Effective fractional wave equations in random multiscale media

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- Wave equation in random media with long- or short-range correlations.
- Explain the apparent attenuation observed in seismic wave propagation.

Geophysics

• Experimental observations for body waves in the earth's crust:
the seismic attenuation factor Q^{-1} defined through the transmission coefficient by

$$|T_L(\omega)|^2 \simeq \exp\Big(-\frac{|\omega|}{c_o} Q^{-1}(\omega) L\Big)$$

has a frequency dependence of the form

$$Q^{-1}(\omega) \approx |\omega|^\alpha$$

α is obtained through a fitting of measured data:
- $\alpha \simeq 0.4$ for low frequencies $< 10^{-3}$Hz [Lekic 2009],
- $\alpha \in (-0.4, 0)$ for mid frequencies $(10^{-3}, 1)$Hz [Choy 1986, Shito 2004],
- $\alpha \simeq -1$ for high frequencies > 1Hz [Choy 1986, Cormier 2011].

• Two possible mechanisms: scattering and intrinsic attenuation (due to friction, viscosity, ...) [Knopoff 1964, Jackson 1970, ..., Ricard 2014 Is there seismic attenuation in the mantle?, ...].

Scalar wave equation in one-dimensional random medium

Scalar wave equation:

$$\frac{\partial^2 p^\varepsilon}{\partial z^2} - \frac{1}{c^\varepsilon(z)^2}\frac{\partial^2 p^\varepsilon}{\partial t^2} = 0$$

A section of random medium is sandwiched in between two homogeneous half-spaces:

$$\frac{1}{c^\varepsilon(z)^2} = \begin{cases} \frac{1}{c_o^2}\Big(1 + \varepsilon \nu\big(\frac{z}{\varepsilon^2}\big)\Big) & \text{for } \ z \in [0, L] \\ \frac{1}{c_o^2} & \text{for } \ z \in (-\infty, 0) \cup (L, \infty) \end{cases}$$

The background velocity is c_o.

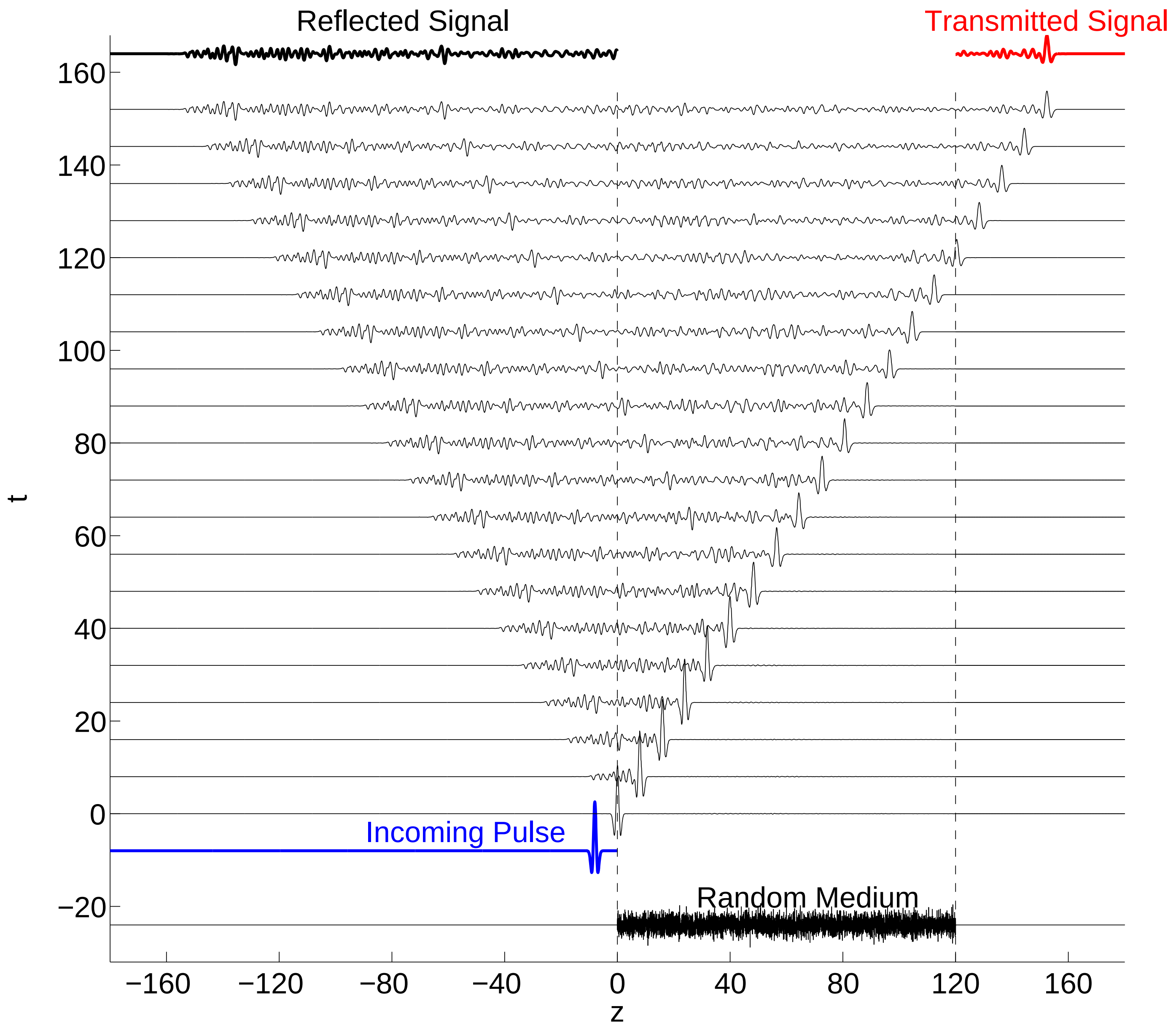
$\nu(z)$ is a zero-mean, stationary, random process with covariance function:

$$\phi(z) := \mathbb{E}[\nu(y)\nu(y+z)]$$

Pulse wave incoming from the left with pulse profile

$$p^\varepsilon(t, z) = p_0\big(\frac{t - z/c_o}{\varepsilon^2}\big), \quad t \ll 0$$

Numerical experiment in one-dimensional random medium



The mixing case

• $\phi(z) = \mathbb{E}[\nu(y)\nu(y+z)]$ decays fast enough at infinity (integrable) and is regular at zero (Lipschitz).

$\hookrightarrow$ The correlation length ℓ_c can be defined by

$$\ell_c = \frac{1}{\phi(0)} \int_0^\infty \phi(z) dz$$

$\hookrightarrow$ The covariance function can be expanded as

$$\phi(z) = \phi(0)\Big(1 - d_c \frac{|z|}{\ell_c} + o\Big(\frac{|z|}{\ell_c}\Big)\Big), \qquad |z| \ll \ell_c$$

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• Example: Binary medium. The process ν is stepwise constant.

$(l_j)_{j\geq 0}$: lengths of the elementary intervals.

