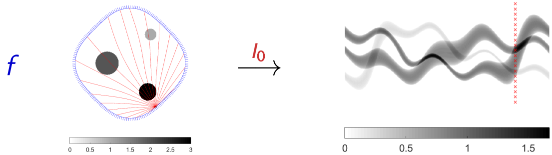
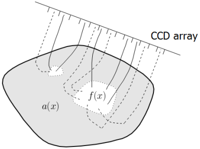
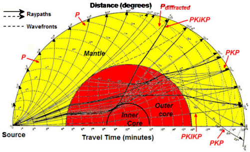


Applications of geodesic X-ray transform

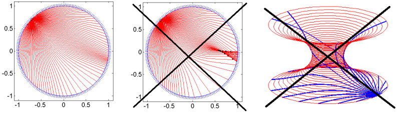
1. Radon transform and X-ray CT
2. SPECT, tomography in media with variable refractive index
3. Seismology, travel-time tomography.





Classical IP questions for simple surfaces

Simple = ∂M stricly convex + no conjugate points + no geodesic of infinite length.



Recovery of f from $I_0 f$ is ...

- Injective over $L^2(M)$ [Mukhometov '75]
- Ill-posed of order $1/2$ [Stefanov-Uhlmann '04], [M.-Nickl-Paternain '19], [Paternain-Salo '20], [M. '20]
- Invertible up to compact error [Pestov-Uhlmann '04] (exact in constant curvature cases)

$$f + \underbrace{K}_{\text{compact}} f = \frac{1}{8\pi} \underbrace{I_\perp^*}_{\text{backproj.}} \underbrace{A_+^* H A_-}_{\text{filter}} I_0 f.$$

$\triangleright$ Great ! let's implement it !

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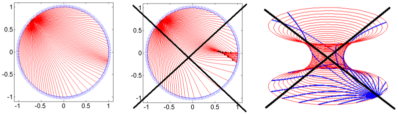

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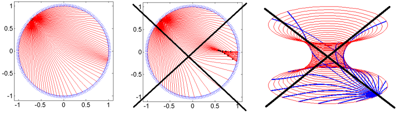

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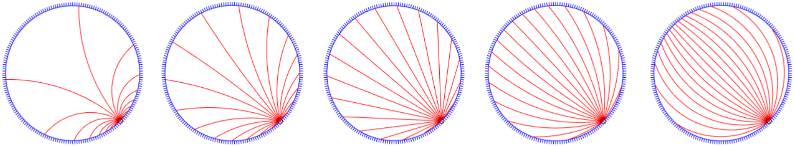
A special family: constant curvature disks

Domain: $\mathbb{D} = \{z \in \mathbb{C}, |z| \le 1\}$
Metric: $g_\kappa(z) = (1+\kappa|z|^2)^{-2}|dz|^2$, for $\kappa \in (-1,1)$ fixed.
Relevant quantities:

$$curv = 4\kappa, \qquad L(\partial\mathbb{D}) = \frac{2\pi}{1+\kappa}, \qquad II = 1-\kappa.$$

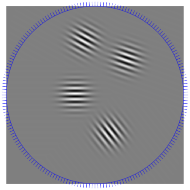
Example with $\kappa = -0.7, -0.3, 0, 0.3, 0.7$.



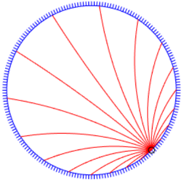
Advantages: rotation-invariant, Pestov-Uhlmann formulas exact, still reaches borderline cases of simplicity.

