Energy Optimization with Orthogonal Potentials on the Sphere

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- If $\mu = \frac{1}{|Z|} \sum_{x \in Z} \delta_x$, then

$$I_F(\mu) = \frac{1}{N^2} \sum_{x,y \in Z} F(\langle x, y \rangle) = E_F(Z).$$

Appearances of Monotonic Potentials

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- Packing Problem:

$$F(\langle x, y\rangle) = \begin{cases} \infty & ||x - y|| < \delta \\ 0 & ||x - y|| \geq \delta \end{cases}.$$

Orthogonal Potentials

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These orthogonal potentials are monotonic in real projective space.

Positive Definiteness

Definition

A function $f : [-1, 1] \to \mathbb{R}$ is **positive definite** on $\mathbb{S}^d$ if, for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in \mathbb{S}^d$, $c_1, \ldots, c_n \in \mathbb{R}$,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j f(\langle x_i, x_j \rangle) \geq 0.$$

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A function f is positive definite iff for all $n \in \mathbb{N}$

$$\hat{f}(n; \frac{d-1}{2}) = a_{d,n} \int_{-1}^{1} f(t) C_n^{\frac{d-1}{2}}(t)(1-t^2)^{\frac{d-2}{2}} dt \geq 0.$$

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Theorem

σ is a minimizer of $I_F(\mu)$ if and only if F is positive definite.

Negative Definiteness

Definition

A function $f : [-1,1] \to \mathbb{R}$ is **negative definite** on $\mathbb{S}^d$ if, for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in \mathbb{S}^d$, $c_1, \ldots, c_n \in \mathbb{R}$,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j f(\langle x_i, x_j \rangle) \leq 0.$$

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Theorem

σ is a maximizer of $I_F(\mu)$ if and only if F is negative definite.

Definition

We call a finite set of unit vectors $\{x_1, ..., x_N\} \subset \mathbb{S}^d$ *a* ***finite unit norm tight frame*** *(FUNTF) if there exists some constant* $A > 0$ *such that for all* $y \in \mathbb{R}^{d+1}$

$$\sum_{j=1}^{N} |\langle y, x_j \rangle|^2 = A||y||^2.$$

Definition

The ***frame potential*** *of* $\{x_i\}_{i=1}^{N}$ *is*

$$FP(\{x_i\}_{i=1}^{N}) = \frac{1}{N^2} \sum_{i,j=1}^{N} |\langle x_i, x_j \rangle|^2,$$

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$$FP(\mu) = \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} |\langle x, y \rangle|^2 d\mu(x) d\mu(y).$$

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Theorem (Benedetto, Fickus (2003))

If $N \geq d+1$, the minimum value of the frame potential is $\frac{1}{d+1}$, and the (local/global) minimizers are precisely the FUNTF's in $\mathbb{R}^{d+1}$.

Fejes Tóth Conjecture

Conjecture (Fejes Tóth (1959))

Let $d \geq 1$, $N = m(d+1) + k$ *with* $m \in \mathbb{N}_0$ *and* $0 \leq k \leq d$, *and*

$$F(\langle x, y\rangle) = \frac{1}{\pi}\arccos(|\langle x, y\rangle|).$$

Then $E_F(Z)$ *is maximized by the point set* $Z = \{z_1, \ldots, z_N\} \subset \mathbb{S}^d$ *with* $z_{p(d+1)+i} = e_i$. *In this case, the energy is*

$$\frac{k(k-1)(m+1)^2 + 2km(d+1-k)(m+1) + (d-k)(d+1-k)m^2}{2N^2}.$$

In particular, if $N = m(d+1)$, *the sum is maximized by* m *copies of the orthonormal basis:*

$$\max_{\substack{Z \subset \mathbb{S}^d \\ \#Z = N}} E_F(Z) = \frac{1}{2} \cdot \frac{d}{d+1}.$$

Bounding the Energy

$$G(t) = \frac{1}{2} - \frac{69}{50\pi}t^2 \geq \frac{1}{\pi}\arccos(|t|) = F(t).$$

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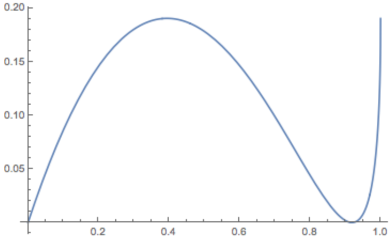

Figure: The graph of the function $G(t) - F(t)$ for $0 \leq t \leq 1$.

