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and Their Foundations in Number Theory

Workshop 3
Discrepancy

Book of Abstracts

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Equidistribution of the Riemann zeta zeros

Christoph Aistleitner
TU Graz

Abstract

A classical theorem of analytic number theory, which goes back to Rademacher, Elliot and Hlawka, asserts that the ordinates of the zeros of the Riemann zeta function are equidistributed modulo one. Since then several refinements have been obtained, including corresponding discrepancy estimates, an explanation for "Rademacher's phenomenon", and multi-dimensional results. We give a survey of known results and of some of the methods used. We also mention recent developments, open problems, and the connection with the "fine-scale" distribution of the zeros predicted by Montgomery's pair correlation conjecture.

The discrepancy of the linear flow on the torus

Bence Borda
Renyi Institute, Budapest

Abstract

In 1989 Drmota proved that there exist continuous curves in the plane whose mod 1 discrepancy is bounded. In fact, the orbits of the linear flow on the 2-dimensional torus enjoy this property provided the irrational direction of the flow satisfies certain Diophantine properties. We have recently been able to generalize this fact to arbitrary dimension if the direction of the flow has algebraic coordinates. In particular, this shows that there is no van Aardenne-Ehrenfest type theorem for the mod 1 discrepancy of continuous curves in any dimension. In addition, we showed that there is a large class of algebraic polytopes which are sets of bounded remainder with respect to the linear flow.

Results and problems old and new in discrepancy theory

William Chen
Macquarie University Sydney

Abstract

We give a brief survey on some of the main results and open problems in discrepancy theory, not just on the classical problem of aligned rectangular boxes but also on the far less studied questions involving other interesting geometric shapes.

The diamond ensemble

Ujué Etayo
Universidad de Cantabria

Abstract

Throughout this talk we will present two challenging problems on uniform distributions on the sphere $\mathbb{S}^2$. The first one verses on the asymptotic expansion of the minimal logarithmic energy on the sphere. The second one consists on finding constructive sets of points for which we can compute analytically the asymptotics of the logarithmic energy.

We present then the Diamond ensemble, a constructive family of points on the sphere $\mathbb{S}^2$ depending on several parameters. The most important property of the Diamond ensemble is that, for some of these parameters, the asymptotic expected value of the logarithmic energy of the points can be computed rigorously and shown to attain very small values, quite close to the conjectured minimal value.

Green and Riesz-s Energies on SO(3)

Damir Ferizović
TU Graz

Abstract

Using the theory of determinantal point processes we are able to give bounds for the Green Riesz-s energy for the rotation group SO(3), for s up to 3. While an algorithm to produce such points seems feasable, we have not pursued this direction, but rather investigated the properties of these points - and as such, this exposition can be used as a blueprint for other spaces. The Green function is computed explicitly, giving means to compare well known uniform point constructions with our bounds.

Joint work with Carlos Beltrán.

Probabilistic discrepancy bounds for Latin hypercube and other types of negatively correlated sampling

Michael Gnewuch
Kiel University

Abstract

Discrepancy measures are commonly used to quantify how uniformly a sample is distributed in the d-dimensional unit cube. Discrepancy theory is, e.g., intimately related to numerical integration and provides the theoretical foundation for quasi-Monte Carlo methods. A very important discrepancy measure is the star discrepancy.

In this talk we consider the star discrepancy, discuss its relation to numerical integration and the known upper and lower bounds for it. Here we focus on probabilistic bounds where the dependence on the sample size and the dimension d is explicitly stated. Such bounds are, e.g., important for high-dimensional and infinite-dimensional numerical integration.

We provide upper discrepancy bounds for Latin hypercube samples that match the probabilistic lower discrepancy bounds from the recent paper [1]. We discuss extensions of our result to other types of negatively correlated sampling.

References

[1] B. Doerr, C. Doerr, M. Gnewuch. Probabilistic lower discrepancy bounds for Latin hypercube samples. *To appear in: J. Dick, F. Y. Kuo, H. Woźniakowski (Eds.), Contemporary Computational Mathematics – a Celebration of the 80th Birthday of Ian Sloan, Springer-Verlag*, pp. 339–350, 2018.

