Photo-Acoustic Imaging Using Nanoparticles

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Workshop 2 "Inverse Problems at Small Scales"

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Outline

- Original Photo-Acoustic imaging modality
- Photo-Acoustic imaging using nanoparticles as contrast agents
- Reconstruction of both the optical and acoustical properties
- Conclusion

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The original Photo-Acoustic Imaging

Photo-acoustic model

$$\begin{cases} \frac{\partial^2 p}{\partial^2 t} - c_s^2(x)\, \rho(x)\, \nabla \left(\rho^{-1} \cdot \nabla p \right) = 0 & \text{in} \quad \mathbb{R}^3 \times \mathbb{R}^+, \\ p(x,0) = f(x) := \Gamma(x)\, \mu_{ab}(x)\, u(x) & \text{in} \quad \mathbb{R}^3, \\ \frac{\partial p}{\partial t}(x,0) = 0 & \text{in} \quad \mathbb{R}^3. \end{cases}$$

- p acoustic pressure
- ρ mass density
- c_s wave speed
- Γ Grueneisen parameter

The light intensity, $u := u^{in} + u^s$, satisfies the Diffusion Approx. RT-Model: [1]

$$\begin{cases} -\nabla \cdot (D \nabla u) + \mu_{ab}(\cdot)\, u = 0 & \text{in} \quad \mathbb{R}^3, \\ u^s \sim \quad L^2 - \text{behavior} & . \end{cases}$$

- $D := \frac{1}{3(\mu_{ab} + \mu_{sc})}$ diffusion coefficient
- μ_{ab} absorption cross-section
- μ_{sc} scattering cross-section

[1] The models above are usually stated in a bounded domain Ω with BC on $\partial\Omega$.

Photo-acoustic Imaging

The acoustic coefficients (ρ and c_s) and the optical coefficients (μ_{ab} and μ_{sc}) are assumed to be known constants outside a bounded domain Ω. We assume also the transfer coefficient Γ to be known.

Goal:

From the measured pressure p on $\partial\Omega \times (0, T)$, generated by few illuminations u^i, reconstruct (few of) the above coefficients.

Naturally, this problem splits into two steps:

- Acoustic Inversion:

$$p(\partial\Omega, (0, T)) \xhookrightarrow{\text{reconstruct}} (f(x) := \Gamma\, \mu_{ab}\, u)\Big|_{\Omega} \text{ or /and } \left(\rho\Big|_{\Omega} \text{ or/and } c_s\Big|_{\Omega}\right).$$

- Optic Inversion: [2]

$$(\mu_{ab}\, u)\Big|_{\Omega} \xhookrightarrow{\text{reconstruct}} \left(\mu_{ab}\Big|_{\Omega} \text{ or/and } \mu_{sc}\,\Big|_{\Omega}\right).$$

[2] Γ being known.

A very brief review of known results:

The mass density ρ is assumed to be constant.

- Acoustic Inversion with known wave speed c_s
  - Ω is a sphere and c_s a constant, via Radon transform. Natterer (01), Finch-Haltmeier-Rakesh (07).
  - Variable c_s or general Ω, use spectral decomposition. Agranovsky-Kuchment (07).
- Acoustic Inversion with unknown variable wave speed c_s
  - Uniqueness and stability of c_s/source. Stefanov-Uhlmann (09, 13) and Stefanov-Yang (17) .
  - Reconstruction of c_s and source term (using many measurements), Kirsch-Scherzer (12).
- Optic Inversion.

  Uniqueness/stability results in Bal-Ren (2011) and Bal-Uhlmann (2010) (non-degeneracy conditions), Alessandrini-DiCristo-Francini-Vessella (2017) (appropriately chosen inputs), Bonnetier-Choulli-Triki (2022) (using point-sources), Naetar-Scherzer (14) (reconstruction of piecewise constant profiles).

More Ref: Ammari, Agranovsky, Arridge, Bal, Choulli, Cox, Finch, Haltmeier, Kuchment, Kunyansky, Natterer, Quinto, Rakesh, Ren, Scherzer, Schotland, Stefanov, Triki, Uhlmann, ...