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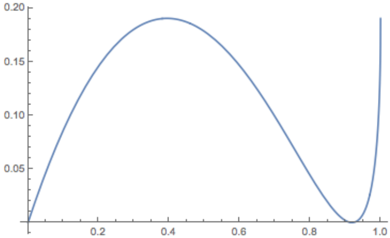

Figure: The graph of the function $G(t) - F(t)$ for $0 \leq t \leq 1$.

From results of Benedetto and Fickus on frame potential, we have

$$\frac{d}{2(d+1)} \leq \max_{\mu \in \mathfrak{B}(\mathbb{S}^d)} I_F(\mu) \leq \max_{\mu \in \mathfrak{B}(\mathbb{S}^d)} I_G(\mu) = \frac{1}{2} - \frac{69}{50\pi(d+1)}.$$

On the Circle

Proofs: Geometric (Fodor, Vígh, Zarnocz), Fourier expansion, Chebyshev expansion.

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For $x \in \mathbb{S}^1$, define the *antipodal quadrants* in the direction of x as $Q(x) = \{y : |\langle x, y\rangle| > \frac{\sqrt{2}}{2}\}$. We have a Quadrant Stolarsky Principle:

Proposition (Bilyk, Matzke (2018))

For an N-point set $Z \subset \mathbb{S}^1$,

$$\left(D_{L^2,quad}(Z)\right)^2 = \int_{\mathbb{S}^1} \left| \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{Q(x)}(z_i) - \sigma\left(Q(x)\right) \right|^2 d\sigma(x)$$
$$= \frac{1}{4} - \frac{1}{\pi} \cdot \frac{1}{N^2} \sum_{i,j=1}^{N} \arccos |\langle z_i, z_j\rangle|.$$

p-Frame Potential

Definition

For $p \in (0, \infty)$, $\mu \in \mathfrak{B}(\mathbb{S}^d)$, *and* $Z \subseteq \mathbb{S}^d$, *we define the* ***p-frame potential*** *of* μ *as*

$$FP(\mu, p) = \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} |\langle x, y \rangle|^p d\mu(x) d\mu(y)$$

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Introduced in the real setting in '12 by Ehler and Okoudjou.

Some p-frame Potential Minimizers

Theorem (Ehler, Oukoudjou (2012))

Let $0 < p < 2$. Then μ is a minimizer of $FP(\mu, p)$ if and only if $\mu(\{e_j, -e_j\}) = \frac{1}{d+1}$ for $1 \leq j \leq d+1$, for some orthonormal basis $\{e_1, ..., e_{d+1}\}$ of $\mathbb{R}^{d+1}$.

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Theorem (Ehler, Okoudjou (2012))

Let p be an even integer. For any probability distribution μ on $\mathbb{S}^d$

$$FP(\mu, p) \geq \frac{1 \cdot 3 \cdot 5 \cdots (p-1)}{(d+1)(d+3)\cdots(d+p-1)},$$

with equality if and only if μ is a tight p-frame, i.e. there exists some constant $A > 0$ such that

$$A||y||^p = \int_{\mathbb{S}^d} |\langle x, y\rangle|^p d\mu(x), \quad \forall y \in \mathbb{R}^{d+1}.$$

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Definition

*A **spherical m-design** is a set of points $\{x_1, ..., x_N\} \subset \mathbb{S}^d$ such that*

$$\int_{\mathbb{S}^d} q(x) d\sigma(x) = \frac{1}{N} \sum_{i=1}^{N} q(x_i)$$

for all polynomials q on $\mathbb{R}^{d+1}$ of degree at most m.

Sharp designs and the 600-cell

Definition

A spherical $(2m+1)$-design is ***sharp*** *if there are $m+2$ inner products between its points.*

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- If C is a sharp, antipodal $2m+1$-design, and $p \in (2m-2, 2m)$, we can find a positive definite polynomial, q, of degree $2m$ that bounds $|t|^p$ from below, with equality at the inner products of C.

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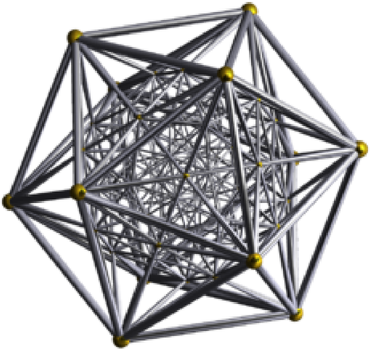
- If C is the 600-cell, and $p \in (8, 10)$, we can find a positive definite polynomial, q, of degree 18, with $\hat{q}(12; 1) = 0$, that bounds $|t|^p$ from below, with equality at the inner products of C.