Optimal order digital nets and sequences

Takashi Goda
University of Tokyo

Abstract

Quasi-Monte Carlo (QMC) integration using higher order digital nets and sequences as quadrature nodes have been proven to achieve the almost optimal rate of convergence for multivariate integration of smooth functions defined over the unit cube. Recently there are some attempts to improve the rate of convergence to be truly optimal for different function spaces. In this talk, we consider non-periodic reproducing kernel Sobolev spaces of arbitrary fixed smoothness $\alpha \in \mathbb{N}$, and prove that QMC integration using order $(2\alpha + 1)$ digital nets and sequences achieves the truly optimal rate of convergence. The key ingredient of the proof is to exploit both the decay and the sparsity of Walsh coefficients of the reproducing kernel simultaneously. This approach is in fact analogous to the one used by Dick and Pillichshammer who prove the optimal L_2-discrepancy bound for order 3 digital nets, which will be highlighted in this talk.

This is joint work with Kosuke Suzuki and Takehito Yoshiki.

Hyperuniformity on the sphere

Peter Grabner
TU Graz

Abstract

The concept of hyperuniformity had been introduced by S. Torquato and F. Stillinger to measure regularity of distributions of infinite particle systems in $\mathbb{R}^d$. An infinite particle system $X \subset \mathbb{R}^d$ is called hyperuniform, if the variance (with respect to the thermodynamic limit) of the number of points in a **large** ball is smaller than "usual":

$$\mathbb{V}\#(X \cap B(\mathbf{x}, R)) = \mathcal{O}(R^{d-1}) \text{ for } R \to \infty.$$

Notice that this variance is of order R^d for Poisson point processes.

We generalise this concept to the sphere and the torus by considering sequences of **finite** point sets $(X_N)_N$ (with $\#X_N = N$). The phenomenon of a "smaller than usual" variance of the point counting function is then be observed for geodesic balls with $N\mathrm{vol}(B_R) \to \infty$ but $\mathrm{vol}(B_R) \to 0$. We will discuss several examples of hyperuniform sequences of point sets.

Asymptotic behaviour of the Sudler product of sines for quadratic irrationals

Sigrid Grepstad
Norwegian University of Science and Technology

Abstract

We study the asymptotic behaviour of the sequence of sine products

$$P_n(\alpha) = \prod_{r=1}^{n} |2 \sin \pi r \alpha|$$

for real quadratic irrationals α. In particular, we study the subsequence P_{q_n}, where q_n is the nth best approximation denominator of α, and show that $(P_{q_n})_{n\geq 1}$ converges to a periodic sequence whose period equals that of the continued fraction expansion of α. This verifies a conjecture recently posed by Mestel and Verschueren in [1]. Moreover, it suggests that Lubinsky's conjecture [2] saying that

$$\liminf_{n\to\infty} P_n(\alpha) = 0$$

for all $\alpha \in \mathbb{R}$ might be false.

If time allows, we discuss possible extensions of our main result to reals α whose continued fraction expansions have certain repetitive structures.

This is joint work with Mario Neumüller.

References

[1] P. Verschueren and B. Mestel, *Growth of the Sudler product of sines at the golden rotation number*, J. Math. Anal. Appl. **433** (2016), 200–226.

[2] D. S Lubinsky, *The size of $(q;q)_n$ for q on the unit circle*, J. Number Theory **76** (1999), 217–247.

A survey of recent results on the dispersion of point sets

Aicke Hinrichs
JKU Linz

Abstract

The dispersion of a point set, which is the volume of the largest axis-parallel box in the unit cube that does not intersect the point set, is an alternative to the discrepancy as a measure for certain (uniform) distribution properties. The computation of the dispersion, or even the best possible dispersion, in dimension two has a long history in computational geometry and computational complexity theory. Given the prominence of the problem, it is quite surprising that, until recently, very little was known about the size of the largest empty box in higher dimensions. In this talk we will give a survey about recent developments.

On bounded remainder intervals for sequences $(\{a_n\alpha\})_{n\geq 1}$

Lisa Kaltenböck
JKU Linz

Abstract

Let $(x_n)_{n\geq 1}$ be an arbitrary sequence in $[0,1)$. An interval $[a,b)\subseteq[0,1)$ is called bounded remainder interval (BRI) for $(x_n)_{n\geq 1}$ if

$$|\#\{1\leq n\leq N : x_n\in[a,b)\}-(b-a)N|\leq c,$$

for all natural numbers N with a constant c independent of N. We give some results on the existence of BRI for sequences of the form $(\{a_n\alpha\})_{n\geq 1}$, where $(a_n)_{n\geq 1}$ - in most cases - is a given sequence of distinct integers. Further we introduce the concept of strongly non-bounded remainder intervals (S-NBRI) and we show for a very general class of polynomial-type sequences that these sequences cannot have any S-NBRI, whereas for the sequence $(\{2^n\alpha\})_{n\geq 1}$ every interval is an S-NBRI.

