Injectivity and stability of inversion of the star transform

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Tomography Across the Scales
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Acknowledgements

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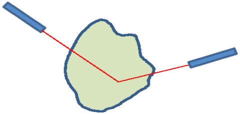
- Partially supported by NSF DMS-1616564

Outline

- Some motivating imaging modalities
- Prior work and terminology
- Definition of the star transform
- Geometric description of its inversion
- Injectivity of the star transform
- Singular directions and stability
- A resolved conjecture from algebraic geometry
- Numerical examples

Single Scattering Optical/X-ray Tomography



Florescu, Markel, Schotland (2009, '10, '11),
Katsevich, Krylov (2013, '15)

- Uses light/X-rays, transmitted and scattered through an object to determine the interior features of that object.
- It is assumed the majority of photons scatter once.
- Collimated emitters and receivers measure the intensity of radiation scattered along various broken rays.
- Need to recover the spatially varying coefficients of absorption and/or scattering.

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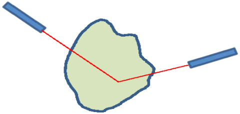

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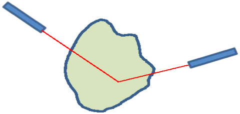

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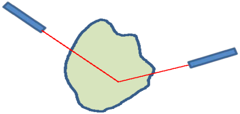

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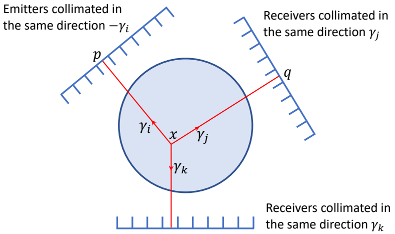
A simple setup of SSXT with multiple detectors

Each collimated emitter-receiver pair (p, q) uniquely identifies the scattering location x inside the body.

The signal generated by emitter i and measured by detector j provides the following information:

$$\phi_{ij}(x) = \mathcal{X}_i f(x) + k_{ij}\mathcal{X}_j f(x) + \eta(x),$$

where f and η represent the attenuation and scattering coefficients, $\mathcal{X}_i f(x) = \int_0^\infty f(x + \phi_i \ell) d\ell$, and k_{ij} is a known constant.



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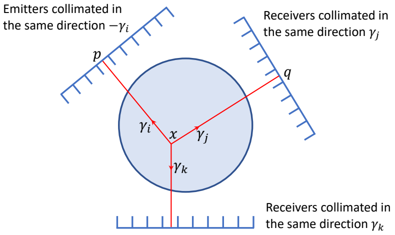

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Taking into consideration Compton scattering effect, it is reasonable to consider the attenuation coefficients at different energy levels (before and after scattering) as f and $k_{ij}f$, $k_{ij} \neq 1$.

If $\eta(x)$ is not known, simultaneous recovery of f and η requires the use of data with at least three distinct directions. One approach is to consider various linear combinations of $\phi_{ij}(x)$, which will eliminate η and produce a weighted star transform of f.

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The star transform (Zhao, Schotland, Markel (2014))

E.g., in a "3-directional" geometry one can use a data function:

$$\Phi \equiv \phi_{12} - \phi_{13} = I_3 - I_2,$$

which is equivalent to the **signed** broken ray transform of μ.

Such an approach was used in the works of Florescu, Markel, Schotland (2009, '10, '11) and Katsevich, Krylov (2013, '15).

Notice, that in the above example we lost information about the integrals of the image function along rays of certain direction. The problem of excluding η, without excluding any ray integrals can be solved by finding coefficients c_{jk} such that

$$(i)\ \sum_{jk} c_{jk} = 0, \quad (ii)\ c_{kk} = 0, \quad (iii)\ c_{jk} = c_{kj}, \quad (iv)\ s_k = \sum_j c_{jk} \neq 0.$$

$$\text{Thus} \quad \Phi \equiv -\frac{1}{2}\sum_{j=1}^{K}\sum_{k=1}^{K} c_{jk}\phi_{jk} = \sum_{k=1}^{K} s_k I_k.$$

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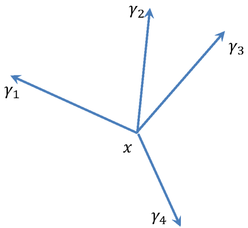
V-line (broken ray) and conical transforms: vertices inside

- G. A., R. Gouia-Zarrad, M. Latifi, R. Mishra, S. Moon, S. Roy
- D. Finch, B. Sherson
- L. Florescu, V. Markel, J. Schotland, F. Zhao
- P. Grangeat, M. Morvidone, M. Nguyen, R. Régnier, G. Rigaud, T. Truong, H. Zaidi
- A. Katsevich, R. Krylov
- P. Kuchment, F. Terzioglu
- J. A. O'Sullivan, M. R. Walker
- V. Palamodov
- . . .