A reconstruction experiment ($\kappa = -0.3$)

true function



geodesics



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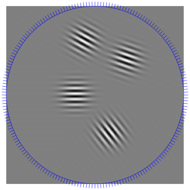

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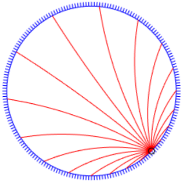
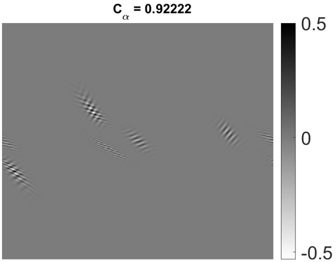
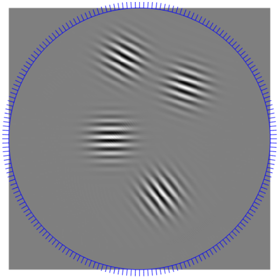


$I_0 f$



$\xrightarrow{I_0^{-1}}$



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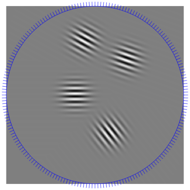

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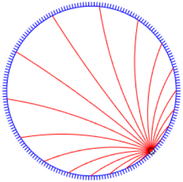

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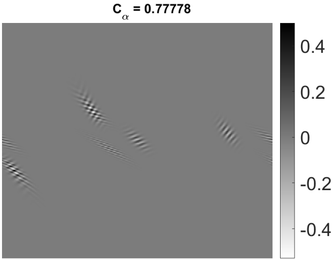

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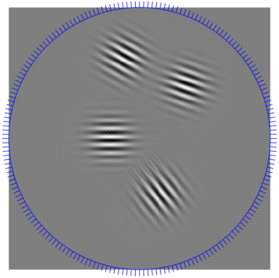

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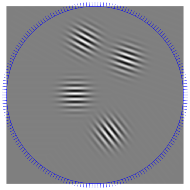

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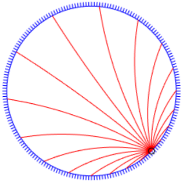

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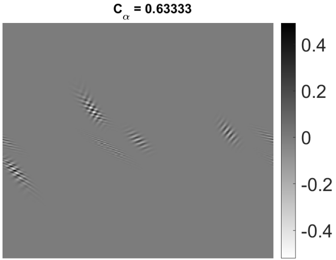

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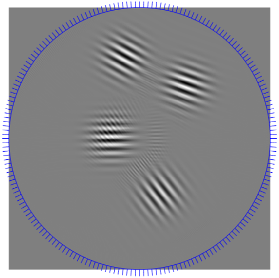

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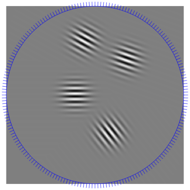

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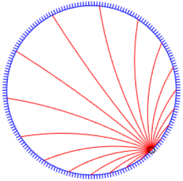

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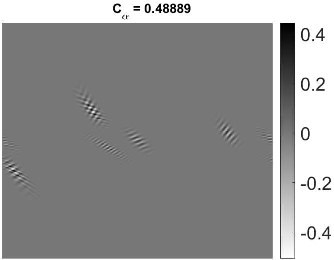

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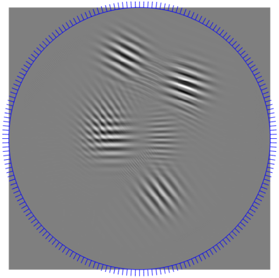

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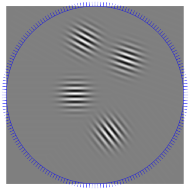

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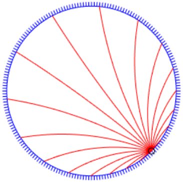

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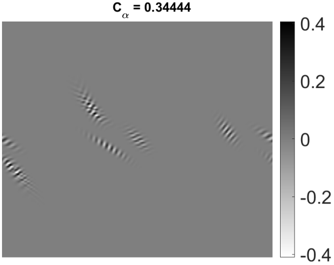

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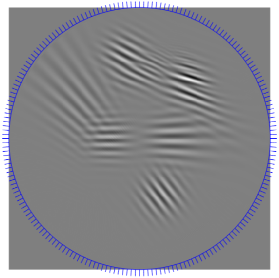