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Photo-Acoustic Imaging using Nanoparticles

Photo-acoustics using nanoparticles as contrast agents [3], [4]

$$\begin{cases} \frac{\partial^2 p}{\partial^2 t} - c_s^2 \, \rho \, \nabla \cdot (\rho^{-1} \, \nabla p) = 0 & \text{in} \quad \mathbb{R}^3 \times \mathbb{R}^+, \\ p(x,0) = \frac{\omega \, \beta_0}{c_p} \, \Im(\varepsilon)(x) \, |u|^2 \, (x) & \text{in} \quad \mathbb{R}^3, \\ \frac{\partial p}{\partial t}(x,0) = 0 & \text{in} \quad \mathbb{R}^3. \end{cases}$$

- c_s wave speed
- ρ mass density
- c_p heat capacity
- β_0 thermal expansion

The electric field, $u := u^{in} + u^s$, satisfies the Maxwell system:

$$\begin{cases} Curl^{(2)}(u) - \omega^2 \, \varepsilon(\cdot) \, \mu \, u = 0 & \text{in} \quad \mathbb{R}^3, \\ u^s \quad \text{satisfies the} \quad S.M.R.C, \\ u^{in} \text{ is an incident plane wave.} \end{cases}$$

- Permittivity:

$$\varepsilon(\cdot) := \begin{cases} \epsilon_\infty & \text{in } \mathbb{R}^3 \setminus \Omega, \\ \epsilon_0(\cdot) & \text{in } \Omega \setminus D, \\ \epsilon_p & \text{in } D. \end{cases}$$

- Permeability: $\mu = C^{te}$

[3] Prost et al.: Photoacoustic generation by gold nanosphere (15). Many other references.

[4] First modeled and analyzed in (Triki-Vauthrin (18)) for the 2D-TE-regime and $c_s = Constant = \rho$

What kind of nanoparticles?

We take $D := z + a\, B$ where B has a volume of order 1 and contains the origin. z is the 'location' of D and a its diameter. It is assume that $a \ll |B|$.

For simplicity of exposition, we take here $\epsilon_0 = \epsilon_\infty$!

The electric field is solution of the Lippmann-Schwinger system of equations

$$u - k^2(\epsilon_p - \epsilon_\infty)\mu\, N_D^k(u) + (\epsilon_p - \epsilon_\infty)\nabla M_D^k(u) = u^{in}, \text{ in } D \tag{1}$$

with the Newtonian potential $N_D^k(u)$ and the Magnetization potential $\nabla M_D^k(u)$:

$$N_D^k(f)(x) := \int_D \Phi_k(x,y)\, f(y)dy \quad \text{and} \quad \nabla M_D^k(f)(x) := \nabla \int_D \underset{y}{\nabla} \Phi_k(x,y)\cdot f(y)dy, \tag{2}$$

where $\Phi_k(x,y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}$.

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What kind of nanoparticles?

Using the change of scale $\tilde{x} := \frac{x-z}{a}$ the system (1) becomes:

$$\tilde{u} - k^2\, a^2\, (\epsilon_p - \epsilon_\infty)\mu\, N_B^{ka}(\tilde{u}) + (\epsilon_p - \epsilon_\infty)\nabla M_B^{ka}(\tilde{u}) = \tilde{u}^{in}, \text{ in } B \quad (3)$$

which we can rewrite as

$$\tilde{u} - k^2\, a^2\, (\epsilon_p - \epsilon_\infty)\mu\, N_B(\tilde{u}) + (\epsilon_p - \epsilon_\infty)\nabla M_B(\tilde{u}) = \tilde{u}^{in} + Controlable(\tilde{u}), \text{ in } B \quad (4)$$

where

$$N_B(\tilde{u})(x) := \int_B \frac{1}{4\pi|x-y|}\tilde{u}(y)dy \qquad \nabla M_B(\tilde{u})(x) := \nabla \int_B \nabla_y\left(\frac{1}{4\pi|x-y|}\right)\cdot \tilde{u}(y)dy. \quad (5)$$

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What kind of nanoparticles?