$(n_j)_{j\geq 0}$: values $\in \{-\sigma, +\sigma\}$ taken by the process over each elementary interval.

The values n_j are independent and identically distributed (i.i.d.) with the distribution

$$\mathbb{P}(n_j = \pm\sigma) = \frac{1}{2}$$

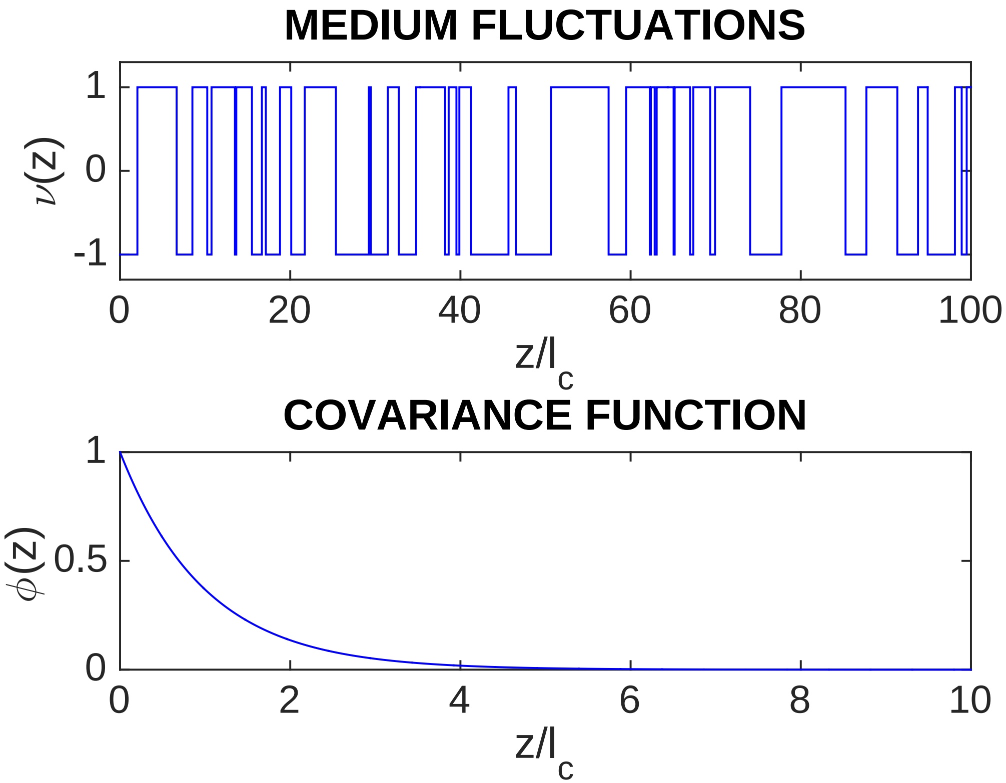
The lengths l_j are i.i.d. with the probability density function (pdf):

$$p_{l_1}(z) = \frac{1}{\ell_c} \exp\Big(-\frac{z}{\ell_c}\Big) \mathbf{1}_{[0,\infty)}(z)$$

$\hookrightarrow$ The covariance function is

$$\phi(z) = \sigma^2 \exp\Big(-\frac{|z|}{\ell_c}\Big)$$

Binary medium - mixing case



Realizations of a binary medium with exponentially distributed intervals.

Effective pulse propagation - mixing case (1/3)

Introduce the random travel time

$$\tau^{\varepsilon}(L) = \frac{L}{c_o} + \varepsilon^2 \tau_c^{\varepsilon}(L), \qquad \tau_c^{\varepsilon}(L) = \frac{1}{2c_o\varepsilon}\int_0^L \nu\left(\frac{z}{\varepsilon^2}\right) dz$$

In the limit $\varepsilon \to 0$, at $z = L$,

1) the *random time shift* $\tau_c^{\varepsilon}(L)$ converges in distribution to a Gaussian random variable $\tau_c(L)$ ($\sim$ Brownian motion) with mean zero and variance:

$$\mathbb{E}\big[\tau_c(L)^2\big] = \frac{\hat{\phi}_c(0)\ell_c}{8c_o^2} L$$

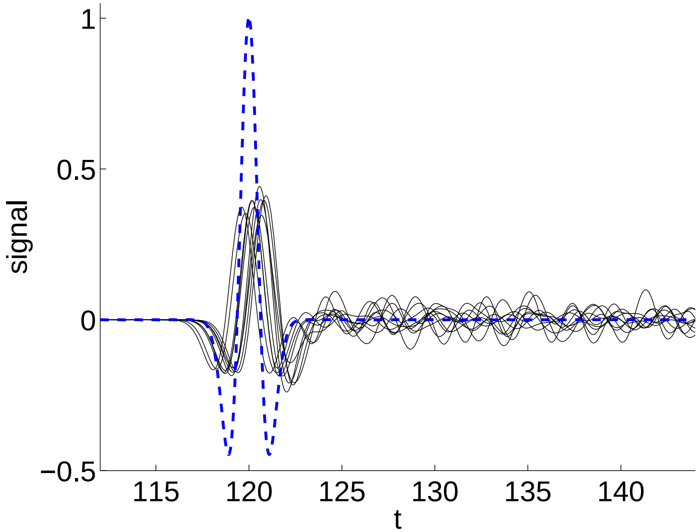
2) the pulse profile converges in probability to a *deterministic profile*:

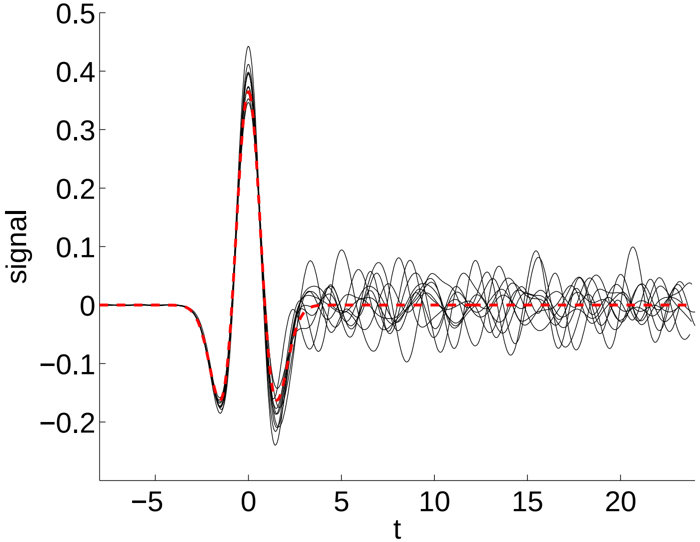
$$p^{\varepsilon}\big(z = L, t = \tau^{\varepsilon}(L) + \varepsilon^2 \tau\big) \stackrel{\varepsilon \to 0}{\longrightarrow} p_L(\tau)$$

$$p_L(\tau) = \frac{1}{2\pi}\int \hat{p}_0(\omega) T_L(\omega) e^{-i\omega\tau} d\omega, \qquad T_L(\omega) = \exp\left(-\frac{\omega^2[\hat{\phi}_c(\omega) + i\hat{\phi}_s(\omega)]}{8c_o^2} L\right)$$

$$\hat{\phi}_c(\omega) := 2\int_0^{\infty} \phi(z)\cos\left(\frac{2\omega z}{c_o}\right) dz, \qquad \hat{\phi}_s(\omega) := 2\int_0^{\infty} \phi(z)\sin\left(\frac{2\omega z}{c_o}\right) dz$$

Effective pulse propagation - mixing case (2/3)





Time profiles of the transmitted wave at $z = L$

Without time shift

Black: ten numerical results obtained with ten realizations of the random medium

Blue: original pulse $p_0(t)$

With time shift

Red: theoretical (deterministic) transmitted pulse shape $p_L(t)$

Effective pulse propagation - mixing case (3/3)

- Remark 1: Known as O'Doherty-Anstey theory in geophysics (70's). Proved in mathematics by Papanicolaou and co (90's) [1].