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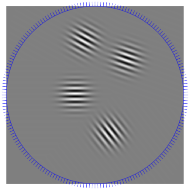

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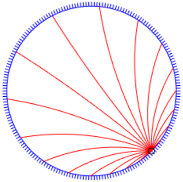

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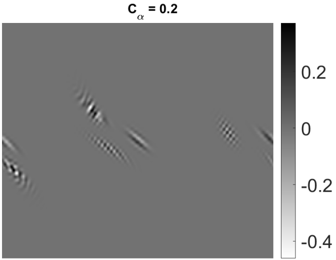

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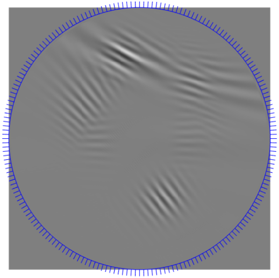

Sampling questions [Stefanov, SIMA '20]

Problem: To reconstruct f from **samples of** Af, with A a linear, injective, (kind of) stable operator. ("$A = id$": classical sampling)

1. Given a bandlimited function f, what are the sampling requirements on Af to "faithfully" reconstruct f ?
2. Given available sampling rates on Af, how to constrain the bandlimit of f ?
3. If data is undersampled,
   (a) can we predict location, orientation and frequency of artifacts ?
   (b) can we reconstruct a blurred yet unaliased version of f ?

- [Stefanov, SIMA '20]: Sharp answers are possible when A is a classical Fourier Integral Operator (such as I_0 !).

Nearby literature:

Fourier-based: Natterer '93, Stefanov '20, Stefanov-Tindel '21

Detecting jump discontinuities: [Katsevich, '17, '20, '21]

Methods exploiting other sparsity: many authors

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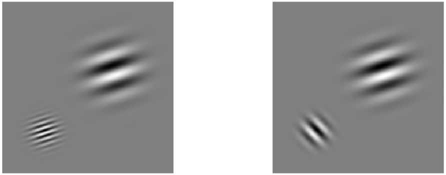
Aliasing illustration when $A = I$ [Stefanov, SIMA, '20]

Nyquist criterion: if $\mathrm{supp}(\hat{f}) \subset [-B, B]^2$, sample at $h < \frac{\pi}{B}$.

When Nyquist is violated (right: twice Nyquist rate):



On the Fourier (modulus) side:

▷ Artifacts at same location, with different frequency and orientation, recovery (generally) impossible

Answering sampling questions for the X-ray transform

Heuristics:

- Bandlimit constraint on $f = f_h$ takes the form "$WF_h(f) \subset \Sigma$" for some compact set $\Sigma \subset T^*M$
- Bandlimit on f translates into bandlimit on $I_0 f$ through the canonical relation of **the FIO** I_0 via

$$WF_h(I_0 f)\backslash\{0\} \subset C_{I_0} \circ WF_h(f)\backslash\{0\}$$

[Stefanov '20, Thm 2.2]

- Recovery of a Σ-bandlimited f requires unaliased sampling of $I_0 f$, which depends on
  - the geometry via C_{I_0} (Jacobi fields, boundary curvature),
  - assuming Cartesian sampling on $\partial_+ SM$, a 'good' choice of coordinate system on $\partial_+ SM$.
- In undersampled situations, aliasing artifacts can be described.