We recall the decomposition

$$(\mathbb{L}^2(B))^3 = \mathbb{H}_{div,0}(B) \oplus \mathbb{H}_{Curl,0}(B) \oplus \nabla \mathcal{H}armonic(B)$$

where

$$\mathbb{H}_{div,0}(B) := \{u \in \mathbb{H}(div)(B);\ div(u) = 0, in\ B,\ \nu \cdot u = 0 \text{ on } \partial B\},$$

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and

$$\nabla \mathcal{H}armonic(B) := \{u = \nabla \phi, \quad \Delta \phi = 0 \text{ in } B\}.$$

Then

- N_B generates bases $(\lambda_n^1, e_n^1)_{n \in N}$ and $(\lambda_n^2, e_n^2)_{n \in N}$ of $H_{div,0}(B)$ and $H_{curl,0}(B)$.
- $\nabla M_B : \nabla Harmonic(B) \to \nabla Harmonic(B)$ has a complete basis $(\lambda_n^3, e_n^3)_{n \in N}$.

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We recall the LS-system of equations

$$\tilde{u} - k^2\ a^2\ (\epsilon_p - \epsilon_\infty)\mu\ N_B(\tilde{u}) + (\epsilon_p - \epsilon_\infty)\nabla M_B(\tilde{u}) = \tilde{u}^{in} + Controlable(\tilde{u}), \text{ in } B \quad (6)$$

We have two possibilities:

1. $\Re(\epsilon_p - \epsilon_\infty) > 0$ and $(\epsilon_p - \epsilon_\infty) \sim a^{-2}$. In this case, we can excite the eigenmodes of N_B, i.e. $(\lambda_n^1, e_n^1)_{n\in N}$ (or $(\lambda_n^2, e_n^2)_{n\in N}$) called Dielectric modes.
2. $\Re(\epsilon_p - \epsilon_\infty) < 0$ and $(\epsilon_p - \epsilon_\infty) \sim 1$. In this case $a^2\ (\epsilon_p - \epsilon_\infty)\mu\ N(\tilde{u}) \ll 1$ and we can excite the eigenmodes of ∇M_B, i.e. $(\lambda_n^3, e_n^3)_{n\in N}$, called Plasmonic modes.

According to these properties, we say that

1. Nanoparticles (D, ϵ_p, μ) for which $\Re(\epsilon_p - \epsilon_\infty) > 0$ and $(\epsilon_p - \epsilon_\infty) \sim a^{-2}$ are called Dielectric nanoparticles.
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Lorentz model and generation of Dielectrics or Plasmonics

We use the Lorentz model for the electric permittivity of the nanoparticle:

$$\epsilon_p(\omega) = \epsilon_\infty \left[1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma_p\omega} \right] \tag{7}$$

where ω_0^2 is the undamped resonance frequency, γ_p is the electric damping parameter, ω_p the electric plasmonic frequency.

Therefore, if we take the incident frequency ω in the regimes:

1. $\omega^2 < \omega_0^2$, $\omega_0^2 - \omega^2 \sim a^2$ and $\gamma_p = o(\omega_0^2 - \omega^2)$, then $\epsilon_p - \epsilon_\infty \sim a^{-2}$. i.e. (D, ϵ_p, μ) behaves as a Dielectric.
2. $\omega_0^2 < \omega^2 < \omega_0^2 + \omega_p^2$ and $\gamma_p = o(\omega_0^2 - \omega^2)$, then $\Re(\epsilon_p - \epsilon_\infty) < 0$ and $\epsilon_p - \epsilon_\infty \sim 1$. i.e. (D, ϵ_p, μ) behaves as a Plasmonic.

Lorentz model and generation of Dielectrics or Plasmonics

We use the Lorentz model for the electric permittivity of the nanoparticle:

$$\epsilon_p(\omega) = \epsilon_\infty \left[1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma_p\omega} \right] \tag{7}$$

where ω_0^2 is the undamped resonance frequency, γ_p is the electric damping parameter, ω_p the electric plasmonic frequency.