- Remark 2: Convergence of p^ε towards p_L in the space of continuous functions with the sup norm (not in L^2).

- If $\frac{\omega \ell_c}{c_o} \ll 1$ (the wavelength $\gg \ell_c$ probes the tail of $\phi(z)$), then:

$$T_L(\omega) = \exp\Big(- \frac{\hat{\phi}_c(0)}{8} \frac{\omega^2 \ell_c^2}{c_o^2} \frac{L}{\ell_c} \Big)$$

$\hookrightarrow$ Effective second-order diffusion (attenuation), no effective dispersion.
Here $\alpha = 1$.

- If $\frac{\omega \ell_c}{c_o} \gg 1$ (the wavelength $\ll \ell_c$ probes the small-z behavior of $\phi(z)$), then:

$$T_L(\omega) = \exp\Big(- \frac{\phi(0) d_c}{16} \frac{L}{\ell_c} - i \frac{\phi(0)}{8} \frac{\omega \ell_c}{c_o} \frac{L}{\ell_c} \Big)$$

$\hookrightarrow$ Effective constant attenuation, no effective dispersion, additional time shift.
Here $\alpha = -1$.

[1] J.-P. Fouque, J. Garnier, G. Papanicolaou, and K. Sølna, Springer, 2007.

Effective pulse propagation - mixing case (3/3)

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Here $\alpha = 1$.

- Remark 3: What happens if $\int_0^\infty \phi(z) dz = \infty$?

- If $\frac{\omega \ell_c}{c_o} \gg 1$ (the wavelength $\ll \ell_c$ probes the small-z behavior of $\phi(z)$), then:

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Here $\alpha = -1$.

- Remark 4: What happens if $\phi(z)$ is not smooth at zero ?

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Long-range and short-range correlations

- Long-range correlations: ϕ is not integrable and has a power decay at infinity:

$$\phi(z) \overset{|z|\to\infty}{\simeq} r_{H'} \left| \frac{z}{\ell_c} \right|^{2H'-2}$$

where $r_{H'} > 0$ and $H' \in (1/2, 1)$ (i.e., $2H' - 2 \in (-1, 0)$).
ℓ_c is the critical length scale beyond which the power law behavior is valid.

- Short-range correlations: ϕ is not smooth at zero:

$$\phi(z) \overset{|z|\to 0}{\simeq} \phi(0)\Big(1 - d_H \left| \frac{z}{\ell_c} \right|^{2H} + O\Big(\left| \frac{z}{\ell_c} \right|\Big)\Big)$$

where $d_H > 0$ and $H \in (0, 1/2)$ (i.e., $2H \in (0, 1)$).
ℓ_c is the critical length scale below which the expansion is valid.

Continuous model 0: Fractional Brownian motion

- Fractional Brownian motion with Hurst index $H \in (0, 1)$:

$$W_H(z)$$

- Gaussian process, with zero mean and covariance:

$$\phi(z) = \mathbb{E}[W_H(y)W_H(y+z)] = \frac{1}{2}\left(|y+z|^{2H} + |y|^{2H} - |z|^{2H}\right)$$

- Stationary increments:

$$\mathbb{E}\left[(W_H(y+z) - W_H(y))^2\right] = |z|^{2H}$$

$H = 1/2$: standard Brownian motion (independent increments).
$H < 1/2$: short-range correlations (negatively-correlated increments). The realizations are continuous but irregular.
$H > 1/2$: long-range correlations (positively-correlated increments). The realizations are continuous and more regular (but not differentiable).

However, the process itself is <u>not stationary</u> !

Continuous model 1: Fractional Ornstein Uhlenbeck (fOU) process

$$\nu(z) := \frac{\sigma}{\sqrt{H\Gamma(2H)\ell_c^H}}\Big[W_H(z) - \frac{1}{\ell_c}\int_{-\infty}^{z} e^{\frac{y-z}{\ell_c}} W_H(y)dy\Big]$$

where W_H is a fBm with Hurst index $H \in (0,1)$.

- The fOU process is a zero-mean, variance σ^2, stationary, Gaussian process and its covariance function is

$$\phi(z) = \frac{\sigma^2}{H\Gamma(2H)\ell_c^{2H}}\Big[\frac{1}{4\ell_c}\int_{-\infty}^{\infty} e^{-\frac{|y|}{\ell_c}}|z+y|^{2H}dy - \frac{1}{2}|z|^{2H}\Big]$$

