The discrepancy of the linear flow on the torus

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The discrepancy of a sequence on the torus $\mathbb{R}^d/\mathbb{Z}^d$ ($= [0,1]^d$ with opposite facets identified)

- cannot be $O(1)$. (van Aardenne–Ehrenfest)
- is $\Omega(\log N)$ if $d=1$ and $\Omega(\log^{d/2} N)$ if $d \geq 2$. (Schmidt, Roth)
  (From now on $a_N = \Omega(b_N)$ means $\limsup_{N\to\infty} |a_N|/b_N > 0$.)

Question: Is there a similar result for the discrepancy of curves?

Given $g : [0,\infty) \to \mathbb{R}^d/\mathbb{Z}^d$ and $T>0$ let

$$\mathrm{discrep}(g,T) = \sup_{R \in \mathcal{R}} \left| \lambda\left(\{0 \leq t \leq T : g(t) \in R\}\right) - T\lambda(R) \right|$$

where $\mathcal{R}$ is the set of axis-parallel boxes in $[0,1]^d$ and λ is the Lebesgue measure. Under natural assumptions (e.g. if g is Lipschitz) we have $\mathrm{discrep}(g,T) = \Omega(1)$. Is there a nontrivial lower estimate?

Curves with bounded discrepancy

Drmota, 1989: Suppose g is continuous and has finite arc length ℓ_T on any $[0, T]$.

- There exist curves for which $\mathrm{discrep}(g, T) = O(1)$ in $d = 2$.
- Conjectured that $\mathrm{discrep}(g, T)/T = \Omega(\log^{d-2-\varepsilon} \ell_T / \ell_T)$ in $d \geq 3$. If g is Lipschitz, $\ell_T = O(T)$ and so $\mathrm{discrep}(g, T) = \Omega(\log^{d-2-\varepsilon} T)$.

We have recently proved that in fact there exist (Lipschitz) curves in any dimension d such that $\mathrm{discrep}(g, T) = O(1)$. As observed, this is best possible up to a constant factor. In particular, **Drmota's conjecture is false**; moreover,

there is no van Aardenne–Ehrenfest type theorem for the discrepancy of curves in any dimension.

Linear flow

Given $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{R}^d$ the linear flow on the torus with direction α is the continuous time dynamical system which maps a point $s \in \mathbb{R}^d/\mathbb{Z}^d$ to $s + t\alpha \pmod{\mathbb{Z}^d}$ at time $t \in \mathbb{R}$. For any function $F : [0,1]^d \to \mathbb{R}$ let

$$\Delta_T(s, \alpha, F) = \int_0^T F(\{s_1 + t\alpha_1\}, \ldots, \{s_d + t\alpha_d\}) \, \mathrm{d}t - T \int_{[0,1]^d} F(x) \, \mathrm{d}x.$$

For a set $A \subseteq [0,1]^d$ let $\Delta_T(s, \alpha, A) = \Delta_T(s, \alpha, \chi_A)$ where χ_A is the characteristic function of A.

The following are equivalent:

- Every orbit is dense (Kronecker's Theorem)
- Every orbit is uniformly distributed (Weyl's Criterion)
- The dynamical system is ergodic
- $\alpha_1, \alpha_2, \ldots, \alpha_d$ are linearly independent over $\mathbb{Q}$

Linear flow in dimension $d=2$

Let $\alpha=(\alpha_1,1)$, $0<\alpha_1<1$ irrational. Let $\|\cdot\|$ denote distance from the nearest integer. Assume $\|n\alpha_1\| \geq C n^{-\gamma}$ for every $n \in \mathbb{N}$ with some $C>0$ and $\gamma \geq 1$.

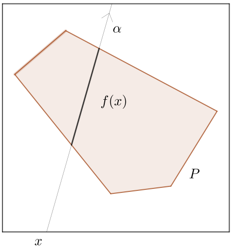
- **Drmota, 1989:** If $\gamma<2$, then $\sup_{R \in \mathcal{R}} |\Delta_T(s,\alpha,R)| = O(1)$. The proof used the Erdős–Turán inequality for curves.
- **Grepstad–Larcher, 2016:** If $\gamma < 5/4$, then $\Delta_T(s,\alpha,P)=O(1)$ for any convex polygon $P \subseteq [0,1]^2$ whose sides are not parallel to α. The proof used an Ostrowski type explicit formula.
- **B, 2018:** If $P \subseteq [0,1]^2$ is a convex polygon with N sides, and $\phi_1, \phi_2, \dots, \phi_N \neq 0, \pi$ are the angles of the sides and α, then

