Rough solutions of the Stochastic Navier-Stokes Equation

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Outline

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The Deterministic Navier-Stokes Equations

- A general incompressible fluid flow satisfies the Navier-Stokes Equation

$$\begin{aligned} u_t + u \cdot \nabla u &= \nu \Delta u - \nabla p \\ u(x,0) &= u_0(x) \end{aligned}$$

with the incompressibility condition

$$\nabla \cdot u = 0,$$

- Eliminating the pressure using the incompressibility condition gives

$$\begin{aligned} u_t + u \cdot \nabla u &= \nu \Delta u + \nabla \Delta^{-1} \mathrm{trace}(\nabla u)^2 \\ u(x,0) &= u_0(x) \end{aligned}$$

- The turbulence is quantified by the dimensionless Taylor-Reynolds number $Re_\lambda = \frac{U\lambda}{\nu}$

The Reynolds Decomposition

- The velocity is written as $U+u$, pressure as $P+p$ U describes the large scale flow, u describes the small scale turbulence
- This is the classical Reynolds decomposition (RANS)

$$U_t + U \cdot \nabla U = \nu \Delta U - \nabla P - \frac{\partial}{\partial x_j} \mathcal{R}_{ij}$$

- The last term the eddy viscosity, where $\mathcal{R}_{ij} = \overline{u_i u_j}$ is the Reynolds stress, describes how the small scale influence the large ones.
- *Closure problem*: compute $\mathcal{R}_{ij}$.

A Stochastic Closure

- Large scale flow

$$
\begin{aligned}
U_t + U \cdot \nabla U &= \nu \Delta U - \nabla P - \frac{\partial}{\partial x_j} \mathcal{R}_{ij} \\
U(x,0) &= U_o(x).
\end{aligned}
$$

- Small scale flow

$$
\begin{aligned}
u_t + u \cdot \nabla u &= \nu \Delta u + \nabla \Delta^{-1} \mathrm{trace}(\nabla u)^2 + \mathrm{Noise} \\
u(x,0) &= u_0(x).
\end{aligned}
$$

- What is the form of the Noise? It will contain both additive noise and multiplicative $u \cdot$ noise.

Stochastic Navier-Stokes with Turbulent Noise

- Adding the two types of additive noise and the multiplicative noise we get the stochastic Navier-Stokes equations describing fully developed turbulence

$$
\begin{aligned}
du = (\nu\Delta u &- (U+u)\cdot\nabla u - u\cdot\nabla U + \nabla\Delta^{-1}\mathrm{tr}(\nabla u)^2)dt \\
&+ \sum_{k\neq 0} c_k^{\frac{1}{2}} db_t^k e_k(x) + \sum_{k\neq 0} d_k|k|^{1/3} dt\; e_k(x) \\
&+ u(\sum_{k\neq 0}^{M} \int_{\mathbb{R}} h_k \bar{N}^k(dt,dz)) \qquad (1) \\
u(x,0) &= u_0(x)
\end{aligned}
$$

- Each Fourier component $e_k = e^{2\pi i k\cdot x}$ comes with its own Brownian motion b_t^k and deterministic bound $|k|^{1/3}dt$

Outline

The Kolmogorov-Obukhov Theory

- In 1941 Kolmogorov and Obukhov [7, 6, 9] proposed a statistical theory of turbulence
- The structure functions of the velocity differences of a turbulent fluid, should scale with the distance (lag variable) l between them, to the power $p/3$

$$E(|u(x,t)-u(x+l,t)|^p) = S_p = C_p l^{p/3}$$

A. Kolmogorov

A. Obukhov

The Kolmogorov-Obukhov Refinded Similarity with She-Leveque Intermittency Corrections

- The Kolmogorov-Obukhov '41 theory was criticized by Landau for including universal constants C_p and later for not including the influence of the intermittency
- In 1962 Kolmogorov and Obukhov [8, 10] proposed a refined similarity hypothesis

$$S_p = C'_p < \tilde{\varepsilon}^{p/3} > l^{p/3} = C_p l^{\zeta_p} \tag{2}$$

l is the lag and ε a mean energy dissipation rate

- The scaling exponents

$$\zeta_p = \frac{p}{3} + \tau_p$$

include the She-Leveque intermittency corrections $\tau_p = -\frac{2p}{9} + 2(1-(2/3)^{p/3})$ and the C_p are not universal but depend on the large flow structure