- If $H \in (1/2,1)$, then the large-z behavior of the covariance function is

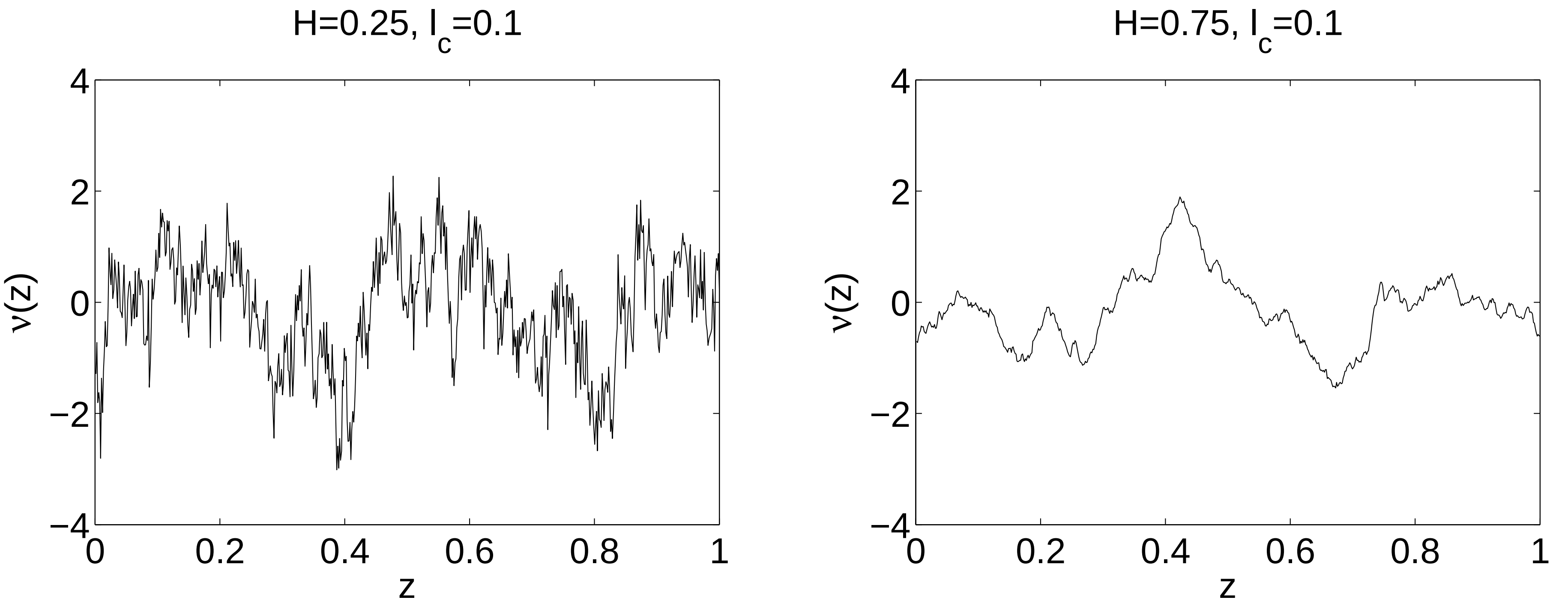
$$\phi(z) \overset{|z|\to\infty}{\simeq} r_H\Big|\frac{z}{\ell_c}\Big|^{2H-2}, \text{ with } r_H = \frac{\sigma^2(2H-1)}{\Gamma(2H)}$$

↪ the fOU has the long-range correlation property.

- If $H \in (0,1/2)$, then the small-z behavior of the covariance function is

$$\phi(z) \overset{|z|\to 0}{\simeq} \sigma^2\Big(1 - d_H\Big|\frac{z}{\ell_c}\Big|^{2H} + O\big(\Big|\frac{z}{\ell_c}\Big|\big)\Big), \text{ with } d_H = 1$$

↪ the fOU has the short-range correlation property.



Realizations of the fOU process with Hurst index H and correlation length ℓ_c.

Binary medium - long-range correlations

The process ν is stepwise constant.

$(l_j)_{j\geq 0}$: lengths of the elementary intervals.

$(n_j)_{j\geq 0}$: values $\in \{-\sigma, +\sigma\}$ taken by the process over each elementary interval.

The values n_j are i.i.d. with the distribution

$$\mathbb{P}(n_j = \pm\sigma) = \frac{1}{2}$$

The lengths l_j are i.i.d. with the pdf $(H' \in (1/2, 1))$:

$$p_{l_1}(z) = \frac{3-2H'}{\ell_{\rm c}} \frac{\ell_{\rm c}^{4-2H'}}{z^{4-2H'}} \mathbf{1}_{[\ell_{\rm c},\infty)}(z)$$

Note: The average length of the intervals is $\frac{3-2H'}{2-2H'}\ell_{\rm c}$ while the variance is infinite.

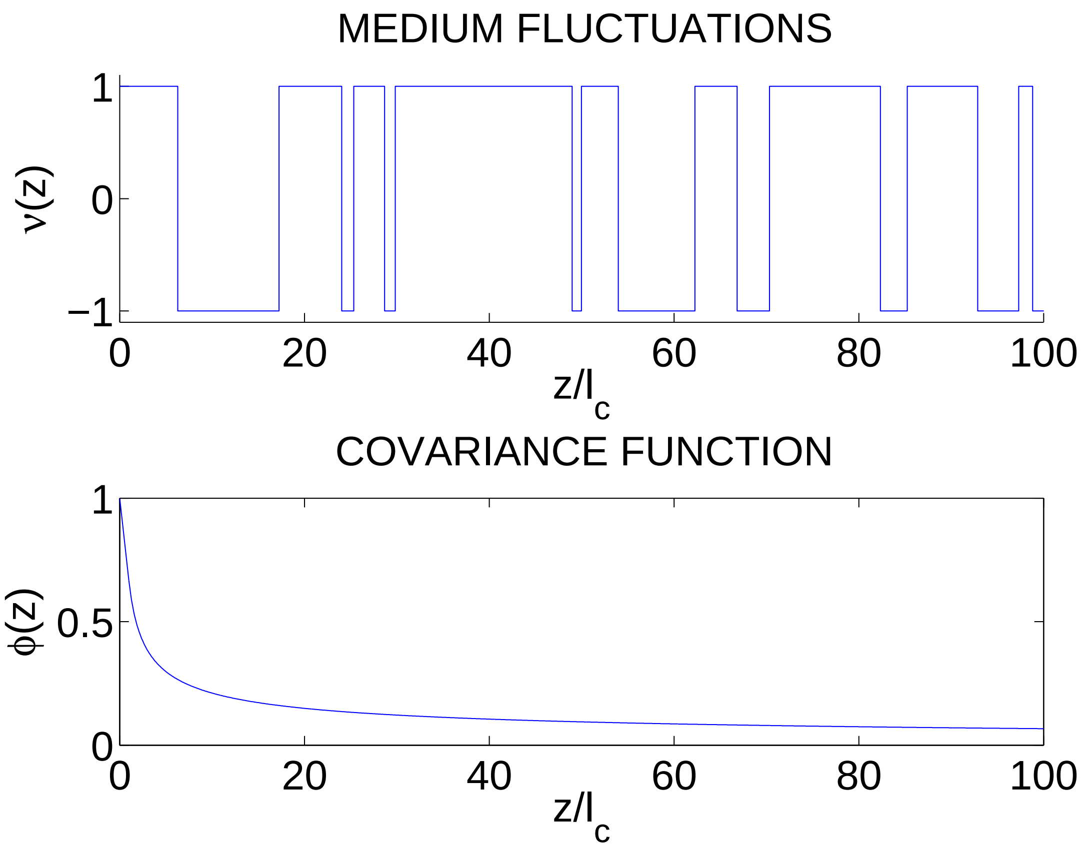
The covariance function is

$$\phi(z) = \sigma^2 \left[\frac{1}{3-2H'} \frac{\ell_{\rm c}^{2-2H'}}{|z|^{2-2H'}} \mathbf{1}_{[\ell_{\rm c},\infty)}(|z|) + \left(1 - \frac{2-2H'}{3-2H'} \frac{|z|}{\ell_{\rm c}}\right) \mathbf{1}_{[0,\ell_{\rm c})}(|z|) \right]$$

We have

$$\phi(z) \overset{|z|\to\infty}{\simeq} r_{H'} \Big| \frac{z}{\ell_{\rm c}} \Big|^{2H'-2}, \text{ with } r_{H'} = \frac{\sigma^2}{3-2H'}$$

Binary medium - long-range correlations



Realizations of a binary medium with the index $H' = 0.75$.

Generation of intervals longer than the average responsible for the long-range correlation property.