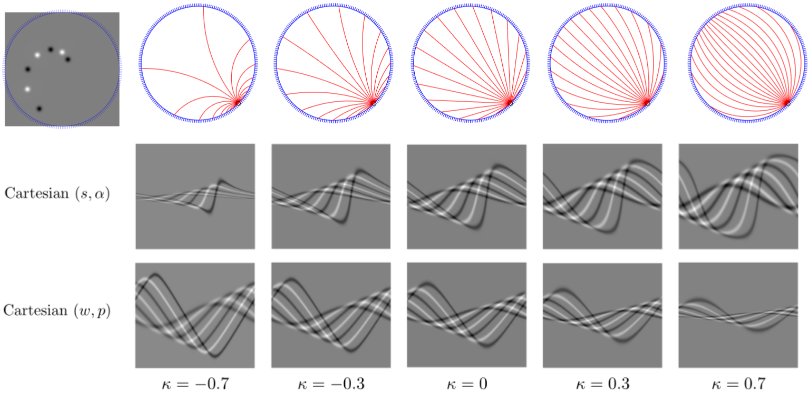
Outline

Coordinate systems

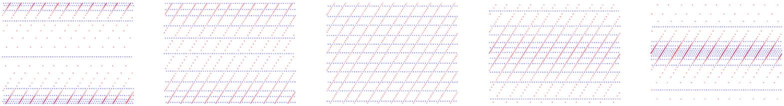
Sampling issues strongly depend on the choice of the coordinate system. Previous coordinate systems: [Assylbekov-Stefanov, '20]

fan-beam (s, α)	parallel (w, p)

X-ray transforms: same function, different coord. systems

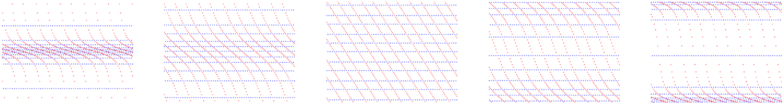


Changes of coordinate systems



$\kappa = -0.7$ $\kappa = -0.3$ $\kappa = 0$ $\kappa = 0.3$ $\kappa = 0.7$

(a) The image of an equispaced Cartesian grid $(s, \alpha) \in [0, L] \times [-\pi/2, \pi/2]$ viewed in $(w, p) \in [0, L] \times [0, L/2]$ (iso-s in red, iso-α in blue).



$\kappa = -0.7$ $\kappa = -0.3$ $\kappa = 0$ $\kappa = 0.3$ $\kappa = 0.7$

(b) the image of an equispaced Cartesian grid $(w, p) \in [0, L] \times [0, L/2]$ viewed in $(s, \alpha) \in [0, L] \times [-\pi/2, \pi/2]$ (iso-w in red, iso-p in blue).

Canonical relation of I_0: geometric description

Using the double fibration picture [Helgason, Guillemin]

$$\partial_+ SM \xleftarrow{F} SM \xrightarrow{\pi} M,$$

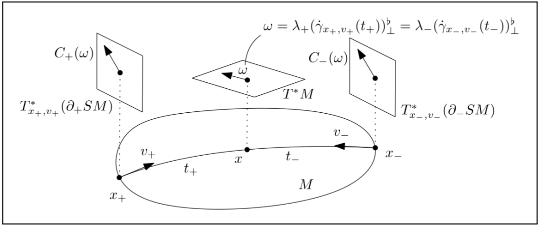
one has the clean composition of FIOs $I_0 = F_* \circ \pi^*$, then:

$$C_{I_0}(\omega) = (C_+(\omega), C_-(\omega)), \qquad \omega \in T^*M \quad \text{(two graphs)}.$$

To find $C_+(\omega)$:

- Let $(x, v) \in \partial_+ SM$ and $t > 0$, $\lambda_+ > 0$ s.t. $\omega = \lambda_+ (\dot\gamma_{x,v}(t))^\flat_\perp$.
- Then $C_+(\omega) = \lambda_+ \eta \in T^*_{(x,v)} \partial_+ SM$, where

$$\eta(V) = b(x, v, t), \qquad \eta(H) = -\mu a(x, v, t) \qquad (a, b : \text{scalar Jacobi field}).$$



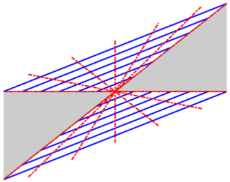
Microlocal range $C_{I_0}(T^*M)$ in coordinate systems