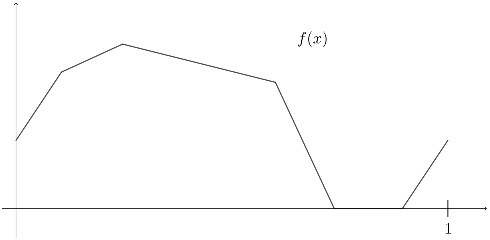
$$|\Delta_T(s,\alpha,P)| \leq 2 + \frac{N+1}{\pi^2 |\alpha|} \max_{1 \leq k < \ell \leq N} |\cot \phi_k - \cot \phi_\ell| \sum_{n=1}^{\infty} \frac{1}{n^2 \|n\alpha_1\|}.$$

Note $\gamma<2$ holds for all algebraic irrationals. The exceptional set has Hausdorff dimension $2/3$.

Question: Is $\gamma<2$ optimal?

Sketch of the proof





- Reduction to a discrete time dynamical system in dimension 1 (irrational rotation by α_1).
- Project the vertices along α to get $0 = c_0 < c_1 < \cdots < c_{N+1} = 1$. $f(x) = a_k x + b_k$ on $[c_{k-1}, c_k]$. Slope a_k depends on $\phi_1, \ldots, \phi_N$.

$$\int_0^1 f(x) e^{-2\pi i n x} \, \mathrm{d}x = \frac{1}{n} \sum_{k=1}^{N+1} (\text{telescoping}) - \sum_{k=1}^{N+1} a_k \int_{c_{k-1}}^{c_k} \frac{e^{-2\pi i n x}}{-2\pi i n} \, \mathrm{d}x$$

from integration by parts. The telescoping sum **cancels** by continuity of f, second sum is $O(1/n^2)$ with explicit implied constant depending only on N and $\phi_1, \ldots, \phi_N$.

Linear flow in dimension $d \geq 3$

Theorem (B, 2018)

Let K be a subfield of $\mathbb{R}$, $\alpha = (\alpha_1, \ldots, \alpha_{d-1}, 1) \in K^d$. Suppose that for any linearly independent linear forms $L_1, \ldots, L_{d-1}$ of $d-1$ variables with coefficients in K there exist $C > 0$ and $\delta < 1$ such that

$$\|n_1\alpha_1 + \cdots + n_{d-1}\alpha_{d-1}\| \prod_{k=1}^{d-1} (|L_k(n)| + 1) \geq C|n|^{-\delta}$$

for all $n \in \mathbb{Z}^{d-1}$, $n \neq 0$. Let $P \subseteq [0,1]^d$ be a polytope with nonempty interior such that every facet has a normal vector $\nu \in K^d$ such that $\langle \nu, \alpha \rangle \neq 0$. Then $\Delta_T(s, \alpha, P) = O(1)$ with implied constant depending only on α and the normal vectors of the facets P.

If $K =$ algebraic reals, we get from Schmidt's Subspace Theorem:

Corollary (B, 2018)

If the coordinates of α are algebraic and linearly independent over $\mathbb{Q}$, then $\sup_{R \in \mathcal{R}} |\Delta_T(s, \alpha, R)| = O(1)$. (Here $\mathcal{R}$ = set of axis parallel boxes.)

Sketch of proof

- Reduction to a discrete time dynamical system in dimension $d-1$ (irrational rotation by $(\alpha_1, \alpha_2, \ldots, \alpha_{d-1})$).
- $f : [0,1]^{d-1} \to \mathbb{R}$ is still "piecewise linear": project P along α to get a partition $P_1, \ldots, P_m$ of $[0,1]^{d-1}$ into polytopes.
- $f(x) = \langle a_k, x \rangle + b_k$ on P_k. Here a_k and the normal vectors of P_k depend only on the normal vectors of P and α.

$$\int_{[0,1]^{d-1}} f(x) e^{-2\pi i \langle n,x \rangle} \, \mathrm{d}x =$$

$$\sum_{k=1}^m \int_{\partial P_k} \frac{-\langle n, \nu(x) \rangle}{2\pi i |n|^2} f(x) e^{-2\pi i \langle n,x \rangle} \, \mathrm{d}x + \sum_{k=1}^m \frac{\langle a_k, n \rangle}{2\pi i |n|^2} \int_{P_k} e^{-2\pi i \langle n,x \rangle} \, \mathrm{d}x$$

  from the Gauss–Ostrogradsky Theorem applied on each P_k.
  The first sum **cancels** by the continuity of f (every facet shows up twice with opposite outer normal vectors $\nu(x)$).
- The Fourier coefficients of f are "smaller than expected" by a factor of $|n|$.