- We solve (1) using the Feynmann-Kac formula, and Girsanov's Theorem
- The solution is

$$\begin{aligned} u &= e^{Kt} e^{\int_0^t \nabla U ds} e^{\int_0^t dq} M_t u^0 \\ &+ \sum_{k \neq 0} \int_0^t e^{K(t-s)} e^{\int_0^{(t-s)} \nabla U dr} e^{\int_s^t dq} M_{t-s} \\ &\times (c_k^{1/2} db_s^k + d_k |k|^{1/3} ds) e_k(x) \end{aligned}$$

- K is the operator $K = \nu\Delta + \nabla\Delta^{-1} tr(\nabla u \nabla)$
- M_t is the Martingale

$$M_t = e^{\{-\int_0^t (U+u)(B_s,s)\cdot dB_s - \frac{1}{2}\int_0^t |(U+u)(B_s,s)|^2 ds\}}$$

- Using M_t as an integrating factor eliminates the inertial terms from the equation (1)

The Feynmann-Kac formula
The computation of the intermittency corrections

- The Feynmann-Kac formula gives the exponential of a sum of terms of the form

$$\int_s^t dq^k = \int_0^t \int_{\mathbb{R}} \ln(1+h_k) N^k(dt,dz) - \int_0^t \int_{\mathbb{R}} h_k m^k(dt,dz),$$

  by a computation similar to the one that produces the geometric Lévy process [1, 2], m^k the Lévy measure.
- The form of the processes

$$e^{\int_0^t \int_{\mathbb{R}} \ln(1+h_k) N^k(dt,dz) - \int_0^t \int_{\mathbb{R}} h_k m^k(dt,dz)} = e^{N_t^k \ln \beta + \gamma \ln |k|} = |k|^{\gamma} \beta^{N_t^k}$$

  was found by She and Leveque [11], for $h_k = \beta - 1$
- It was pointed out by She and Waymire [12] and by Dubrulle [5] that they are log-Poisson processes.

KOSL Scaling of the Structure Functions, higher order $Re_\lambda \sim 16,000$

Comparison of Theory and Experiments

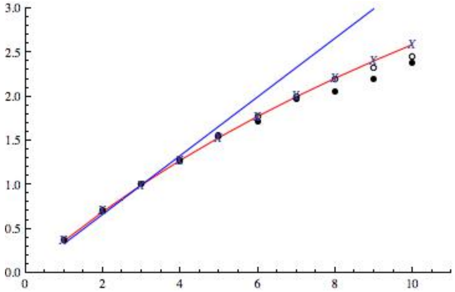

Figure: The exponents of the structure functions as a function of order, theory or Kolmogorov-Obukov-She-Leveque scaling (red), experiments (disks), dns simulations (circles), from [4], and experiments (X), from [11]. The Kolomogorov-Obukhov '41 scaling is also shown as a blue line for comparion.

KOSL Scaling of the Structure Functions, low order $Re_\lambda \sim 16,000$

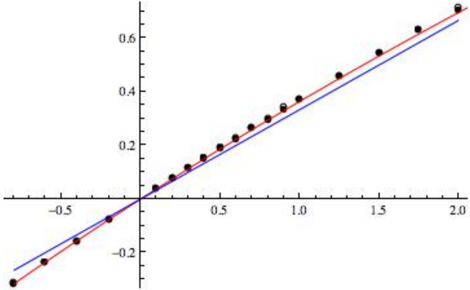

Figure: The exponents of the structure functions as a function of order $(-1, 2]$, theory or Kolmogorov-Obukov-She-Leveque scaling (red), experiments (disks), dns simulations (circles), from [4]. The Kolmogorov-Obukov '41 scaling is also shown as a blue line for comparion.