Fan-beam:

$$\eta = \lambda(\eta_s(t)\ ds + \eta_\alpha(t)\ d\alpha)$$

$$\eta_s = II(s)b(s,\alpha,t) - \cos\alpha a(s,\alpha,t)$$
$$\eta_\alpha = b(s,\alpha,t).$$

Sample cotangent fiber (Eucl. disk):



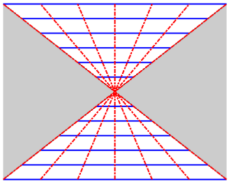
Sensitive to: Jacobi fields

Parallel:

$$\eta = \lambda(\eta_w(t)\ dw + \eta_p(t)\ dp)$$

$$\eta_w = \mu(S_A(w,p))\frac{b(w,p,t)}{b(w,p,\tau)} - \mu(w,p)\frac{b(S_A(w,p),t)}{b(S_A(w,p),\tau)}$$
$$\eta_p = \mu(S_A(w,p))\frac{b(w,p,t)}{b(w,p,\tau)} + \mu(w,p)\frac{b(S_A(w,p),t)}{b(S_A(w,p),\tau)}$$

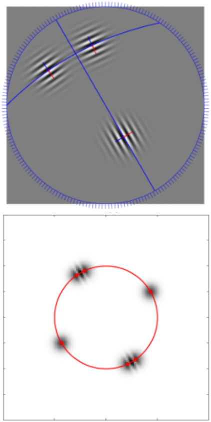
Sample cotangent fiber (Eucl. disk):