Binary medium - short-range correlations

The process ν is stepwise constant.

$(l_j)_{j\geq 0}$: lengths of the elementary intervals.

$(n_j)_{j\geq 0}$: values $\in \{-\sigma, +\sigma\}$ taken by the process over each elementary interval.

The values n_j are i.i.d. with the distribution

$$\mathbb{P}(n_j = \pm\sigma) = \frac{1}{2}$$

The lengths l_j are i.i.d. with the pdf ($H \in (0, 1/2)$, $\ell_{\rm i} \ll \ell_{\rm c}$):

$$p_{l_1}(z) = \frac{1-2H}{\ell_{\rm c}[(\ell_{\rm i}/\ell_{\rm c})^{2H-1} - 1]} \frac{\ell_{\rm c}^{2-2H}}{z^{2-2H}} \mathbf{1}_{[\ell_{\rm i},\ell_{\rm c}]}(z)$$

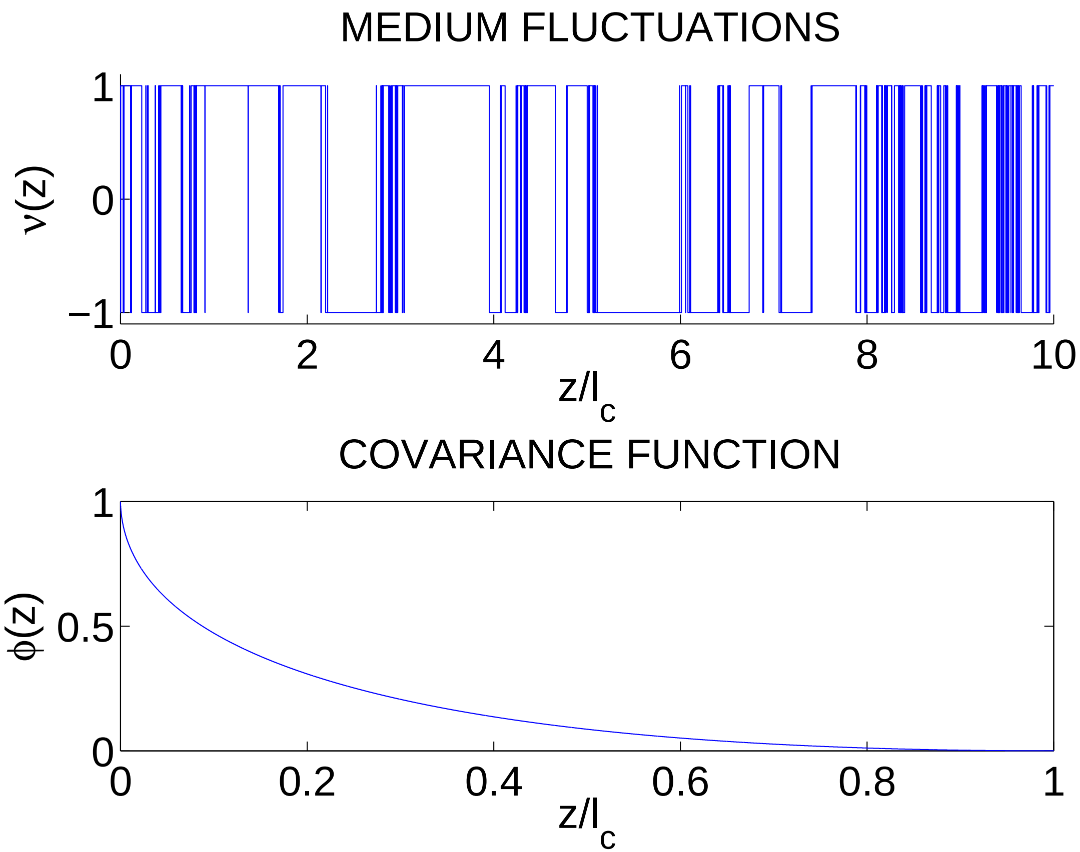
The covariance function is

$$\begin{aligned}\phi(z) &= \frac{\sigma^2}{1-\delta^{2H}}\Big(1 - \frac{1}{1-2H}\frac{|z|^{2H}}{\ell_{\rm c}^{2H}} + \frac{2H}{1-2H}\frac{|z|}{\ell_{\rm c}}\Big)\mathbf{1}_{(\ell_{\rm i},\ell_{\rm c}]}(|z|) \\ &\quad +\sigma^2\Big(1 - \frac{2H}{1-2H}\frac{\delta^{2H}-\delta}{1-\delta^{2H}}\frac{|z|}{\ell_{\rm i}}\Big)\mathbf{1}_{(0,\ell_{\rm i}]}(|z|)\end{aligned}$$

with $\delta = \ell_{\rm i}/\ell_{\rm c}$. We have

$$\phi(z) \overset{\ell_{\rm i}<|z|\ll\ell_{\rm c}}{\simeq} \sigma^2\Big(1 - d_H\Big|\frac{z}{\ell_{\rm c}}\Big|^{2H} + O\big(\Big|\frac{z}{\ell_{\rm c}}\Big|\big)\Big), \text{ with } d_H = \frac{1}{1-2H}$$

Binary medium - short-range correlations



Realizations of a binary medium with the index $H = 0.25$.

Accumulation of very small intervals responsible for the short-range correlation property.

Binary medium - short- and long-range correlations

The process ν is stepwise constant.

$(l_j)_{j\geq 0}$: lengths of the elementary intervals.

$(n_j)_{j\geq 0}$: values $\in \{-\sigma, +\sigma\}$ taken by the process over each elementary interval.

The values n_j are i.i.d. with the distribution

$$\mathbb{P}(n_j = \pm\sigma) = \frac{1}{2}$$

The lengths l_j are i.i.d. with the pdf:

$$\begin{aligned} p_{l_1}(z) &= \Big(1 - a\delta^{1-2H}\Big) \frac{1-2H}{\delta^{2H-1}-1} \frac{\ell_{\rm c}^{1-2H}}{z^{2-2H}} \mathbf{1}_{[\ell_{\rm i},\ell_{\rm c}]}(z) \\ &\quad + a\delta^{1-2H}(3-2H') \frac{\ell_{\rm c}^{3-2H'}}{z^{4-2H'}} \mathbf{1}_{[\ell_{\rm c},\infty)}(z)\,, \end{aligned}$$

where $H \in (0,1/2)$, $H' \in (1/2,1)$, $a \in (0,1)$, $\delta = \ell_{\rm i}/\ell_{\rm c}$, and $0 < \ell_{\rm i} \ll \ell_{\rm c}$.

The pdf is continuous at $\ell_{\rm c}$ if we choose a as

$$\frac{1}{a} = \frac{3-2H'}{1-2H} - 2\delta^{1-2H} \frac{1+H-H'}{1-2H}\,,$$

but it is not required.