L_p *discrepancy of digital nets with non-singular upper-triangle (NUT) generating matrices*

Ralph Kritzinger
JKU Linz

Abstract

We study the discrepancy function of digital $(0,m,2)$-nets and digital $(0,1)$-sequences with the aid of Haar functions.

- We consider nets which are determined by a nonsingular-upper-triangle (NUT) $(m\times m)$-matrix. It is known by a result of Larcher and Pillichshammer in 2001 that the L_2 discrepancy of symmetrized versions of such nets is always of optimal order $\sqrt{\log N}/N$, where N is the number of elements of the point set. We ask whether the symmetrization technique is necessary in order to obtain the optimal L_2 discrepancy rate and determine NUT-matrices, for which this is not the case.
- Then we consider infinite $(0,1)$-sequences generated by an infinite $(\mathbb{N}\times\mathbb{N})$-NUT-matrix. We conjecture that for such sequences the L_2 discrepancy exceeds $(\log N)/N$ for a constant $c>0$ and infinitely many N, which is not optimal. We show this conjecture for special instances of NUT-matrices.

Our results also hold for the L_p discrepancy for all $p\in[1,\infty)$.

This talk is based on a joint work with Friedrich Pillichshammer.

On pair correlation of sequences

Gerhard Larcher
JKU Linz

Abstract

The introduction and the analysis of the concept of pair correlation of sequences in the unit-interval was motivated by certain conjectures in quantum physics, and was started – from the mathematical point of view – in several papers by Rudnick, Sarnak and Zaharescu in the 1990's. The pair correlation of a sequence in some sense measures the distribution of "small distances" of points of the sequence.
In the last years there grew new interest in this topic by several research papers in which a strong connection of the concept of pair correlation to questions and concepts from additive combinatorics and from uniform distribution theory was discovered.
In this talk we give a survey on the concept, on basic and on new properties and results, and we state several open problems. Further we give a multi-dimensional concept of pair correlation and its basic properties.

Energy optimization with orthogonal potentials on the sphere

Ryan W Matzke
University of Minnesota - Twin Cities

Abstract

The majority of energy minimization problems on the sphere involve potentials $P : \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{R}$ such that $P(x, y)$ is maximized if $x = y$ and is minimized if $x = -y$. One often expects σ, the uniform measure on the sphere, or similar "nicely" distributed measures, to provide minimizers for the energy

$$I_P(\mu) = \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} P(x, y) d\mu(x) d\mu(y).$$

What if instead we had a potential that was minimized when $\langle x, y\rangle = 0$, and was maximized when $x = \pm y$? The frame potential, introduced by J. Benedetto and M. Fickus, and the more general p-frame potential, introduced by M. Ehler and K. Okoudjou, are instances of such potentials. In such cases, it more often seems to be the case that minimizers of the energy are measures that avoid being "too" uniformly distributed.

Asymptotic behaviour of the Sudler product of sines for quadratic irrationals

Mario Neumüller
JKU Linz

Abstract

We study the growth of the following sequence $P_n(\alpha) = \prod_{r=1}^{n} |2 \sin(\pi r \alpha)|$ for real, irrational numbers α. More precise, we study the case where α is a quadratic irrational i.e. the continued fraction expansion of α is of the form $\alpha = [a_0; a_1, \ldots, a_h, \overline{a_{h+1}, \ldots, a_{h+l}}]$.
Similar products appear in many different fields of mathematics such as partition theory, Padé approximation, dynamical systems and recently also in connection with the analysis of the discrepancy of certain hybrid sequences.
In particular, we study the subsequence $(Q_n(\alpha))_{n\geq 1}$ of $(P_n(\alpha))_{n\geq 1}$ defined as

$$Q_n(\alpha) := \prod_{r=1}^{q_n} |2 \sin(\pi r \alpha)| .$$

Here $(q_n)_{n\geq 0}$ are the best approximation denominators of α. We show that this subsequence converges to a periodic sequence with period equal to that of the continued fraction expansion of α. This verifies a recently posed conjecture by Mestel and Verschueren in 2016.