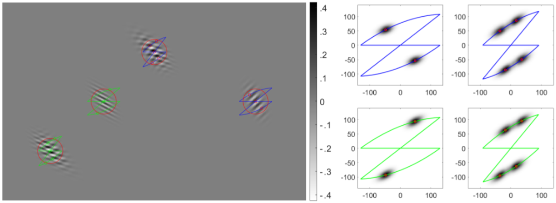
Sensitive to: simplicity

Numerical example $\kappa = 0.3$

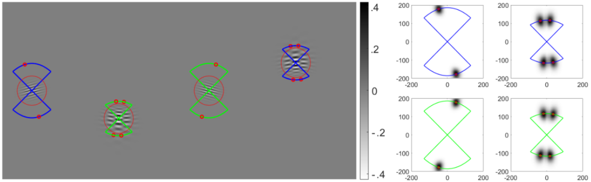
function + FT



$WF_h(I_0 f_h) \subset C_{I_0} \circ WF_h(f_h)$



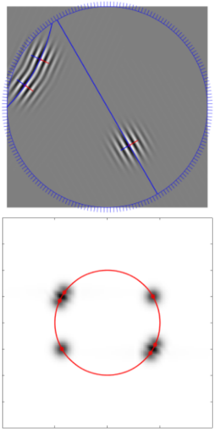
(a) (s, α) coordinates



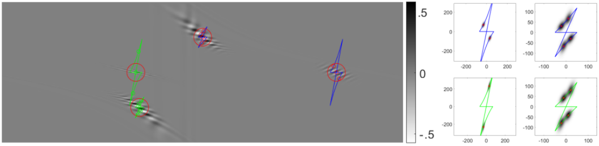
(b) (w, p) coordinates

Numerical example $\kappa = -0.3$

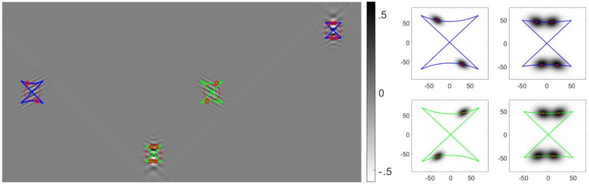
function +
FT



$$WF_h(I_0 f_h) \subset C_{I_0} \circ WF_h(f_h)$$



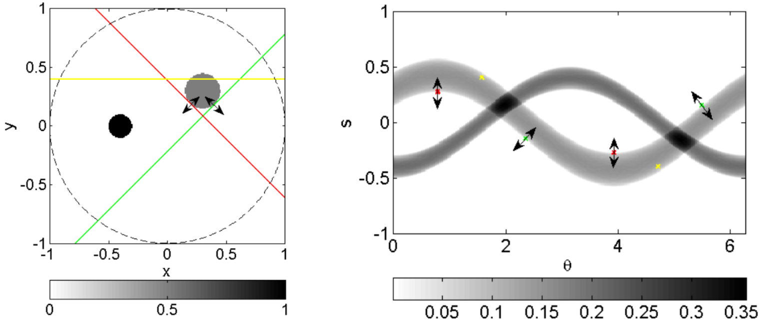
(a) (s, α) coordinates



(b) (w, p) coordinates

A classical comparison

The classical picture: $WF(I_0 f) \subset C_{I_0} \circ WF(f)$



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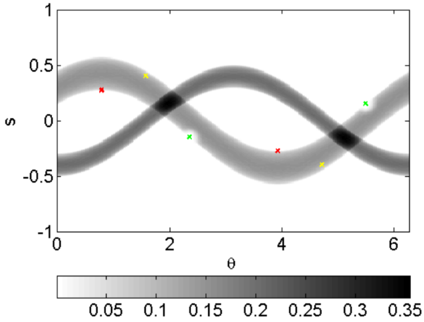

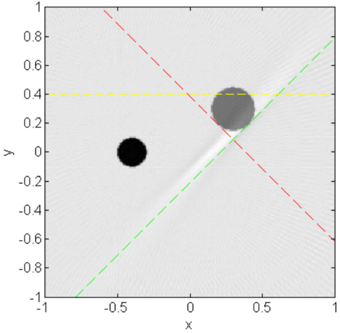

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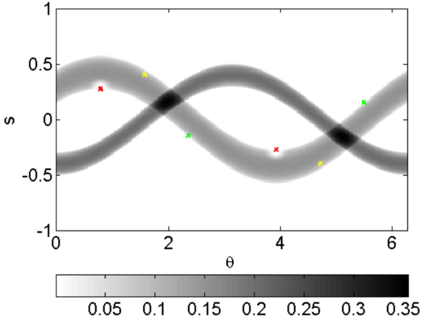
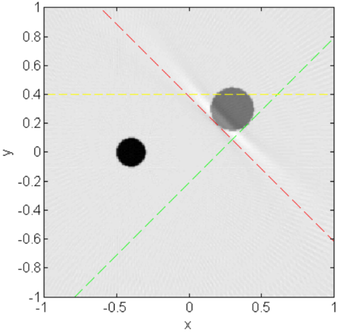




Outline

Indicators of sharp sampling rates

Q: Given a bandlimit on f, how to predict sampling rates on $I_0 f$?
A: If f is $\lambda B^* M$-bandlimited, visualize $\lambda C_{I_0}(B^* M)$ (fiber-dependent) and fit it in a Nyquist box !

Geometries: l. to .r, $\kappa = -0.7, -0.3, 0, 0.3, 0.7$
Coordinates: fan-beam (top), parallel (bottom)