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From now on, we focus only on Plasmonic Nanoparticles

Forward problems-Electromagnetic waves

The electromagnetic scattering problem

$$\begin{cases} Curl^{(2)}(u) - \omega^2 \, \varepsilon(\cdot) \, \mu \, u = 0 \quad \text{in} \quad \mathbb{R}^3, \\ u = u^{in} + u^s, \text{ and } u^s \text{ satisfies the } S.M.R.C, \\ u^{in} \text{ is an incident plane wave.} \end{cases}$$

- Permittivity:

$$\varepsilon(\cdot) := \begin{cases} \epsilon_\infty & \text{in } \mathbb{R}^3 \setminus \Omega, \\ \epsilon_0(\cdot) & \text{in } \Omega \setminus D, \\ \epsilon_p & \text{in } D. \end{cases}$$

- Permeability: $\mu = C^{te}$

is well-posed.

Regularity of the electric field u

The electric field u is in $L^4_{loc}(\mathbb{R}^3)$ if ϵ_0 is of class $\mathcal{C}^1$. [5]

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Forward problems: Acoustic waves

We have existence and uniqueness of a weak solution of

$$\begin{cases} \frac{\partial^2 p}{\partial^2 t} - c_s^2 \, \rho \, \nabla \cdot (\rho^{-1} \, \nabla p) = 0 & \text{in} \quad \mathbb{R}^3 \times \mathbb{R}^+, \\ p(x,0) = \frac{\omega \, \beta_0}{c_p} \, \Im(\varepsilon)(x) \, |u|^2 \, (x) & \text{in} \quad \mathbb{R}^3, \\ \frac{\partial p}{\partial t}(x,0) = 0 & \text{in} \quad \mathbb{R}^3. \end{cases}$$

- c_s wave speed
- ρ mass density
- c_p heat capacity
- β_0 thermal expansion

with [6]

$$p(\cdot,\cdot) \in \mathcal{C}\left([0;M];\mathbb{L}^2\left(\mathbb{R}^3\right)\right) \cap \mathcal{C}^1\left([0;M];\mathbb{H}^{-1}\left(\mathbb{R}^3\right)\right), \tag{8}$$

where $\rho > 0$ is of class $W^{1,\infty}$ (and M is such that $c_s(\cdot) \geq M^{-1}$).

Integral representation, and regularity, of the pressure p

$$p(x,t) = \frac{\omega \, \beta_0}{c_p} \, \partial_t \int_\Omega G(y,t,x) \, \Im(\varepsilon)(y) \, |u|^2 (y) \, dy. \tag{9}$$

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The Green function $G(x,t,y,s) = G(x,t-s,y)$ satisfies

$$\begin{cases} \partial_t'' G(x,t,y,s) - c^2(x)\,\rho(x)\underset{x}{\nabla}\,(\rho^{-1}\nabla G(x,t,y,s)) = \underset{y}{\delta}(x)\,\underset{s}{\delta}(t), \quad \text{in} \quad \mathbb{R}^3 \times \mathbb{R}^+, \\ G(x,0) = 0, \quad \text{in} \quad \mathbb{R}^3, \\ \partial_t G(x,0) = 0, \quad \text{in} \quad \mathbb{R}^3. \end{cases}$$

Assumptions to derive the singularity analysis of G

Assume that both ρ, c_s and Ω are infinitely smooth. In addition, we state the following geometric conditions.

The metric τ, where τ is the travel time function, and also the families of induced geodesics $\Gamma(\cdot,\cdot)$, are taken satisfying the following properties:

1. Any two points of the domain Ω are connected by a unique geodesic $\Gamma(\cdot,\cdot)$, of the metric τ, contained in Ω and with ends points on the boundary $\partial\Omega$.
2. The boundary $\partial\Omega$ is convex relative to these geodesics.