Joint work with Sigrid Grepstad.

Tusnády's problem and the discrepancy of boxes
Aleksandar Nikolov
University of Toronto

Abstract

Tusnády's problem is a classical question in combinatorial discrepancy theory, which asks for the smallest number $\Delta_d(n)$ for which any set P of n points in d-dimensional space can be colored with -1 and $+1$ so that the sum of colors in any axis-aligned box is at most $\Delta_d(n)$ in absolute value. We show that $\Delta_d(n) = O((\log n)^{d-0.5})$, which is only a $O(\sqrt{\log n})$ factor away from the best known lower bound.

As an application of this combinatorial result, we show an upper bound on the geometric discrepancy of axis-aligned boxes with respect to an arbitrary Borel probability measure μ on $[0,1]^d$. In particular, we show that for any n there exists an n-point set P in $[0,1]^d$ so that for any axis-aligned box B the discrepancy $\left|\frac{|P \cap B|}{n} - \mu(B)\right|$ is bounded by $O((\log n)^{d-0.5}/n)$. This bound is only an $O(\sqrt{\log n})$ factor away from the best known upper bound when μ is the Lebesgue measure.

Part of the talk is based on joint work with Christoph Aistleitner and Dmitriy Bilyk.

Stolarsky's invariance principle for projective spaces
Maxim M. Skriganov
St. Petersburg Departement of Steklov Mathematical Institute

Abstract

We show that Stolarsky's invariance principle, known for point distributions on the Euclidean spheres, can be extended to the real, complex, and quaternionic projective spaces and the octonionic projective plane.

This talk is based on the recent publication arXiv:1805.03541

Numerical integration errors, probabilistic and deterministic point sets on sphere
Tania Stepaniuk
TU Graz

Abstract

For the classical Sobolev spaces $\mathbb{H}^s(\mathbb{S}^d)$ $(s > \frac{d}{2})$ upper and lower bounds for the worst case integration error of numerical integration on the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$, $d \geq 2$, have been obtained by Brauchart, Hesse and Sloan. We investigate the case when $s \to \frac{d}{2}$ and introduce the spaces $\mathbb{H}^{\frac{d}{2},\gamma}(\mathbb{S}^d)$ of continuous functions on $\mathbb{S}^d$ with an extra logarithmic weight. For these spaces we obtain estimates for the worst case integration error.

We make a comparison between certain probabilistic and deterministic point sets and show that some deterministic constructions (spherical t-designs, minimizing point-sets) are better or as good as probabilistic ones. In particular the asymptotic equalities for the discrete Riesz s-energy of N-point sequence of well separated t-designs on the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$, $d \geq 2$ are found.

Characterization of digital $(0, m, 3)$-nets and $(0, 2)$-sequences in base 2

Kosuke Suzuki
Hiroshima University

Abstract

The notions of a (t, m, s)-net and a (t, s)-sequence in base b represent how uniformly distributed the points from the net or the sequence are in the b-adic elementary intervals in $[0, 1]^s$. A powerful method to construct (t, m, s)-nets and (t, s)-sequences is the digital construction. Our purpose is to characterize digital nets and sequences with best possible quality parameter t. We give a characterization of all $m \times m$ generating matrices (A, B, C) which generate a digital $(0, m, 3)$-net in base 2 and all infinite matrices (B, C) which generate a digital $(0, 2)$-sequence in base 2. This talk is based on a joint work with Roswitha Hofer and one with Makoto Matsumoto and Hiroki Kajiura.

Pseudorandomness and discrepancy

Robert Tichy
TU Graz

Abstract

We follow a suggestion of Donald Knuth who introduced various statistical tests for pseudorandomness of sequences. These tests are based on distribuion properties such as normality and discrepancy. We discuss computational and probabilistic properties of normal numbers and limit laws for lacunary and super-lacunary sequences. The focus lies on laws of the iterated logarithm for discrepancy and permutation invariance of these laws.

Integer points in convex bodies with isolated flat points

Giancarlo Travaglini
Università di Milano-Bicocca

Abstract

Let $B \subset \mathbb{R}^d$ be a convex body. We study different norms for the discrepancy between the volume of a large dilated copy of B and the number of integer points therein. We focus on convex bodies having isolated (possibly non-smooth) flat points on their boundaries. These bodies are, in a sense, intermediate between polyhedra and the case where ∂B is smooth with everywhere positive Gaussian curvature.

