Reconstruction of the profile of a wave guide at locally resonant frequencies

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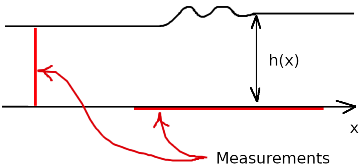
1. Introduction



- We are interested in monitoring the shape of a waveguide from measurements of the fields on a part of its surface, or from measurements in the section
- Applications include corrosion, quality control of semi-conducting structures,...
- We are particularly interested in how one can use locally resonant frequencies, which are finely tuned to the local geometric features
- Our work is motivated by experiments made by Claire Prada and her collaborators at Institut Langevin, who subject an elastic plate to laser excitation to measure the plate thickeness

There has been a lot of previous work concerning the detection of defects in waveguides

- Reconstructions based on linear sampling at a fixed frequency
  [Dediu-McLaughlin, Bourgeois-Fliss, Bourgeois-Lunéville, Monk-Selgas-Yang, Borcea-Cakoni-Meng,...]

- Asymptotic studies of the fields when the defects are small [Ammari-Iakovleva-Kang,...]

- Inverse problems using multi-frequency measurements [Bao-Triki, Isakov-Lu, Sini-Thành,...]

2. The forward problem : modal decomposition and asymptotic structure of the solution near resonant frequencies

In a perfectly straight acoustic waveguide $\Omega = \mathbb{R} \times (0,1)$, the Helmholtz equation

$$\begin{cases} \Delta u + k^2 u = f \quad \text{in } \Omega \\ \partial_\nu u = b_1 \quad \text{on } \mathbb{R} \times \{y = 1\} \\ \partial_\nu u = b_0 \quad \text{on } \mathbb{R} \times \{y = 0\} \\ u \text{ is outgoing} \end{cases}$$

has a unique solution (under regularity assumptions on the source terms), which can be represented as a series

$$u(x,y) = \sum_{n \geq 0} u_{k,n}(x) \varphi_n(y)$$

$$u_{k,n}(x) = \frac{i}{2k_n} \int_{\mathbb{R}} \left[f_n(x') + b_1(x')\varphi_n(1) + b_0(x')\varphi_n(0) \right] e^{ik_n|x-x'|} dx'$$

where $(\varphi_n)_{n\geq 1}$ is the basis of $L^2(0,1)$ defined by

$$\varphi_0(y) = 1 \qquad \text{and} \qquad \varphi_n(y) = \sqrt{(2)}\cos(n\pi y),\ n \geq 1$$

The expansion holds assuming that for all $n \geq 0$,

$$k_n^2 = k^2 - n^2\pi^2/h^2 \neq 0$$

and expressing the radiation condition in the form

$$\forall\ n \geq 0, \quad \left| \frac{d}{dx}(< u_k, \varphi_n >)\frac{x}{|x|} - ik_n < u_k, \varphi_n > \right| \quad \to \quad 0 \quad \text{as } |x| \to \infty$$

Let $\Omega_r = (-r, r) \times (0,1)$. The solution operator is continuous

$$||u_k||_{H^2(\Omega_r)} \quad \leq \quad C_{sol}\Big(||f||_{L^2(\Omega_r)} + ||b_1||_{\tilde{H}^{1/2}(-r,r)} + ||b_0||_{\tilde{H}^{1/2}(-r,r)}\Big)$$

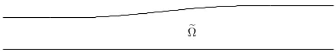
where C_{sol} depends on r and on $\text{dist}(k, n\pi),\ n \geq 0$

We consider a waveguide with variable thickness h, which is a perturbation of a straight wavewguide, under the smallness assumption that

$$\forall\, x \in \mathbb{R}, \quad 0 < h_{\min} \le h(x) \le h_{\max} < \infty$$

$$||h'||_\infty < \eta \qquad ||h''||_\infty < \eta^2 \qquad \mathrm{supp} h' \subset (-R/\eta, R/\eta)$$

and to simplify the analysis, we assume that the bottom of the waveguide is flat and that h is increasing from $h_{\min}$ to $h_{\max}$