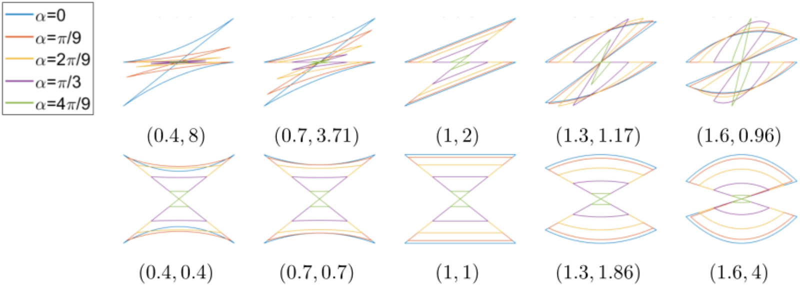
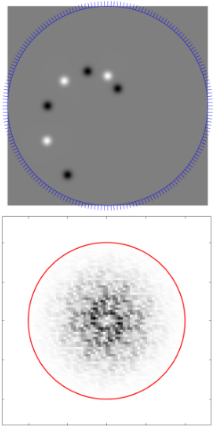
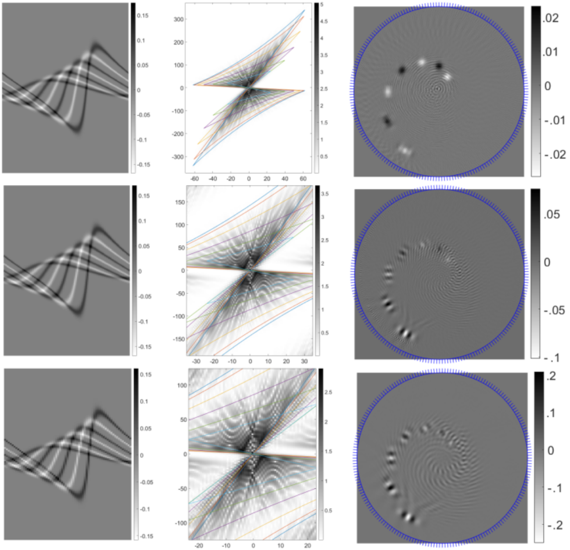


Illustration of sharp rates. $\kappa = -0.3$

function + FT

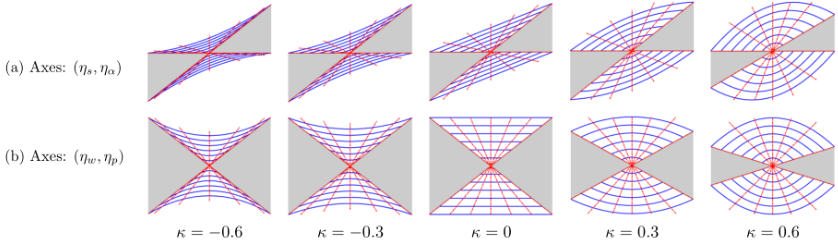


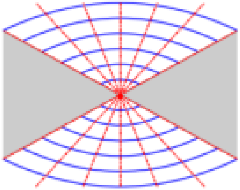


How to read and predict aliasing artifacts 1/2

Let's return to the canonical relation of I_0.

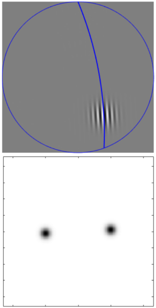
Partitioning of **one** fiber of $T^*(\partial_+ SM)$ ($\alpha = 0$ or $p = 0$):



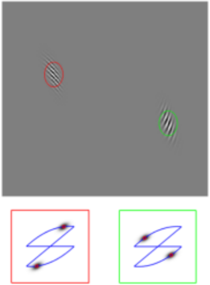


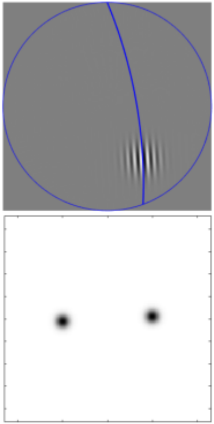
How to read and predict aliasing artifacts 2/2

Original function + FT



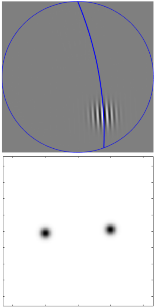
Geometry:
$(R, \kappa) = (1, 0.4)$
Coordinates:
fan-beam





