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Absolute Value Information from IBC perspective

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Solution operator:

$$S : F \to G,$$

where F linear space, G normed space with $\|\cdot\|$.

Approximation:

$$S(f) \sim A_n(f) = \varphi(\mathbf{y})$$

where $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ is *information* about f,

$$\begin{aligned} y_i &= L_i(f) && \text{(nonadaptive)} \\ y_i &= L_i(f; y_1, \ldots, y_{i-1}) && \text{(adaptive)} \end{aligned}$$

$L_i(\cdot; y_1, \ldots, y_{i-1}) \in \Lambda$ a class of functionals on F.

Classes of information

Classes Λ of information in IBC:

- Λ^{all} all linear functionals,
- Λ^{std} function values only.

Different class in *phase retrieval*:

- $|\Lambda| = \{\, |L| : \ L \in \Lambda \,\}$ for given Λ.

Information $|\Lambda|$ was used in exact recovery in Hilbert spaces, up to the phase shift, e.g., Cahill, Casazza, Daubechies (2016).

(Applications in signal reconstruction, audio processing...)

Algorithm errors

For a set $\mathcal{F} \subset F$ of problem instances,

$$e(A_n) = \sup_{f \in \mathcal{F}} d\big(S(f), A_n(f)\big).$$

In standard IBC:

$$d^{\mathrm{std}}(g_1, g_2) = \|g_1 - g_2\|.$$

In phase retrieval:

$$d^{\mathrm{mod}}(g_1, g_2) = \inf_{|z|=1} \|g_1 - z\, g_2\|.$$

Information ε-complexity

We want to compare the powers of

$$\Lambda \quad \text{and} \quad |\Lambda|$$

in terms of information ε-complexities:

$$n^{\text{std}}(S, \Lambda; \varepsilon)$$
$$= \min \left\{ n : \text{ there is } A_n \text{ using } \Lambda \text{ s.t. } e^{\text{std}}(A_n) \leq \varepsilon \right\},$$

$$n^{\text{mod}}(S, |\Lambda|; \varepsilon)$$
$$= \min \left\{ n : \text{ there is } A_n \text{ using } |\Lambda| \text{ s.t. } e^{\text{mod}}(A_n) \leq \varepsilon \right\}.$$

Result for $\Lambda = \Lambda^{\mathrm{all}}$ (positive)

Theorem

Let

- *the solution operator $S : F \to G$ be linear, and*
- *the class $\mathcal{F} \subset F$ be convex and balanced.*

Then

$$n^{\mathrm{std}}\big(S, \Lambda^{\mathrm{all}}, 4\varepsilon\big) \;\le\; n^{\mathrm{mod}}\big(S, |\Lambda^{\mathrm{all}}|, \varepsilon\big) \;\le\; 3\, n^{\mathrm{std}}\big(S, \Lambda, \tfrac{1}{2}\varepsilon\big).$$

Hence, $|\Lambda^{\mathrm{all}}|$ and Λ^{all} are roughly of the same power.

Remark

The theorem holds for $\Lambda \subseteq \Lambda^{\mathrm{all}}$ satisfying the following.

If $L_1, L_2 \in \Lambda$ then

- in real case: $L_1 + L_2 \in \Lambda$,
- in complex case: $L_1 + L_2 \in \Lambda$, $L_1 + \mathrm{i}L_2 \in \Lambda$.

Observe that this holds for Λ^{all}, but not for Λ^{std}.

Result for $\Lambda = \Lambda^{\text{std}}$ (negative)

Theorem

Let

- *F be a linear space of (real or complex valued) functions,*
- *the class $\mathcal{F} \subset F$ be convex and balanced,*
- *the solution operator $S : F \to G$ be linear.*

Suppose there are two functions $f_1, f_2 \in \mathcal{F}$ such that

$$f_1, f_2 \notin \ker S \qquad \textit{and} \qquad f_1 * f_2 = 0.$$

Then there is $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$

$$n^{\text{mod}}\left(S, |\Lambda^{\text{std}}|; \varepsilon\right) = +\infty.$$

Recovery of polynomials

Let $\mathcal{F} = F = \mathcal{P}_k$ be real (algebraic) polynomials f on $\mathbb{R}$ with

$$\deg f \leq k - 1.$$

The problem of exact recovery of $f \in \mathcal{P}_k$ can be solved using:

- k evaluations of f for Λ^{std},
- $2k - 1$ evaluations of f for $|\Lambda^{\mathrm{std}}|$.

Note that the assumptions of the last theorem are not satisfied, since $f_1 * f_2 \neq 0$ whenever $f_1, f_2 \neq 0$.

Another way...

For

$$S : F \to G$$

and $\mathcal{F} \subset F$, the problem can be re-defined as recovery of the multi-valued mapping

$$\mathbf{S} : \mathcal{F} \to 2^G$$

given by

$$\mathbf{S}(f) = \left\{ S(f_1) : \ f_1 \in \mathcal{F}, \ |L(f_1)| = |L(f)| \text{ for all } L \in \Lambda \right\}.$$

Algorithm $\mathbf{A}_{n,m}$ using n functionals from $|\Lambda|$ returns subsets of G of cardinality m,

Another way...

Algorithm error:

$$e^H(\mathbf{A}_{n,m}) = \sup_{f\in\mathcal{F}} d^H\big(\mathbf{S}(f), \mathbf{A}_{n,m}(f)\big)$$

where d^H is the *Hausdorff distance*,

$$d^H(W,Z) = \max\left\{ \sup_{w\in W} \inf_{z\in Z} \|w-z\|, \sup_{z\in Z} \inf_{w\in W} \|z-w\| \right\}.$$

ε-complexity:

$$n^H(S, |\Lambda|; \varepsilon)$$
$$= \min\left\{n+m : \text{ there is } \mathbf{A}_{n,m} \text{ using } |\Lambda| \text{ s.t. } e^H(A_{n,m}) \leq \varepsilon\right\}.$$

IBC for approximation of multi-valued operators?