Thanks to (V. G. Romanov-09), under these conditions, we have

$$G(x,t,y) := \sum_{k=-1}^{+\infty} \alpha_k(x,y)\; \Theta_k\left(t^2 - \tau^2(x,y)\right), \quad x \neq y, \quad t \geq 0, \tag{10}$$

where $\Theta_{-1}(t) = \underset{0}{\delta}(t)$, Dirac-distribution, $\Theta_0(t)$, the Heaviside function, and $\Theta_k(t) = \frac{t^k}{k!}\,\Theta_0(t),\; k \geq 1$.
Moreover,

$$\alpha_{-1}(x,y) := \frac{1}{2\,\pi}\left(\det\, \frac{\partial}{\partial\, x}\left(\frac{-1}{2}\left(\underset{y}{\nabla}\tau^2(x,y)\right)^{tr}\right)\right)^{\frac{1}{2}} \exp\left(\frac{1}{2}\int_{\Gamma(x,y)} \langle \underset{\xi}{\nabla}\log(\rho(\xi)); d\xi\rangle\right) \tag{11}$$

and

$$\begin{aligned}\alpha_k(x,y) \quad &:= \quad \frac{\alpha_{-1}(x,y)}{4\;\tau(x,y)^{k+1}}\int_{\Gamma(x,y)} \frac{c^2(\xi)\,\underset{\xi}{\Delta}\,\alpha_{k-1}(\xi,y)}{\alpha_{-1}(\xi,y)}\,\tau(\xi,y)^k\, d\tau(\xi,y) \\ &- \quad \frac{\alpha_{-1}(x,y)}{4\;\tau(x,y)^{k+1}}\int_{\Gamma(x,y)} \frac{c^2(\xi)\,\nabla\log(\rho(\xi))\cdot\underset{\xi}{\nabla}\,\alpha_{k-1}(\xi,y)}{\alpha_{-1}(\xi,y)}\,\tau(\xi,y)^k\, d\tau(\xi,y),\;\; k \geq 0,\end{aligned} \tag{12}$$

where the function $\Gamma(x,y)$ represents the geodesic, in the metric $\tau(\cdot,\cdot)$, connecting the points x and y. Finally, the point $\xi = (\xi_1, \xi_2, \xi_3)$, on $\Gamma(x,y)$, represents the Riemannian coordinates and it is given by $\xi = -\frac{1}{2}\, c^2(y)\, \underset{y}{\nabla}\tau^2(x,y)$. [7]

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- We assume that $\gamma := \|\Im(\epsilon_0)\|_{L^\infty(\Omega)} \ll 1$. (This condition can be removed!)

- Recall that the electric damping frequency of the nanoparticle is small, i.e. $\gamma_p \ll 1$.

For $z \in \Omega$, we define (recalling that $\epsilon_p(\omega) = \epsilon_\infty[1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma_p\omega}]$)

$$f_n(\omega, z) := \epsilon_0(z) - (\epsilon_0(z) - \epsilon_p(\omega))\, \lambda_n^3$$

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Under the assumptions above on γ and γ_p, the dispersion equation $f_n(\omega, z) = 0$ has one and only one solution in the complex plan, with the dominant part of its real part in the interval $\left(\omega_0; \;\; \sqrt{\omega_0^2 + \omega_p^2}\right)$.
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Estimation of the pressure fields

We assume the conditions on Ω, D, c_s, ρ, ϵ_0 to be satisfied and $\gamma \ll a^3\, \gamma_p$. Let the used incident frequency ω be such that

$$\omega^2 - \omega_{n_0}^2 \sim a^h, \quad h \in [0,1).$$

For $x \in \partial\Omega$, we have the following approximations of the average pressure:[8]

1. Before the entrance time, i.e. $s < \tau_1(x,z) := \left(\underset{y \in D}{Inf}\, \tau(x,y) \right) - a$,

$$p^\star(x,z,s,\omega) := \int_0^s 2r \int_0^r p(x,z,t,\omega) dt\, dr = \mathcal{O}(\gamma). \tag{13}$$

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$$\Psi_2(x,z,s) := \Im\left(\epsilon_p\right)\ \alpha_{-1}(x,z) \left| \underset{y}{\nabla} \tau(x,z) \right| + \Im\left(\epsilon_p\right) \int_{\tau_2(x,z)}^s 2\, r \sum_{k=0}^{+\infty} \alpha_k(x,z) \frac{\left(r^2 - \tau^2(x,z)\right)^k}{k!}\, dr.$$

[8] Ghandriche-S arxiv 2022.