The Kolmogorov-Obukhov-She-Leveque scaling
The scaling of the structure functions is

$$S_p \sim C_p |x-y|^{\zeta_p},$$

where

$$\zeta_p = \frac{p}{3} + \tau_p = \frac{p}{9} + 2(1-(2/3)^{p/3})$$

$\frac{p}{3}$ being the Kolmogorov scaling and τ_p the intermittency corrections. The scaling of the structure functions is consistent with Kolmogorov's $4/5$ law. Let $\tilde{S}_p$ denote structure function without the absolute value, then

$$\tilde{S}_3 = -\frac{4}{5}\varepsilon|x-y|$$

to leading order, were $\varepsilon = \frac{d\mathcal{E}}{dt}$ is the energy dissipation

Can we compute the Reynolds number dependence of the structure functions?

John Kaminsky in his Ph.D. thesis

$$S_1(x,y,t) = \frac{2}{C} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{|d_k|(1-e^{-\lambda_k t})}{|k|^{\zeta_1} + \frac{4\pi^2\nu}{C}|k|^{\zeta_1+\frac{4}{3}}} |\sin(\pi k \cdot (x-y))|.$$

We get a stationary state as $t \to \infty$, and for $|x-y|$ small,

$$S_1(x,y,t) \sim \frac{2\pi^{\zeta_1}}{C} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{|d_k|}{1 + \frac{4\pi^2\nu}{C}|k|^{\frac{4}{3}}} |x-y|^{\zeta_1}.$$

where $\zeta_1 = 1/3 + \tau_1 \approx 0.37$. Similarly,

$$S_2(x,y,t) = \frac{4}{C^2} \sum_{k \in \mathbb{Z}^3} \left\{ \frac{\frac{C}{2} c_k (1-e^{-2\lambda_k t})}{|k|^{\zeta_2} + \frac{4\pi^2\nu}{C}|k|^{\zeta_2+\frac{4}{3}}} + \frac{d_k^2(1-e^{-\lambda_k t})}{|k|^{\zeta_2} + \frac{8\pi^2\nu}{C}|k|^{\zeta_2+\frac{4}{3}} + \frac{16\pi^4\nu^2}{C^2}|k|^{\zeta_2+\frac{8}{3}}} \right\} |\sin^2(\pi k \cdot (x-y))|,$$

where $\zeta_2 = 2/3 + \tau_2 \approx 0.696$.

Higher order structure functions

The third and pth structure functions are:

$$S_3(x,y,t) = \frac{8}{C^3}\sum_{k\in\mathbb{Z}^3}\Big[\Big\{\frac{\frac{C}{2}c_k|d_k|(1-e^{-2\lambda_k t})(1-e^{-\lambda_k t})}{|k|^{\zeta_3}+\frac{8\pi^2\nu}{C}|k|^{\zeta_3+\frac{4}{3}}+\frac{16\pi^4\nu^2}{C^2}|k|^{\zeta_3+\frac{8}{3}}}+$$

$$\frac{|d_k|^3(1-e^{-\lambda_k t})^3}{|k|^{\zeta_3}+\frac{12\pi^2\nu}{C}|k|^{\zeta_3+\frac{4}{3}}+\frac{48\pi^4\nu^2}{C^2}|k|^{\zeta_3+\frac{8}{3}}+\frac{64\pi^6\nu^3}{C^3}|k|^{\zeta_3+4}}\Big\}\Big]$$

$$\times|\sin^3(\pi k\cdot(x-y))|.$$

$$S_p(x,y,t) = \frac{2^p}{C^p}\sum_{k\neq 0}A_p\times|\sin^p[\pi k\cdot(x-y)]|,$$

$$A_p = \frac{2^{\frac{p}{2}}\Gamma(\frac{p+1}{2})\sigma_k^p {}_1F_1(-\frac{1}{2}p,\frac{1}{2},-\frac{1}{2}(\frac{M_k}{\sigma_k})^2)}{|k|^{\zeta_p}+\frac{p_k\pi^2\nu}{C}|k|^{\zeta_p+\frac{4}{3}}+O(\nu^2)},$$

and $M_k = |d_k|(1-e^{-\lambda_k t})$, and $\sigma_k = \sqrt{(\frac{C}{2}c_k(1-e^{-2\lambda_k t}))}$.