How to read and predict aliasing artifacts 2/2

Original function + FT



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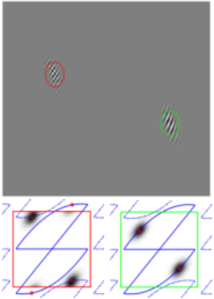
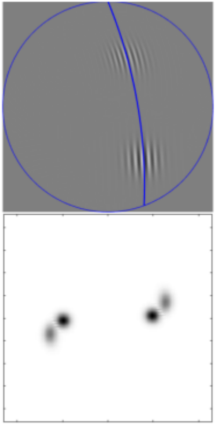
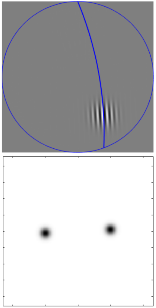




How to read and predict aliasing artifacts 2/2

Original function + FT



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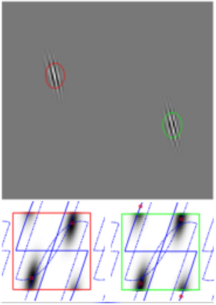
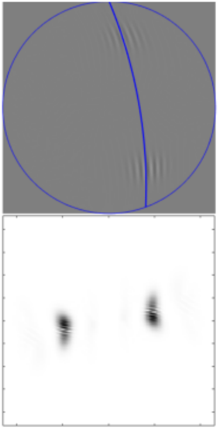
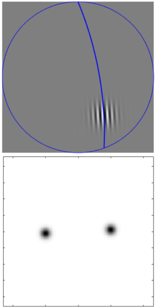




How to read and predict aliasing artifacts 2/2

Original function
+ FT



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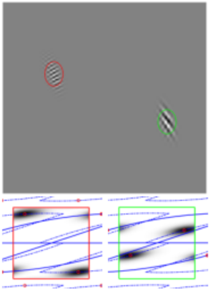
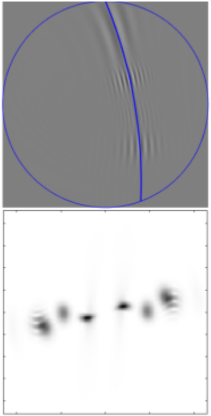
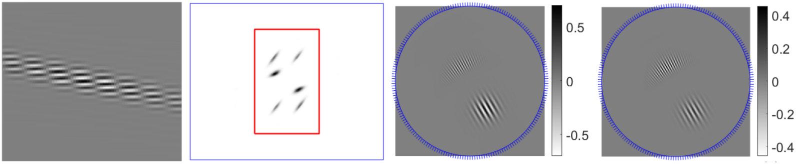




How to beat the 'box-based' Nyquist rate ? [Natterer]

Vertically subsampled $I_0 f$ - Fourier transform:



(a) Upsampling method based on $\mathcal{P}$. The singularity remains aliased after upsampling. Reconstruction has aliasing artifacts.

How to beat the 'box-based' Nyquist rate ? [Natterer]

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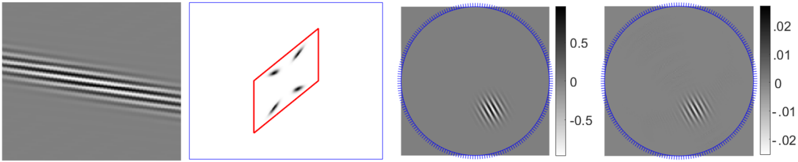

(b) Upsampling method based on $\mathcal{B}$. The singularity is properly recovered after upsampling. Reconstruction has no aliasing artifacts.

Conclusions

• Injectivity, stability and reconstruction formulas at the continuous level still do not address a variety of issues that can occur on the discretization side.

• The sampling of FIOs can lead to new artifacts compared to the classical sampling problem:

- Artifacts can be at a different orientation, frequency **and location**.
- Unlike in classical sampling, undersampling can lead to **higher**-frequency reconstructions.

• Addressing sampling issues for FIOs requires a good understanding of their *canonical relation* and a good choice of *coordinate system* in the data space.

Thank you !

Reference: F.M. and P. Stefanov, Sampling the X-ray transform on simple surfaces, preprint (2021). arxiv:2110.05761

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