Estimation of the electric fields

The electric field $u(\cdot)$ satisfies the following approximation [9]

$$\int_D |u|^2(x)\, dx = \frac{a^3\, |\epsilon_0(z)|^2\, \left|\langle u_0(z); \int_B e_{n_0}(x)\, dx\rangle\right|^2}{\left|\epsilon_0(z) - (\epsilon_0(z) - \epsilon_p(\omega))\, \lambda^3_{n_0}\right|^2} + \mathcal{O}\left(a^{\min(3;4-3h)}\right). \tag{15}$$

Here, u_0 is the electric field generated in the absence of the nanoparticle.

We see that

$$\int_D |u|^2(x)\, dx \sim a^{3-2h} + \mathcal{O}\left(a^{\min(3;4-3h)}\right), \quad h \in [0,1). \tag{16}$$

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Simultaneous reconstruction of ρ, c_s and ϵ_0

We need the measurements of $p(x, z, t, \omega)$ for

1. a single point x on $\partial\Omega$,
2. $t \in (0, \;\; M\, Diam(\Omega))$, recalling that $M^{-1} < \inf_{x \in \Omega} c_s(x)$, (A)
3. $\omega \in \left(\omega_{min} := \omega_0, \;\; \omega_{max} := \sqrt{\omega_0^2 + \omega_p^2}\right)$, (B)
4. and moving z in Ω. (C)

These measurements are used as follows:

1. With (A) and (C), we reconstruct c_s and ρ.
2. With (B) and (C), we reconstruct ϵ_0.

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Reconstruction procedure for $c_s(\cdot)$

We have shown that for x, z and ω fixed:

1. For $s < \tau_1(x,z) := \left(\underset{y \in D}{Inf}\ \tau(x,y) \right) - a$,

   $p^\star(x,z,s,\omega) := \int_0^s 2r \int_0^r p(x,z,t,\omega) dt\, dr = \mathcal{O}(\gamma)$.

2. $s > \tau_2(x,z) := \left(\underset{y \in D}{Sup}\ \tau(x,y) \right) + a$,

   $p^\star(x,z,s,\omega) \sim \gamma_p\, a^{3-2h} + \mathcal{O}(\gamma)$. (Remember that $\gamma \ll a^3\, \gamma_p$)

Therefore, from the map $s \to p^\star(x,z,s,\omega)$, with x, z, ω fixed, we can recover $\tau(x,z)$ with an error of the order a. Moving z in Ω, we get $\tau(x,\cdot)$ in Ω. We conclude with the Eikonal equation

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Schematic view of the behavior in terms of time

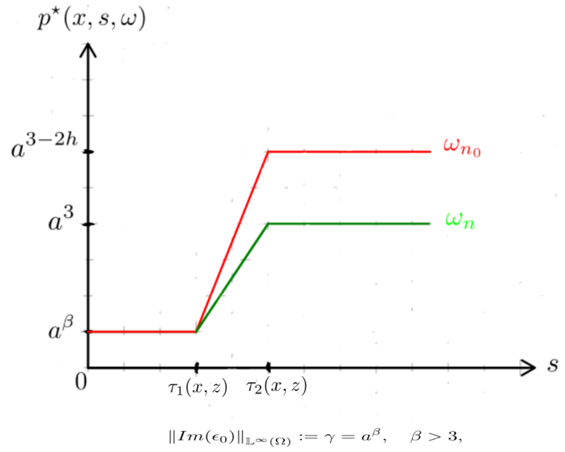

Figure: Schematic representation for the average pressure $s \to p^\star(\omega, x, s)$. The case when we are away from the resonance is marked with green color. In this case we need $\beta > 3$. The case when we are close to one resonance is marked with red color. In this case we can take $\beta > 3 - h$.