Questions

We needed with some $C > 0$, $\delta < 1$

$$\|n_1\alpha_1 + \cdots + n_{d-1}\alpha_{d-1}\| \prod_{k=1}^{d-1} (|L_k(n)| + 1) \geq C|n|^{-\delta}.$$

- Is $\delta < 1$ optimal?
- If $K = \mathbb{R}$, is the theorem true for almost every $\alpha \in \mathbb{R}^d$?
  The problem is that in the proof $L_1, \dots, L_{d-1}$ not only depend on the normal vectors of P, but also on α.

In $d = 2$:

- If $\gamma < 5/4$, the discrepancy with respect to balls is $O(1)$. Convex sets with C^2 boundary of positive curvature are also sets of bounded remainder. (Grepstad–Larcher, 2016)
- If $\gamma = 1$ (α_1 is badly approximable), the discrepancy with respect to all convex sets is $O(\log T)$. Best possible up to a constant factor. $f(x)$ is BV, reduces to Koksma's inequality. (Beck, unpublished)
- Are there similar results in higher dimensions?

Discrete analogues

For the discrete time dynamical system $s \mapsto s + t\alpha \pmod{\mathbb{Z}^d}$, $t \in \mathbb{Z}$ the following are equivalent:

- Every orbit is dense
- Every orbit is uniformly distributed
- The dynamical system is ergodic
- $1, \alpha_1, \alpha_2, \dots, \alpha_d$ are linearly independent over $\mathbb{Q}$

But the quantitative results are very different.

Niederreiter, 1972: If $1, \alpha_1, \dots, \alpha_d$ are algebraic and linearly independent over $\mathbb{Q}$, then every orbit (=Kronecker sequence) has discrepancy $O(N^\varepsilon)$ for any $\varepsilon > 0$.

Beck, 1994: Every orbit (=Kronecker sequence) has discrepancy $O(\log^d N \varphi(\log \log N))$ for a.e. $\alpha \in \mathbb{R}^d$ if and only if $\sum_{n=1}^\infty 1/\varphi(n) < \infty$. (Here $\varphi(n) > 0$, increasing.)

Sets of bounded remainder

Halász, 1976: Let $(\Omega, \mathcal{F}, \mu, T)$ be a discrete time, ergodic dynamical system with $\mu(\Omega) = 1$.

- There exists a set of bounded remainder of measure $0 \leq m \leq 1$ if and only if $e^{2\pi i m}$ is an eigenvalue of the system (that is, there exists a measurable function g, not a.e. zero with $g(Tx) = e^{2\pi i m} g(x)$ a.e.)
- If non-atomic, for any $2 \leq \varphi(0) \leq \varphi(1) \leq \cdots \leq \varphi(n) \to \infty$ and any $0 \leq m \leq 1$ there exists $A \in \mathcal{F}$ with $\mu(A) = m$ such that

$$\left| \sum_{i=1}^{n} \chi_A(T^i x) - n\mu(A) \right| \leq \varphi(n) \qquad \text{a.e.}$$

In particular, for the discrete time system $s \mapsto s + t\alpha \pmod{\mathbb{Z}^d}$, $t \in \mathbb{Z}$ sets of bounded remainder have measure of the form $n_0 + n_1\alpha_1 + \cdots + n_d\alpha_d$ where $n_0, n_1, \ldots, n_d \in \mathbb{Z}$. In continuous time any measure $0 \leq m \leq 1$ is possible (e.g. any axis-parallel box is a set of bounded remainder).

Measurable test functions

Both **discrete and continuous time**: given α (with the appropriate linear independence property) and $F \in L^1([0,1]^d)$, we have $\Delta_T(s, \alpha, F) = o(T)$ for a.e. $s \in [0,1]^d$. (Birkhoff)

What if $s \in [0,1]^d$ and F are given, and we want metric results in α?

In **discrete time** there exists an open set $A \subseteq [0,1]$ with $\Delta_T(0, \alpha, A) = \Omega(T)$ for every α. (Marstrand's counterexample to Khinchin's Conjecture)

In **continuous time** for any $F \in L^2([0,1]^d)$ we have

$$\Delta_T(0, \alpha, F) = O(T^{1/2-1/(2d-2)} \log^{3+\varepsilon} T)$$

for a.e. $\alpha \in \mathbb{R}^d$. Optimal up to powers of $\log T$. (Beck, 2015)

Question: Estimate optimal up to a constant factor? This would be most interesting in $d = 2$ when $\Delta_T(0, \alpha, F) = O(\log^{3+\varepsilon} T)$ for a.e. $\alpha \in \mathbb{R}^2$.

Thank you!

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