The Helmholtz equation in $\widetilde{\Omega} = \{x \in \mathbb{R}, 0 < x < h(x)\}$

$$\begin{cases} \Delta \tilde{u} + k^2 \tilde{u} = \tilde{f} \quad \text{in } \Omega \\ \partial_\nu \tilde{u} = \tilde{b}_{\text{top}} \quad \text{on } \mathbb{R} \times \{y = h(x)\} \\ \partial_\nu \tilde{u} = \tilde{b}_{\text{bot}} \quad \text{on } \mathbb{R} \times \{y = 0\} \\ \tilde{u} \text{ is outgoing} \end{cases} \tag{1}$$

can be mapped to a PDE on a straight waveguide using the change of variable $(x, y) \to (x, y/h(x))$

$$\begin{cases} \Delta_h u + k^2 u = f \quad \text{in } \Omega \\ \partial_\nu u - D_h u = b_{\text{top}} \quad \text{on } \mathbb{R} \times \{y = 1\} \\ \partial_\nu u - D_h u = b_{\text{bot}} \quad \text{on } \mathbb{R} \times \{y = 0\} \\ u \text{ is outgoing} \end{cases} \tag{2}$$

Δ_h, D_h are differential operators that involve h and its derivatives

Under the smallness assumption, we may neglect the terms in δ_h, D_h that are small and consider approaching the previous system by the simpler

$$\begin{cases} \partial_{xx}v + \dfrac{1}{h(x)^2}\partial_{yy}v + k^2 v \;=\; f \quad \text{in } \Omega \\ \partial_\nu v \;=\; b_{\text{top}} \quad \text{on } \mathbb{R}\times\{y=1\} \\ \partial_\nu v \;=\; b_{\text{bot}} \quad \text{on } \mathbb{R}\times\{y=0\} \\ u \text{ is outgoing} \end{cases}$$

for which wee seek a solution in the form of a series

$$v(x,y) \quad = \quad \sum_{n\geq 0} v_n(x)\varphi_n(y)$$

where the v_n's are outgoing and solve the 1D equation

$$v_n''(x) + k_n(x)^2 v_n(x) \quad = \quad f_n(x) - \varphi_n(1)b_{\text{top}}(x) - \varphi_n(0)b_{\text{bot}}(x), \qquad x \in \mathbb{R}$$

Here the 'local wavenumber' is $\quad k_n(x)^2 \;=\; k^2 - \dfrac{n^2\pi^2}{h(x)^2}$

We assume that $\quad \delta = \inf\left(\left|k^2 - \frac{n^2\pi^2}{h_{\max}{}^2}\right|^{1/2}, \left|k^2 - \frac{n^2\pi^2}{h_{\min}{}^2}\right|^{1/2}\right) > 0$

We use results by F. Olver on the Schrödinger equation :

1. If $k_n(x)^2 > 0$ (resp. $k_n^2 < 0$) the mode is called propagative (resp. evanescent)

   One can change variable from x to $z(x) = \int^x |k_n|$, so that $w_n = \sqrt{z} v_n$ solves

   $$\partial_{zz} w_n \pm w_n = \zeta(x,z) w_n \qquad \text{with} \quad ||\zeta||_\infty = O(\eta)$$

   then w_n can be expressed as the sum of 2 exponential functions plus an error term

2. If $k_N(x_*) = 0$ then the mode is called locally resonant (given our assumption that h is increasing, there is only one such value)