Reconstruction procedure for $\epsilon_0(\cdot)$

We use the map $\omega \to p^\star(x, z, s, \omega)$, for $x \in \partial\Omega$, $z \in \Omega$ and $s > \tau_2(x, z)$ fixed, to reconstruct the permittivity function ϵ_0.

This map reaches its maximum at the zeros of

$$f_{n_0}(\omega) := \epsilon_0(z) - (\epsilon_0(z) - \epsilon_p(\omega))\ \lambda_{n_0}^3.$$

We show that with

$$\boldsymbol{\omega_{n_0}} := \left(\omega_0^2 + \frac{\omega_p^2\, \lambda_{n_0}^3\, \epsilon_\infty}{\lambda_{n_0}^3\, \epsilon_\infty + (1 - \lambda_{n_0})\, \Re(\epsilon_0(z))}\right)^{\frac{1}{2}},$$

we have

$$f_{n_0}\left(\boldsymbol{\omega_{n_0}}, z\right) = \mathcal{O}(\gamma) + \mathcal{O}(\gamma_p).$$

Therefore by plotting the curve $\omega \to p^\star(x, z, s, \omega)$ in $(\omega_{min};\ \ \omega_{max})$, we can estimate $\boldsymbol{\omega_{n_0}}$ and hence reconstruct $\epsilon_0(z)$, with error $\sim \mathcal{O}(\gamma) + \mathcal{O}(\gamma_p)$, as

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Schematic view of the behavior in terms of the frequency

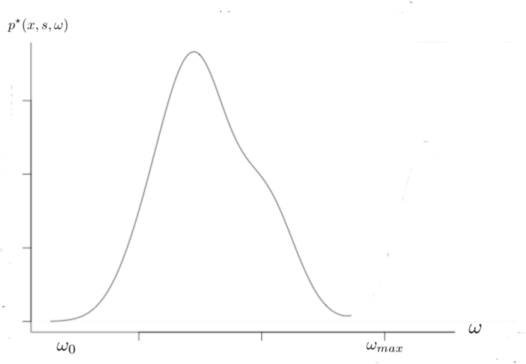

Figure: A schematic representation of the function $\omega \to p^\star(\omega, x, s)$. The peak is reached for ω near $\boldsymbol{\omega_{n_0}}$.

Reconstruction procedure for $\rho(\cdot)$

For $s > \tau_2(x,z)$, we have the expression

$$\begin{aligned}\int_0^s 2r \int_0^r p(x,t)dt\,dr \quad &= \quad \Im\left(\epsilon_p\right)\ \alpha_{-1}(x,z)\ \left|\nabla_y \tau(x,z)\right| \int_D \left|u_1\right|^2(y)dy \\ &+ \quad \Im\left(\epsilon_p\right) \int_{\tau_2(x,z)}^s 2\,r \sum_{k=0}^{+\infty} \alpha_k(x,z)\, \frac{\left(r^2 - \tau^2(x,z)\right)^k}{k!}\ dr \int_D \left|u_1\right|^2(y)dy + Remainder_1.\end{aligned}$$

Now, in particular, when s is close to $\tau_2(x,z)$ the previous expression is reduced to

$$\int_0^s 2r \int_0^r p(x,t)dt\,dr = \Im\left(\epsilon_p\right)\ \alpha_{-1}(x,z)\ \left|\nabla_y \tau(x,z)\right| \int_D \left|u_1\right|^2(y)dy + Remainder_2.$$

Therefore, we can reconstruct $\Im\left(\epsilon_p\right)\ \alpha_{-1}(x,z)\ \left|\nabla \tau(x,z)\right| \int_D \left|u_1\right|^2(y)dy$. Hence, we reconstruct the function $\alpha_{-1}(\cdot,x)$ inside Ω. Recall that

$$\alpha_{-1}(x,y) := \frac{1}{2\,\pi}\left(\det \frac{\partial}{\partial\,x}\left(\frac{-1}{2}\left(\nabla_y \tau^2(x,y)\right)^{tr}\right)\right)^{\frac{1}{2}} \exp\left(\frac{1}{2}\int_{\Gamma(x,y)} \langle \nabla_\xi \log(\rho(\xi)); d\xi\rangle\right) \qquad (17)$$