Binary medium - short- and long-range correlations

The covariance function has both H-short-range and H'-long-range properties:

- if $|z| \geq \ell_c$, then

$$\phi(z) \simeq \frac{a\sigma^2}{(2-2H')\left(\frac{1-2H}{2H} + a\frac{3-2H'}{2-2H'}\right)} \frac{\ell_c^{2-2H'}}{|z|^{2-2H'}}$$

which shows it has the H'-long-range property in the range $|z| \in [\ell_c, \infty)$.

- if $\ell_i \ll |z| \ll \ell_c$, then

$$\phi(z) \simeq \sigma^2 \left\{ 1 - \frac{1}{1 - 2H + a2H\frac{3-2H'}{2-2H'}} \frac{|z|^{2H}}{\ell_c^{2H}} \right\}$$

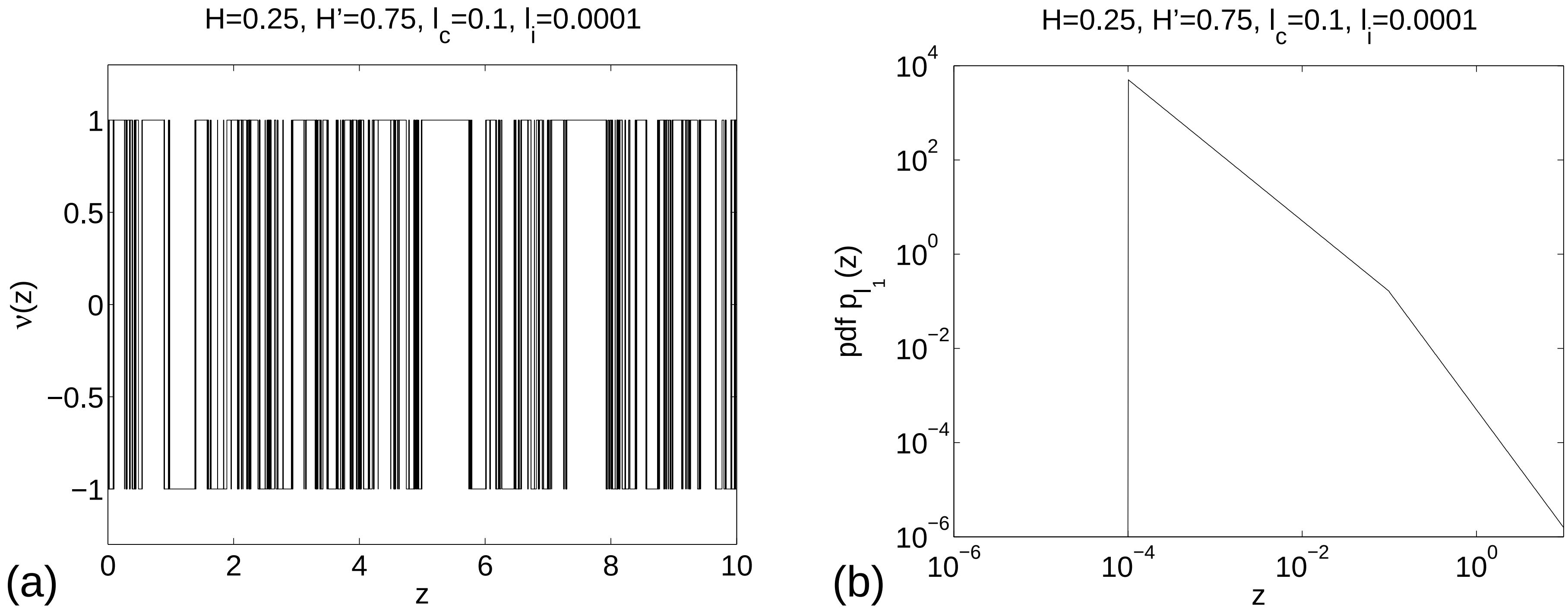
which shows it has the H-short-range property in the range $|z| \in (\ell_i, \ell_c)$.

- if $|z| \leq \ell_i$, then

$$\phi(z) \simeq \sigma^2 \left\{ 1 - \frac{1}{\left(\frac{1-2H}{2H} + a\frac{3-2H'}{2-2H'}\right)\delta^{1-2H}} \frac{|z|}{\ell_c} \right\}$$

which shows it has the regular property in the range $|z| \in [0, \ell_i]$.

Binary medium - short- and long-range correlations



Realizations of a binary medium long- and short-range properties with $H = 0.25$ and $H' = 0.75$.

Accumulation of very small intervals responsible for the short-range correlation property and generation of intervals longer than the average responsible for the long-range correlation property

Analysis of pulse propagation (1/2)

Write wave equation as a hyperbolic system:

$$\begin{cases} \dfrac{\partial p^\varepsilon}{\partial z} + \dfrac{\partial u^\varepsilon}{\partial t} = 0 \\ \dfrac{\partial u^\varepsilon}{\partial z} + \dfrac{1}{c^\varepsilon(z)^2}\dfrac{\partial p^\varepsilon}{\partial t} = 0 \end{cases}$$

Introduce the right- and left-going wave amplitudes:

$$\begin{bmatrix} A^\varepsilon(t,z) \\ B^\varepsilon(t,z) \end{bmatrix} = \begin{bmatrix} c^\varepsilon(z)^{-1/2} p^\varepsilon(t,z) + c^\varepsilon(z)^{1/2} u^\varepsilon(t,z) \\ -c^\varepsilon(z)^{-1/2} p^\varepsilon(t,z) + c^\varepsilon(z)^{1/2} u^\varepsilon(t,z) \end{bmatrix}$$

The mode amplitudes satisfy

$$\frac{\partial}{\partial z}\begin{bmatrix} A^\varepsilon \\ B^\varepsilon \end{bmatrix} = -\frac{1}{c^\varepsilon(z)}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\frac{\partial}{\partial t}\begin{bmatrix} A^\varepsilon \\ B^\varepsilon \end{bmatrix} + \frac{\partial_z c^\varepsilon(z)}{2c^\varepsilon(z)}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} A^\varepsilon \\ B^\varepsilon \end{bmatrix}$$

Incoming wave:

$$A^\varepsilon(t,z) = 2c_o^{-1/2} p_0\Big(\frac{t - z/c_o}{\varepsilon^2}\Big), \qquad B^\varepsilon(t,z) = 0, \qquad t < 0$$

Main phenomena:

- transport along the random characteristics with the local sound speed $c^\varepsilon(z)$.

- coupling between the right- and left-going modes.

Analysis of pulse propagation (2/2)

$$\frac{\partial}{\partial z}\begin{bmatrix} A^\varepsilon \\ B^\varepsilon \end{bmatrix} = -\frac{1}{c^\varepsilon(z)}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\frac{\partial}{\partial t}\begin{bmatrix} A^\varepsilon \\ B^\varepsilon \end{bmatrix} + \frac{\partial_z c^\varepsilon(z)}{2c^\varepsilon(z)}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} A^\varepsilon \\ B^\varepsilon \end{bmatrix}$$

Main steps of the analysis:

- rewrite the evolution equations of the modes by centering along the characteristic of the right-going mode: we obtain an upper-triangular system
- apply limit theorems [based on G. Samorodnitsky and M. S. Taqqu, Stable non-Gaussian random processes, Chapman and Hall, New York, 1994] to this system in order to establish an effective equation for the wave front.