Broken ray and conical transforms: vertices on boundary

- M.Allmaras, D.Darrow, Y.Hristova, G.Kanschat, P.Kuchment
- M. Cree, P. Bones and R. Basko, G. Zeng, G. Gullberg
- M. Haltmeier, D. Schiefeneder
- C. Jung, K. Kwon, S. Moon
- V. Maxim
- D. Nguyen, L. Nguyen
- L. Parra and M. Hirasawa, T. Tomitani
- B. Smith
- Y. Zhang
- ...

The Star Transform



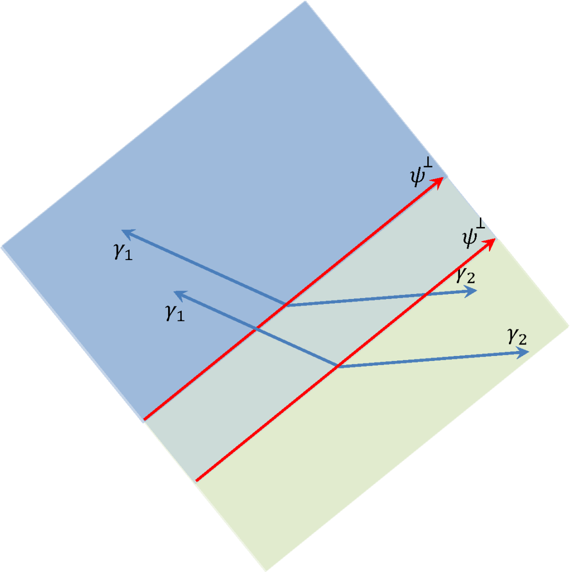
Definition

The (weighted) star transform $\mathcal{S}$ of f at $x \in \mathbb{R}^2$ is defined as:

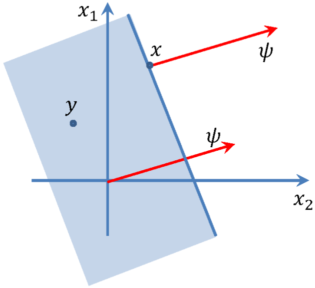
$$\mathcal{S}f(x) = \sum_{i=1}^{m} c_i \, \mathcal{X}_{\gamma_i} f(x) = \sum_{i=1}^{m} c_i \int_0^\infty f(x + t\gamma_i) \, dt, \qquad (1)$$

where each γ_i is a unit vectors and $\mathcal{X}_{\gamma_i}$ is the divergent beam transform in the direction of γ_i.

Geometric Description of Inversion



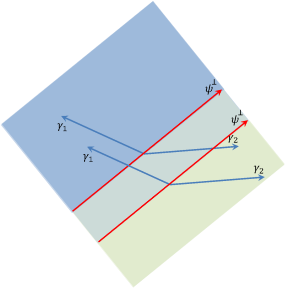
Notations



Let $l(\psi, s) = \{x \in \mathbb{R}^2 | \langle x - s\psi, \psi \rangle = 0\}$ be the line normal to the vector ψ and of distance s from the origin.

Define $F(x) = \int\limits_{\langle y,\psi\rangle \leq \langle x,\psi\rangle} f(y)\, d\mu$, and $F_\psi(s) = F(s\psi)$.

Composition of the Radon and Ray Transforms



Lemma

Assume that $f \in C_c(\mathbb{R}^2)$. *If* $\langle \psi, \gamma \rangle \neq 0$, *then*

$$\frac{d}{ds}\mathcal{R}(\mathcal{X}_\gamma f)(\psi, s) = -\frac{1}{\langle \psi, \gamma \rangle}\frac{dF_\psi(s)}{ds} = -\frac{1}{\langle \psi, \gamma \rangle}\mathcal{R}f(\psi, s).$$

Inversion of the Star Transform

Theorem

Let $\mathcal{S} = \sum_{i=1}^{m} c_i \, \mathcal{X}_{\gamma_i}$ *be the weighted star transform and let*

$$q(\psi) = \frac{-1}{\sum_{i=1}^{m} \frac{c_i}{\langle \psi, \gamma_i \rangle}}.$$