Then, we can recover

$$g_x(y) := \int_{\Gamma(x,y)} \langle \nabla_\xi \log(\rho(\xi)); d\xi\rangle = \int_{\Gamma(x,y)} d_\xi \log(\rho(\xi)) = \log(\rho(y)) - \log(\rho(x)), \qquad (18)$$

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Conclusion

We described an approach to solve the inverse problem for PAT using electromagnetic nanoparticles as contrast agents (Plasmonics or Dielectrics). The key arguments are:

1. Looking at the behavior of the pressure in terms of time, solely, allows us to estimate the internal travel function $\tau(x, z)$, for $x \in \partial\Omega$ and $z \in \Omega$, from which we recover the sound speed via the Eikonal equation.

2. Looking at the behavior of the pressure in terms of the incident frequencies, solely, allows us to localize the Plasmonic (resonant) frequencies from which we recover the permittivity with an explicit formula.

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Expected Results

Wave-Model	Particle	Assumption-particle	Reconstruction
Acoustics	micro-bubble	small ρ_1 & small k_1	$\rho_0(\cdot)$ & $k_0(\cdot)$
Electromagnetism	plasmonic nano-particle	$\Re(\epsilon_p) < 0$ & small $\Im(\epsilon_p)$	$\epsilon_0(\cdot)$
Electromagnetism	dielectric nano-particle	large $\Re(\epsilon_p)$ & moderate $\Im(\epsilon_p)$	$\epsilon_0(\cdot)$
Elasticity	slow-SP cavity	large (ρ_p) & moderate (λ_p, μ_p)	$\rho_0(\cdot)$
Elasticity	moderate-P cavity	large (ρ_p, λ_p) & moderate (μ_p)	$\rho_0(\cdot)$ & $c_P(\cdot)$
Elasticity	moderate-SP cavity	large $(\rho_p, \lambda_p, \mu_p)$	$\rho_0(\cdot), \lambda_0(\cdot), \mu_0(\cdot)$

Regarding elasticity, we recall that

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \text{ and } \mu = \frac{E}{(1+\nu)}$$

where E is the Young modulus, which is positive, and ν is the Poisson ratio, with $\nu \in [-1, \frac{1}{2}]$.

We are interested in small scaled cavities with high mass density:

1. With ν in $(-1, \frac{1}{2})$ and away from -1 and $\frac{1}{2}$, we can get slow-SP cavities.
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Electromagnetism	plasmonic nano-particle	$\Re(\epsilon_p) < 0$ & small $\Im(\epsilon_p)$	$\epsilon_0(\cdot)$
Electromagnetism	dielectric nano-particle	large $\Re(\epsilon_p)$ & moderate $\Im(\epsilon_p)$	$\epsilon_0(\cdot)$
Elasticity	slow-SP cavity	large (ρ_p) & moderate (λ_p, μ_p)	$\rho_0(\cdot)$
Elasticity	moderate-P cavity	large (ρ_p, λ_p) & moderate (μ_p)	$\rho_0(\cdot)$ & $c_P(\cdot)$
Elasticity	moderate-SP cavity	large $(\rho_p, \lambda_p, \mu_p)$	$\rho_0(\cdot), \lambda_0(\cdot), \mu_0(\cdot)$

Regarding elasticity, we recall that

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \text{ and } \mu = \frac{E}{(1+\nu)}$$

where E is the Young modulus, which is positive, and ν is the Poisson ratio, with $\nu \in [-1, \frac{1}{2}]$.

We are interested in small scaled cavities with high mass density:

1. With ν in $(-1, \frac{1}{2})$ and away from -1 and $\frac{1}{2}$, we can get slow-SP cavities.
2. With ν close to $\frac{1}{2}$ we can get moderate-P cavities.
3. With ν close to -1, we can get moderate-SP cavities.

References

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Thank you