Effective pulse propagation - long-range correlations (1/4)

- Covariance function:

$$\phi(z) \overset{|z|\to\infty}{\simeq} r_{H'} \left| \frac{z}{\ell_c} \right|^{2H'-2}, \quad H' \in (1/2, 1)$$

- Introduce the random travel time

$$\tau^\varepsilon(L) = \frac{L}{c_o} + \varepsilon^{3-2H'} \tau_c^\varepsilon(L), \qquad \tau_c^\varepsilon(L) = \frac{1}{2c_o\varepsilon^{2-2H'}} \int_0^L \nu\left(\frac{z}{\varepsilon^2}\right) dz$$

- In the limit $\varepsilon \to 0$, at $z = L$,

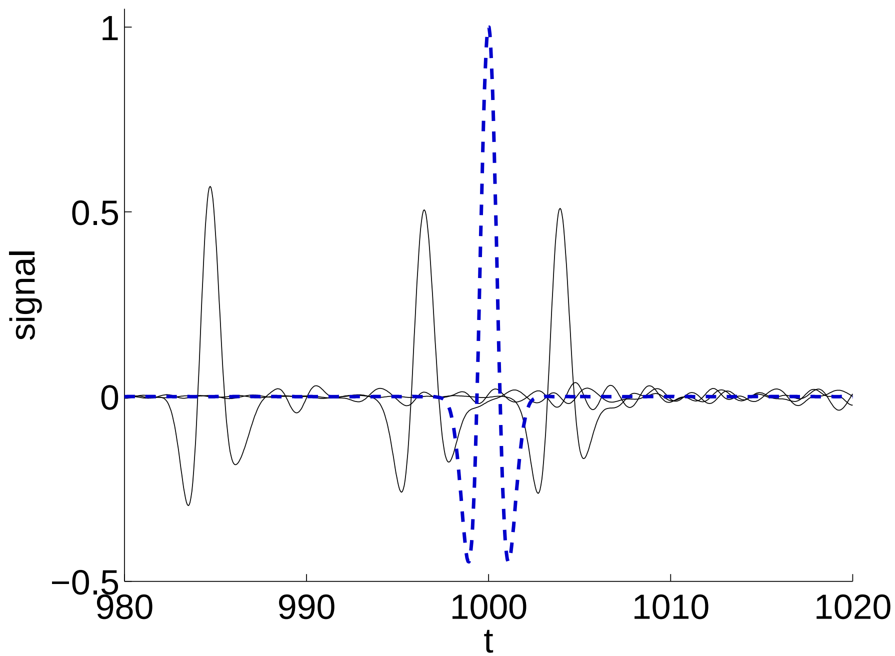
1) the *random time shift* $\tau_c^\varepsilon(L)$ converges in distribution to a zero-mean random variable with variance of order one,

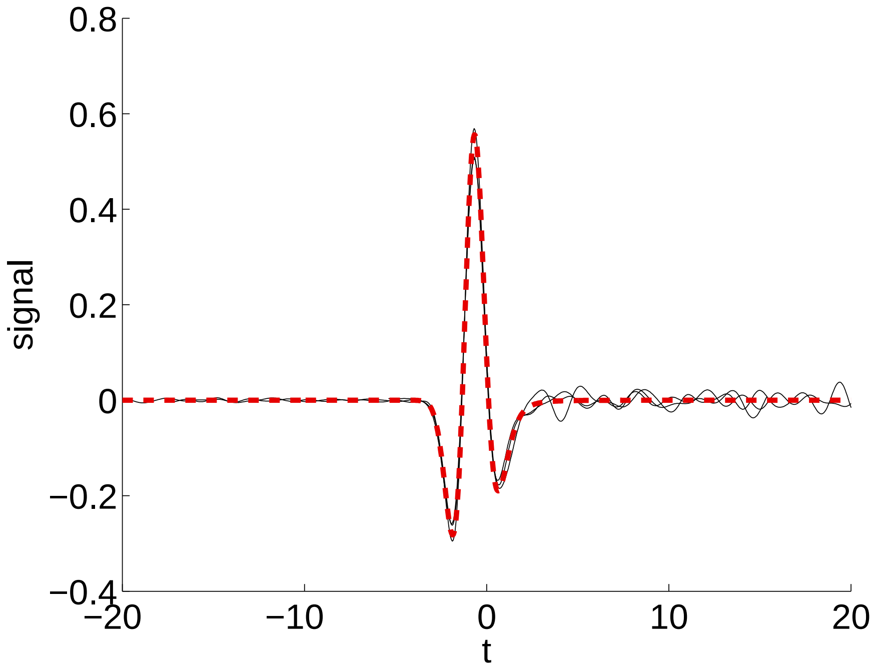
2) the pulse profile converges in probability to a *deterministic profile*:

$$p^\varepsilon\left(z = L, t = \tau^\varepsilon(L) + \varepsilon^2\tau\right) \xrightarrow{\varepsilon\to 0} p_L(\tau) = \frac{1}{2\pi} \int \hat{p}_0(\omega) T_L(\omega) e^{-i\omega\tau} d\omega$$

Note that the time shift is of order $\varepsilon^{3-2H'} \gg \varepsilon^2$ (here $H' > 1/2$, i.e. $3 - 2H' < 2$):
the random time shift is larger than the deterministic deformation.

Effective pulse propagation - long-range correlations (2/4)





Blue: original pulse profile $p_0(t)$

Red: theoretical transmitted profile $p_L(t)$

Black: results of three numerical simulations (with centering in the right picture).

Here $H' = 0.875$.

Effective pulse propagation - long-range correlations (3/4)

More precisely:

1) the random time shift $\tau_c^\varepsilon(L)$ converges in distribution to a Gaussian random variable $\tau_c(L)$ with mean zero and variance ($\sim$ fractional Brownian motion):

$$\mathbb{E}\big[\tau_c(L)^2\big] = \frac{\ell_c^{2-2H'}}{c_o^2} \frac{r_{H'}}{4H'(2H'-1)} L^{2H'}$$

2) If $\frac{\omega \ell_c}{c_o} \ll 1$, then

$$T_L(\omega) = \exp\Big(-\frac{r_{H'}}{2} \frac{\Gamma(2H'-1)}{2^{2H'}} \cos\Big((H'-\frac{1}{2})\pi \Big) \Big(\frac{|\omega|\ell_c}{c_o} \Big)^{3-2H'} \frac{L}{\ell_c} - i \frac{r_{H'}}{2} \frac{\Gamma(2H'-1)}{2^{2H'}} \sin\Big((H'-\frac{1}{2})\pi \Big) \Big(\frac{|\omega|\ell_c}{c_o} \Big)^{2-2H'} \frac{\omega \ell_c}{c_o} \frac{L}{\ell_c} \Big)$$

$\hookrightarrow$ Effective fractional diffusion (attenuation) $\sim |\omega|^{3-2H'}$, $3-2H' \in (1,2)$.
$\hookrightarrow$ Effective fractional dispersion $\sim |\omega|^{2-2H'}\omega$.
Here $\alpha = 2-2H' \in (0,1)$.