Then the following is true for any ψ *in the domain of* q

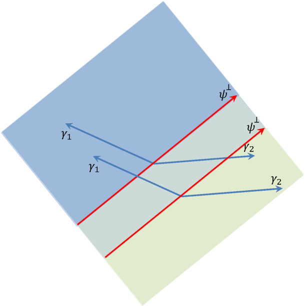
$$\mathcal{R}f(\psi, s) = q(\psi) \frac{d}{ds} \mathcal{R}(\mathcal{S}f)(\psi, s).$$

Inversion of the V-line Transform

Corollary

An inversion formula for the V-line transform with ray directions γ_1, γ_2 is given by

$$f = \mathcal{R}^{-1}\left(\frac{-\langle\psi,\gamma_1\rangle\langle\psi,\gamma_2\rangle}{\langle\psi,\gamma_1\rangle+\langle\psi,\gamma_2\rangle}\frac{d}{ds}\mathcal{R}(\mathcal{S}f)(\psi,s)\right).$$



Non-invertible Configurations

- In certain configurations of the star transform, the function $q(\psi)$ is not defined for any ψ. For example, this happens when $m = 2$, $c_1 = c_2 = 1$, and $\gamma_1 = -\gamma_2 = (1, 0)$. In this case $\langle \psi, \gamma_1 \rangle + \langle \psi, \gamma_2 \rangle \equiv 0$ for any $\psi \in \mathbb{S}^1$.

- Consider a similar configuration with more rays: $m = 4$, $c_1 = \ldots = c_4 = 1$, $\gamma_1 = -\gamma_3$ and $\gamma_2 = -\gamma_4$. The star transform is again not injective, since it provides less information than the ordinary Radon transform restricted to two directions.

- Interestingly enough, such configurations are the only ones, for which the star transform is not injective.

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Symmetric and Regular Star Transforms

Definition

A star transform is called **symmetric**, if $m = 2k$ for some $k \in \mathbb{N}$ and (after possible re-indexing) $\gamma_i = -\gamma_{k+i}$ with $c_i = c_{k+i}$ for all $i = 1, \ldots, k$. The corresponding star is also called **symmetric**.

Definition

A star transform is called **regular**, if $c_1 = \ldots = c_m = 1$ and the ray directions γ_i, $i = 1, \ldots, m$ correspond to the radius vectors of the vertices of a regular m-gon. The corresponding star is also called **regular**.

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Injectivity of the Star Transform

Theorem

The star transform $\mathcal{S} = \sum_{i=1}^{m} c_i \mathcal{X}_{\gamma_i}$ *is invertible* **if and only if** *it is not symmetric.*

Corollary

Any star transform with an odd number of rays is invertible.

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Elementary Symmetric Polynomials

Consider the elementary symmetric polynomial of degree $m-1$ in m variables $y = (y_1, \ldots, y_m)$

$$e_{m-1}(y_1, \ldots, y_m) = \sum_{i=1}^{m} \prod_{j \neq i} y_j. \tag{2}$$

Using the above notation, one can re-write the formula for $q(\psi)$ as

$$q(\psi) = \frac{c_1^{-1} \ldots c_m^{-1}}{\dfrac{e_{m-1}\left(\langle \psi, c_1^{-1}\gamma_1 \rangle, \ldots, \langle \psi, c_m^{-1}\gamma_m \rangle\right)}{\langle \psi, \gamma_1 \rangle \ldots \langle \psi, \gamma_m \rangle}}. \tag{3}$$

The proof of the injectivity theorem is based on the description of zero sets of e_{m-1} and the fact that the star transform $\mathcal{S}$ is invertible if $e_{m-1}(\langle \psi, c_1^{-1}\gamma_1 \rangle, \ldots, \langle \psi, c_m^{-1}\gamma_m \rangle)$ is not identically zero as a function of ψ.

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Singular Directions of the Star Transform

Definition

We call

$$\mathcal{Z}_1 = \bigcup_{j=1}^{m} \{\psi : \langle \psi, \gamma_j \rangle = 0\} \tag{4}$$

the set of singular directions of Type 1 for $\mathcal{S}$, and

$$\mathcal{Z}_2 = \left\{ \psi : \sum_{j=1}^{m} c_j \prod_{i \neq j} \langle \psi, \gamma_j \rangle = 0 \right\} \tag{5}$$

the set of singular directions of Type 2 for $\mathcal{S}$.