Outline

- The data comes from the Max Planck Institute for Dynamical and Self-Orgranization, in Göttingen, Germany (E. Bodenschatz). It was generated by the variable density turbulence tunnel (VDTT).
- The pressurized gases circulate in the VDTT in an upright, closed loop. At the upstream end of two test sections, the free stream is disturbed mechanically.
- The data in the current paper is generated by a fixed grid, but the gas stream can also be disturbed by an active grid resulting in even higher Reynolds number turbulence.
- In the wake of the grid the resulting turbulence evolves down the length of the tunnel without the middle region being substantially influenced by the walls of the tunnel.

The Variable Density Turbulent Tunnel (VDTT)

- The test sections are about 8 meters long so the turbulence to evolve through at least one eddy turnover time, around 1 second.
- This means that the turbulence can be observed over the time that it takes the energy to cascade all the way from the large eddies to the dissipate scale, see [3].
- Measurements were taken from Taylor Reynolds Numbers 110, 264, 508, 1000, and 1450.
- One might think that the system length is the square root of the cross sectional value of the tunnel $\sqrt{\mathcal{A}}$, but the relevant system length is the grid size D of the grid.

Outline

Fitting the data

- We have to fit the system size and bring the largest measurements into the range of the structure functions, r/η, where η is the Kolmogorov dissipative scale.
- The largest eddies may be influenced by the system size and need to be modeled.
- The large eddies should scale $c_k \sim b^{-1}$ and $d_k \sim a^{-1}$ for k small.
- The small eddies should scale with k, $c_k \sim k^{-m}$ and $d_k \sim k^{-m}$, for k large.
- The constants C and D should measure the norm of u and the system length, for different Reynolds numbers.

Results of fits

Taylor Reynolds Number	a	b	D
110	11.6425	0.0161237	1.56917
264	9.58075	0.0523598	1.76897
508	8.31406	0.0650384	1.51799
1000	3.79242	0.0924666	1.32014
1450	2.68367	0.409223	1.3

Table: The fitted values for *a*, *b*, and *D* and *C* below.

Taylor Reynolds Number	110	264	508	1000	1450
Second	2.79532	3.31462	4.20662	7.61993	21.0531
Third	1.40022	1.92759	1.48768	2.7192	3.58878
Fourth	1.0749	1.01212	1.1907	2.35552	5.99954
Sixth	1.15286	1.28604	1.34263	1.73144	2.48915
Eighth	0.615824	.5316486	.596233	1.16513	2.84003