   The change of variable

   $$\xi(x) = \begin{cases} \left(-3i/2 \int_x^{x^*} k_N(s)\,ds\right)^{2/3} & \text{if } x < x^* \\ -\left(3/2 \int_{x^*}^{x} k_N(s)\,ds\right)^{2/3} & \text{if } x > x^* \end{cases}$$

   shows that $w_N(\xi) = -(\sqrt{k_N} \xi(x)^{-1/4}) v_N(x)$ solves the Airy equation

   $$\partial_{\xi\xi} w_N - \xi w_N = \zeta(\xi) w_N, \qquad \text{with} \quad ||\zeta||_\infty = O(\eta)$$

   and w_N can be expressed as a combination of Airy functions plus an error term

Theorem :

Let $f \in L^2(\Omega), b_{\text{top}}, b_{\text{bot}} \in H^{1/2}(\mathbb{R})$ with compact support in $|x| < r$ for some $r > 0$

Assume that there is a single locally resonant mode N, associated with a single point x^*

There exists $\eta_0 = \eta_0(h_{\min}, h_{\max}, R, r, \delta)$ such that for $\eta < \eta_0$ the Helmholtz equation (2) has a unique solution $u \in H^2_{\text{loc}}(\Omega)$

Moreover u can be approximated by

$$u^{\text{app}}(x,y) = \sum_{n\geq 0} \Big(\int_{\mathbb{R}} G^{\text{app}}(x,s)\Big(-f_n(s) + \varphi_n(1) b_{\text{top}}(s) + \varphi_n(0) b_{\text{bot}}(s)\, ds\Big) \varphi_n(y)$$

$$G^{\text{app}}(x,s) = \begin{cases} \dfrac{i}{2\sqrt{k_n(s)k_n(x)}} \exp\Big(i\Big|\displaystyle\int_s^x k_n\Big|\Big) & n < N \\ \dfrac{i}{2\sqrt{|k_n|(s)|k_n|(x)}} \exp\Big(-\Big|\displaystyle\int_s^x |k_n|\Big|\Big) & n > N \\ \dfrac{\pi\big(\xi(s)\xi(x)\big)^{1/4}}{\sqrt{k_n(s)k_n(x)}}(i\mathcal{A}+\mathcal{B})(\xi(s))\mathcal{A}(\xi(x)) \quad \text{if } x < s & \\ & n = N \\ \dfrac{\pi\big(\xi(s)\xi(x)\big)^{1/4}}{\sqrt{k_n(s)k_n(x)}}(i\mathcal{A}+\mathcal{B})(\xi(x))\mathcal{A}(\xi(s)) \quad \text{if } x > s & \end{cases}$$

Finally, one can infer existence of a solution to the original Helmholtz equation under the assumption of smallness of h and define an approximate solution

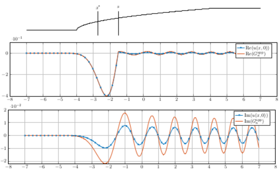
$$\tilde{u}^{\mathrm{app}}(x,y) \quad = \quad u^{\mathrm{app}}(x, \frac{y}{h(x)})$$

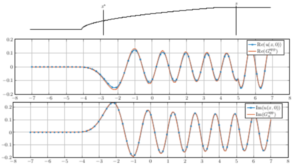
Remarks :

- This construction is a Born approximation
- The (stringent) assumptions show modes coupling is weak, inspite of the x-dependance of h
- The construction can be extended to a non-increasing profile h which is the union of a finite number of sections where h is strictly monotonous

  (one has to assume that they are no trapped modes)

Comparisons between FEM approximation of u and u^{app}





Red : $G_n^{\text{app}}(x,0) \sim u^{\text{app}}$ when $f \sim \delta_0(x)$

Blue : $\tilde{u}(x,0)$ computed via a FEM approximation, using PML's

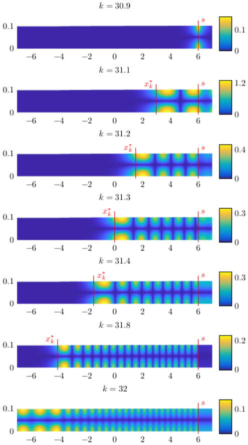
Relative L^2 error between 5 and 10%

Representation of the wavefield for a monochromatic source

evanescent mode

locally resonant modes

propagative mode



3. Reconstruction of the profile of the waveguide

We use a monochromatic source $f(x,y) = f_N(x)\varphi_N(y)$ where n is a locally resonant frequency