Singular Directions of the Star Transform

- The singular directions of Type 2 correspond to "division by zero" of the processed data $\frac{d}{dt}\mathcal{R}(\mathcal{S}f)$, while those of Type 1 correspond to "multiplication by zero". Hence, it is natural to expect that singular directions of Type 2 will create instability and adversely impact the reconstruction. Our numerical experiments confirm these expectations.
- The (totally different) algorithm for inversion of the star transform obtained in [ZSM'14] produces a relation equivalent to the defining relation of $\mathcal{Z}_2$ as a necessary and sufficient condition for the instability of that algorithm.
- While the geometric meaning of singular directions of Type 1 is obvious for any m, there is no easy interpretation of set $\mathcal{Z}_2$ for $m \geq 3$. However, the singular directions of Type 2 are more crucial for the quality of reconstruction.

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Star Transforms with Uniform Weights

Theorem

The regular star transform $\mathcal{S}$ with $m = 2k + 1$ rays does not have a singular direction of Type 2.

Theorem

Consider the star transform $\mathcal{S} = \sum_{i=1}^{m} \mathcal{X}_{\gamma_i}$ with uniform weights.

1. *If m is even, $\mathcal{S}$ must contain a singular direction of Type 2.*
2. *When m is odd, there exist configurations of $\mathcal{S}$ that contain singular directions of Type 2, as well as configurations that do not contain them.*

Star Transforms with Uniform Weights

Theorem

The regular star transform $\mathcal{S}$ with $m = 2k + 1$ rays does not have a singular direction of Type 2.

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Star Transforms with Non-Uniform Weights

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Theorem

Consider a star transform $\mathcal{S} = \sum_{i=1}^{m} c_i \mathcal{X}_{\gamma_i}$, where $m = 2k + 1$.

(a) For any set of specified weights $c_1 \ldots, c_m$, there exist $\gamma_1, \ldots, \gamma_m$ such that $\mathcal{S}$ contains singular directions of Type 2. In other words, for any set of weights there are configurations of $\mathcal{S}$ with unstable inversion.

(b) Let $m = 3$. For any set of specified weights c_1, c_2, c_3, there exist $\gamma_1, \gamma_2, \gamma_3$ such that $\mathcal{S}$ does not contain singular directions of Type 2. In other words, for any set of weights there are configurations of $\mathcal{S}$ with a stable inversion.

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Zeros of $e_{m-1}(y_1, \ldots, y_m)$ for Odd m.

The following conjecture was formulated in "A. Conflitti, *Zeros of real symmetric polynomials*, Applied Mathematics E-Notes, 6 (2006), pp 219-224":

Conjecture

If r is even then $e_r^{-1}(0)$ contains no real vector subspace of dimension r.

Furthermore, it is stated there that one of the extreme cases is "the case $e_{m-1}(y_1, ..., y_m)$, $m \equiv 1 \pmod 2$, which becomes a task quite hard to tackle".

Our proof of the theorem about regular star transforms includes a proof of the aforementioned extreme case. Namely,

Theorem

Let $m = 2k + 1$ for some $k \in \mathbb{N}$. Then $e_{m-1}^{-1}(0)$ contains no real vector subspace of dimension $m - 1$.

Zeros of $e_{m-1}(y_1, \ldots, y_m)$ for Odd m.

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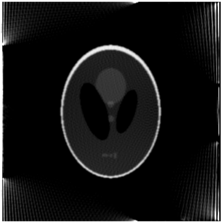
Numerical Examples



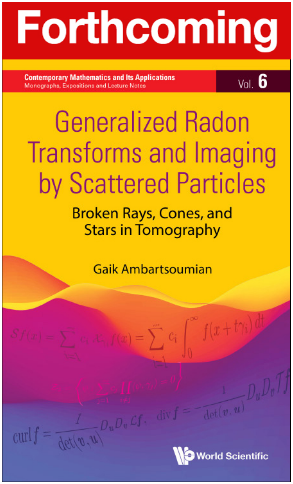
Reconstruction using the star transform with directions $\gamma_1 = (1,0)$, $\gamma_2 = (cos(3\pi/4), sin(3\pi/4))$, $\gamma_3 = (cos(5\pi/4), sin(5\pi/4))$

Numerical Examples

Reconstructions using the star transform with γ's directed along polar angles of: (0,72,144,216,288), (0,120) and (0,90,135) degrees.

Thanks for Your Attention!

Upcoming Book



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