Effective pulse propagation - long-range correlations (4/4)

Effective fractional wave equation in the "original" frame (up to random time correction):

$$\frac{\partial^2 p}{\partial z^2} - \frac{1}{c_o^2}\frac{\partial^2 p}{\partial t^2} = \frac{r_{H'}\ell_c^{2-2H'}}{2^{2H'}c_o^{4-2H'}}\int_0^\infty \frac{1}{s^{2-2H'}}\frac{\partial^3 p}{\partial t^3}(t-s)ds$$

Same form as wave equations used in sound propagation in lossy media;

→ slightly different from the standard models [M. Caputo, Geophys. J. R. Astron. Soc. 1, 529 (1967), T. L. Szabo, J. Acoust. Soc. Am. 96, 491 (1994)];

→ equivalent to the model proposed in [W. Chen and S. Holm, J. Acoust. Soc. Am. 114, 2570 (2003)];

→ respects causality (dispersion relation satisfies Kramers-Kronig relation).

Effective pulse propagation - short-range correlations (1/2)

$$\phi(z) \overset{|z|\to 0}{\simeq} \phi(0)\Big(1 - d_H\Big|\frac{z}{\ell_c}\Big|^{2H} + O\Big(\Big|\frac{z}{\ell_c}\Big|\Big)\Big), \quad H \in (0, 1/2)$$

In the limit $\varepsilon \to 0$, at $z = L$, the pulse profile converges in probability to a deterministic profile:

$$p^\varepsilon\big(z = L, t = \frac{L}{c_o} + \varepsilon^2\tau\big) \overset{\varepsilon\to 0}{\longrightarrow} p_L(\tau) = \frac{1}{2\pi}\int \hat{p}_0(\omega) T_L(\omega) e^{-i\omega\tau} d\omega$$

Note that the random time shift is vanishing.

If $\frac{\omega \ell_c}{c_o} \gg 1$, then

$$\begin{aligned} T_L(\omega) \quad &= \quad \exp\Big(-\frac{\phi(0) d_H}{8}\frac{\Gamma(1+2H)}{2^{2H}}\sin(H\pi)\Big(\frac{|\omega|\ell_c}{c_o}\Big)^{1-2H}\frac{L}{\ell_c} \\ &\qquad -i\frac{\phi(0)}{8}\frac{\omega\ell_c}{c_o}\frac{L}{\ell_c} + i\frac{\phi(0) d_H}{8}\frac{\Gamma(1+2H)}{2^{2H}}\cos(H\pi)\Big(\frac{|\omega|\ell_c}{c_o}\Big)^{-2H}\frac{\omega\ell_c}{c_o}\frac{L}{\ell_c}\Big) \end{aligned}$$

$\hookrightarrow$ Deterministic time shift (proportional to L).

$\hookrightarrow$ Effective fractional diffusion (attenuation) $\sim |\omega|^{1-2H}$, $1 - 2H \in (0, 1)$.

$\hookrightarrow$ Effective fractional dispersion $\sim |\omega|^{-2H}\omega$.

Here $\alpha = -2H \in (-1, 0)$.

Effective pulse propagation - short-range correlations (2/2)

- Remark 1: Effective fractional wave equation in the "original" frame:

$$\frac{\partial^2 p}{\partial z^2} - \frac{1}{c_o^2}\frac{\partial^2 p}{\partial t^2} = \frac{H d_H}{2^{1+2H} c_o^{2-2H} \ell_{\rm c}^{2H}} \int_0^\infty \frac{1}{s^{1-2H}} \frac{\partial^2 p}{\partial t^2}(t-s) ds$$

Effective pulse propagation - short- and long-range correlations

$$\frac{1}{L}\ln|T_L(\omega)| \approx \begin{cases} -\left(\frac{\omega \ell_c}{c_o}\right)^{3-2H'} \frac{\Gamma(2H'-1)\cos((H'-1/2)\pi)}{2^{2H'}(3-2H')(1+\mathcal{C})} & , \quad \frac{\omega}{c_o} \ll \frac{1}{\ell_c}, \\ -\left(\frac{\omega \ell_c}{c_o}\right)^{1-2H} \frac{\Gamma(1+2H)\sin(H\pi)}{2^{2+2H}(1-2H)(1+\mathcal{C}^{-1})} & , \quad \frac{1}{\ell_c} \ll \frac{\omega}{c_o} \ll \frac{1}{\ell_i}, \\ -\frac{H\delta^{2H-1}}{4(1-2H)(1+\mathcal{C}^{-1})} & , \quad \frac{\omega}{c_o} \gg \frac{1}{\ell_i}, \end{cases}$$

with $\mathcal{C} = \sigma^2 \frac{(1-2H)(2-2H')(1-a\delta^{1-2H})(1-\delta^{2H})}{a2H(3-2H')(1-\delta^{1-2H})}$.

- Summary:

- For low frequencies $\frac{\omega}{c_o} \ll \frac{1}{\ell_c}$, the H'-long-range correlation property gives an exponent $\alpha = 2 - 2H' \in (0,1)$.
- For mid frequencies $\frac{1}{\ell_c} \ll \frac{\omega}{c_o} \ll \frac{1}{\ell_i}$, the H-short-range correlation property gives an exponent $\alpha = -2H \in (-1,0)$.
- For high frequencies $\frac{\omega}{c_o} \gg \frac{1}{\ell_i}$, the regular property gives an exponent $\alpha = -1$.

- Remember the experimental findings:

- $\alpha \simeq 0.4$ for low frequencies $< 10^{-3}$Hz [Lekic 2009],
- $\alpha \in (-0.4, 0)$ for mid frequencies $(10^{-3}, 1)$Hz [Choy 1986, Shito 2004],
- $\alpha \simeq -1$ for high frequencies > 1Hz [Choy 1986, Cormier 2011].

Conclusions

- In a random medium with a covariance function that decays at infinity as $|z|^{2H'-2}$, $H' \in (1/2, 1)$, the attenuation factor at low frequency is $Q^{-1}(\omega) \sim |\omega|^{\alpha}$, with $\alpha = 2 - 2H' \in (0, 1)$.

- In a random medium with a covariance function that behaves at zero like $1 - d_H |z|^{2H}$, $H \in (0, 1/2)$, the attenuation factor at high frequency is $Q^{-1}(\omega) \sim |\omega|^{\alpha}$, with $\alpha = -2H \in (-1, 0)$.

- A special frequency-dependent phase is associated to the frequency-dependent attenuation and it ensures that causality and Kramers-Kronig relations are respected.

- Effective fractional wave equations can be written that have the form of equations studied in the literature in the context of wave propagation in lossy media.

- A simple binary medium with short- and long-range correlation properties can explain the frequency dependence of the apparent attenuation observed for seismic body waves.