The measurements are

$$u(x,0) \simeq u_N^{\mathrm{app}}(x)\varphi_N(0) \simeq \int_{\mathbb{R}} G_N^{\mathrm{app}}(x,s) f(s)\, ds$$

where $G_N^{\mathrm{app}}(x,s) = q(s)\dfrac{-\xi(x)^{1/4}}{\sqrt{k_n(x)}}\mathcal{A}\big(\xi(x)\big)$

One can show that

$$\mathrm{data}(x) = u(x,0) = z\mathcal{A}\big(\alpha(x^* - x)\big) + \text{error term}$$

$$z = \Big(\frac{2N^2\pi^2 h'(x^*)}{h(x^*)^3}\Big)^{-1/6} \int_{x^*+R}^{\infty} q(s) f_N(s)\, ds \qquad \alpha = \Big(\frac{2N^2\pi^2 h'(x^*)}{h(x^*)^3}\Big)^{1/3}$$

Setting $F(z, \alpha, \beta) = z\mathcal{A}(\beta - \alpha x)$, the parameters z, α, β are estimated from the data points by a least square fit

And one obtains an estimate for $x^* = \beta/\alpha$

On the other hand, if the frequency k is locally resonant at x^*

$$k_N^2 = 0 \quad \Rightarrow \quad k = \frac{N\pi}{h(x^*)}$$

In this way, one can infer both x^* and $h(x^*)$ from the data and recover the profile of the waveguide

Reconstruction method

1. Find an approximation of the support of h' and a range of frequencies $[k_{min}, k_{max}]$ that contain locally resonant frequencies

2. Choose a discrete set of frequencies $K \subset [k_{min}, k_{max}]$, choose source terms f, b_{bot}. Measure the wavefield $u(x, 0)$ for each chosen frequency

3. Filter the data, eliminating components corresponding to propagative frequencies in the Fourier transform of the responses

4. Find the point x_{max} where the response $|u(x, 0)|$ is maximal and choose points in $[x_{max} - R, x_{max} + R]$. The data is then

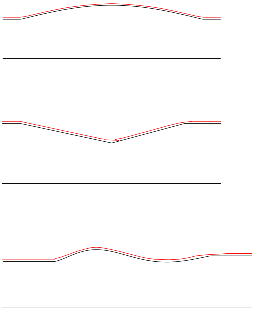
$$data(i) = u(x_i, 0)$$

5. For every frequency k in K, compute $h(x^*(k)) = N\pi/k$. Minimize

$$\sum_i |u(x_i, 0) - F(z, \alpha, \beta)(x_i)|^2$$

to obtain an estimate of $x^*(k)$

Examples of reconstruction



4. Conclusions

- We studied the propagation of waves in a waveguide with a slowly and smoothly varying profile

- One can map the setting to that of a straight waveguide and construct a solution as a Born approximation

- We analyzed the wavefield for locally resonant frequency

  Under our assumptions, mode mixing does not perturb too much the structure of the solution

  For a locally resonant mode, the modeal equation can be cast as an Airy equation

- We have obtain an asymptotic form of the wavefield, from which one can extract information on the geometric features of the waveguide and proposed a reconstruction method

Many questions remain open :

- Can one relax the smoothness assumptions on the geometry ? In particular, what about kinks ?
- Can one find a method to show existence of solutions that does not rely on a modal expansion ?
- Extension to elasticity : in the case of plates, Lamb modes allow for a representation of the displacement field

  However, the associated wave structure is more complex and there are several types of resonances

  In her experiments, Claire Prada uses ZGV modes (wavenumbers for which $\partial_\omega k = 0$) what is the geometrical information contained in the asymptotic structure of these modes ?