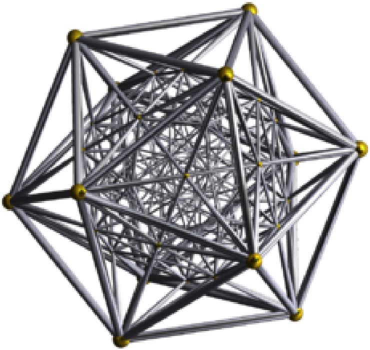
600-cell

- In $\mathbb{S}^3$



600-cell

- In $\mathbb{S}^3$
- 120 vertices, 600 tetrahedral cells



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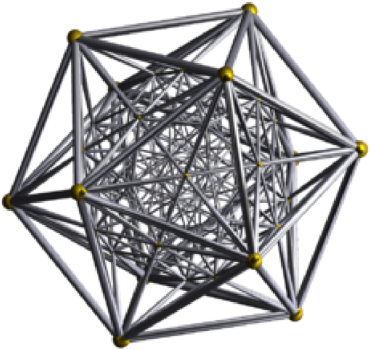
- Has finite reflection group H_4.



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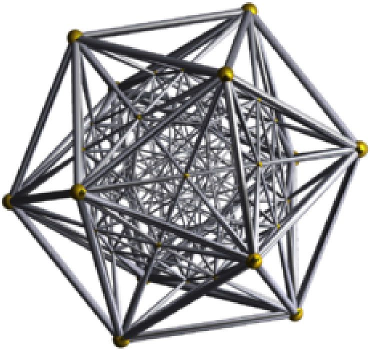
- Is exact for C_n^1, for $n \in \{1, ..., 11, 13, ...19\}$



600-cell

- In $\mathbb{S}^3$
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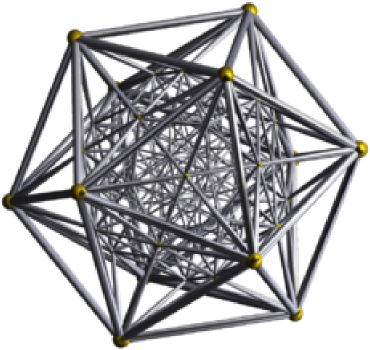
- Is exact for C_n^1, for $n \in \{1, ..., 11, 13, ...19\}$
- Has 9 inner products.



Sharp designs and the 600-cell

Theorem (Bilyk, Glazyrin, M, Park, Vlasiuk)

If C is a sharp antipodal design or the 600-cell and $p \in (2m-2, 2m)$, then $\mu = \frac{1}{|C|}\sum_{x\in C}\delta_x$ is a minimizer of $FP(\mu, p)$ on $\mathbb{S}^d$.

d	$\lvert C\rvert$	$2m+1$	Inner Products	Configuration
d	$2d+2$	3	$0, \pm 1$	cross polytope
1	$2k$	$2k-1$	$\cos(\frac{\pi j}{k})$ $(0 \le j \le k)$	$2k$-gon
2	12	5	$\pm\frac{1}{\sqrt{5}}, \pm 1$	icosahedron
3	120	11	$0, \pm\frac{1}{2}, \frac{\pm 1 \pm \sqrt{5}}{4}, \pm 1$	600-cell
7	240	7	$0, \pm\frac{1}{2}, \pm 1$	E_8 roots
6	56	5	$\pm\frac{1}{3}, \pm 1$	kissing configuration
23	196560	11	$0, \pm\frac{1}{4}, \pm\frac{1}{2}, \pm 1$	Leech lattice
22	4600	7	$0, \pm\frac{1}{3}, \pm 1$	kissing configuration
22	552	5	$\pm\frac{1}{5}, \pm 1$	equiangular lines

Possible minimizers

d	p	N	Configuration
2	$(6,8)$	32	Union of a regular icosahedron and its dual dodecahedron
2	$(8,10)$	50	An octohedral weighted 11-design (McLaren)
3	$(4,6)$	48	Union of dual 24-cells
4	$(2,4)$	32	all permutations of $\frac{1}{\sqrt{30}}(5,-1,-1,-1,-1,-1)$, $\frac{1}{\sqrt{30}}(-5,1,1,1,1,1)$, and $\frac{1}{\sqrt{6}}(1,1,1,-1,-1,-1)$
5	$(2,4)$	44	Union of a cross polytope and a hemicube contained by its dual cube
5	$(4,6)$	126	Union of minimal vectors of E_6 and E_6^*

Thank you!

[0]This work is in collaboration with Dmitriy Bilyk, Alexey Glazyrin, Josiah Park, and Alexander Vlasiuk, and was supported in part by NSF GRFP grant 00039202