The proofs depend on suitable estimates for the decay of the Fourier transform of the characteristic function χ_B of B.

(Joint work with L. Brandolini, L. Colzani, B. Gariboldi, G. Gigante)

Deterministic constructions of point sets with small dispersion

Mario Ullrich
FSU Jena

Abstract

Based on deep results from coding theory, we present an deterministic algorithm that constructs a point set with dispersion at most ϵ in dimension d of size $poly(1/\epsilon) \log(d)$, which is optimal with respect to the dependence on d. The running time of the algorithms is, although super-exponential in $1/\epsilon$, only logarithmically in d.

Γ-convergence of hypersingular Riesz energies and its applications

Oleksandr Vlasiuk
Florida State University

Abstract

We characterize asymptotic behavior of a class of Riesz energy functionals in terms of Γ-convergence. As a consequence, a unified expression for the limiting convex functional on probability measures is obtained, giving a simple proof of the uniqueness of the weak* limit of empirical measures corresponding to energy minimizers.

The applications of this property to discretization of continuous distributions are also discussed. The talk is based on a joint work in progress with Douglas Hardin and Edward Saff.

Volumes of convex and non-convex bodies

Jan Vybiral
Czech Technical University

Abstract

Estimates of volumes of bodies in finite-dimensional spaces are known to play an important role in high-dimensional geometry, entropy, approximation theory, and signal processing. We review some recent results (partially still in progress) on volumes of unit balls of Lebesgue, mixed Lebesgue, and Lorentz spaces.

Interval exchange transformation as source for low-discrepancy sequences

Christian Weiss
Hochschule Ruhr West University of Applied Sciences

Abstract

Kronecker sequences build an important class of one dimensional low-discrepancy sequences. At the the same time they can be realized as orbits of circle rotations, which are in turn the simplest class of interval exchange transformations (IETs). It is natural to ask whether there exist more general IETs which also yield low-discrepancy sequences. In the case of n=3 intervals, the criteria for circle rotations can easily be carried over. The dynamics of IETs with more than three intervals is much more complex. Although, some abstract results are known if IETs are treated as systems of rank one, these can hardly be applied in practice and to our best knowledge did so far not allow any concrete examples but circle rotations. In this talk, it is shown how to construct examples of IETs with an arbitrary number of intervals whose orbits are indeed low-discrepancy sequences. The construction does not involve systems of rank one.

Negative dependence in numerical integration and discrepancy

Marcin Wnuk
Kiel University

Abstract

Let $s \in \mathbb{N}$. We say that a randomized point set $\mathcal{P} \subset [0,1)^s$ is *pairwise negative dependent* if for every $p_1, p_2 \in \mathcal{P}, p_1 \neq p_2$ the point p_1 is uniformly distributed in $[0,1)^s$ and for every $q, r \in [0,1)^s$ if we put $Q = [0,q), R = [0,r)$ we have

$$\mathbb{P}(p_1 \in Q, p_2 \in R) \leq \mathbb{P}(p_1 \in Q)\mathbb{P}(p_2 \in R).$$

In a recent paper C. Lemieux showed how quadratures based on pairwise negative point sets lead to variance reduction. We show that appropriately randomized Rank-1-Lattice Rules are pairwise negatively dependent.

More involved notions of negative dependence had also been studied by M. Gnewuch and N. Hebbinghaus. We give some examples of negatively dependent point sets and talk about their relevance for discrepancy theory.

Metric discrepancy with respect to fractal measures

Agamemnon Zafeiropoulos
TU Graz

Abstract

Let $(n_k)_{k=1}^{\infty}$ be a lacunary sequence of integers. In joint work with N. Technau (University of York), we show that if μ is a probability measure on $[0,1)$ such that $|\widehat{\mu}(t)| \leq c|t|^{-\eta}$, then for μ-almost all x, the discrepancy $D_N(n_k x)$ satisfies

$$\frac{1}{4} \leq \limsup_{N\to\infty} \frac{N D_N(n_k x)}{\sqrt{N \log\log N}} \leq C,$$

for some constant $C > 0$ depending on the choice of the sequence. This enables us to improve a previous result of Haynes, Jensen, Kristensen on products of the form $q\|q\alpha\|\|q\beta - \gamma\|$.