Structure functions for Taylor-Reynolds number 110

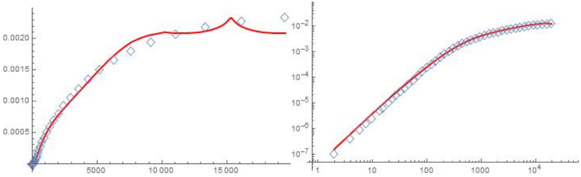

Figure: Second Structure Function, Normal Scale and log-log scale, T-R 110

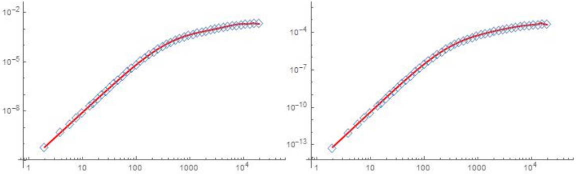

Figure: Third and Fourth Structure function, log-log scale, T-R 110

Structure functions for Taylor-Reynolds number 110 and 1450

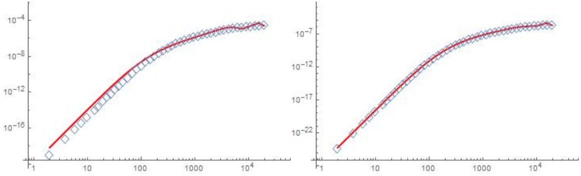

Figure: Sixth and Eighth Structure Function, T-R 110

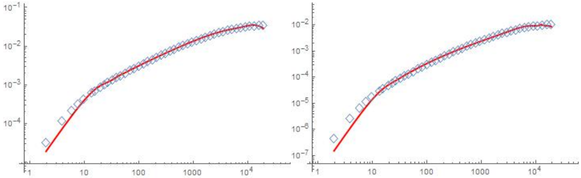

Figure: The Second and Third Structure Function, T-R 1450

Structure functions for Taylor-Reynolds number 1450

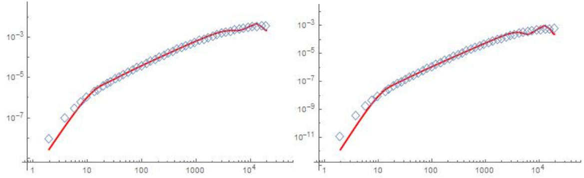

Figure: The Fourth and Sixth Structure Function, log-log scale, T-R 1450

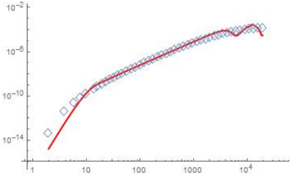

Figure: The Eight Structure function, log-log scale, T-R 1450

Onsager's Observation

- The velocity u, lies in Sobolev space H^s, where $s = \frac{11}{6}$ when intermittency is not taken into account and $s = \frac{29}{18}$ when it is.
- This, in turn, implies that ∇u lies in Sobolev space H^s, where $s = \frac{5}{6}$ without intermittency and $s = \frac{11}{18}$ with intermittency, now $H^s \subset L^p$.
- This follows, by the Sobolev inequality, provided that

$$|\nabla u|_p \leq C \|\nabla u\|_s,$$

- or

$$\frac{5}{6} \geq \frac{3}{2} - \frac{3}{p}.$$

- This is true for $p = 2$, $p = 3$, and $p = 4$, but does not hold for $p = 6$ and $p = 8$.

The Divergence of the Sixth and Eight Structure Functions

The data tell us how rough the fluid velocity is

Taylor Reynolds Number	110	264	508	1000	1450
Second	2.09081	1.49402	1.31448	1.07963	.984291
Third	1.79012	1.41339	1.05553	.822192	.730565
Fourth	1.6408	1.09179	.920749	.687336	.595942
Sixth	1.65727	1.08667	.91658	.681818	.592901
Eighth	1.66164	1.06728	.901549	.662111	.577724

Table: The fitted values for m, uncorrected structure functions.

- The low value of m is due to the divergence of the sine series for the Sixth and the Eight Structure Functions.
- The Fourth structure function sine series diverges with intermittency present.

Outline

Do the Structure Functions with the Taylor-Reynolds number give a better fit?

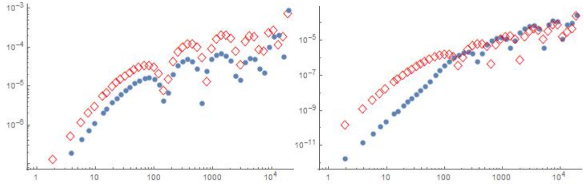

Figure: Second and Third Structure Function Error, with T-R 110

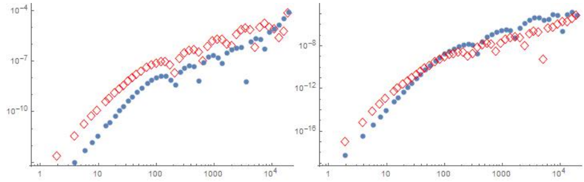

Figure: The Fourth and Sixth Structure Function Error, T-R 110

Structure Function error for Taylor-Reynolds number 1450

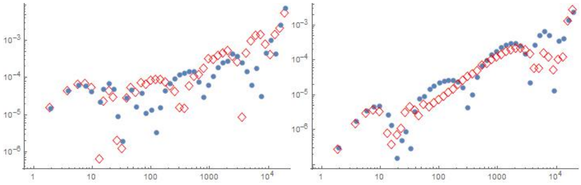

Figure: The Second and Third Structure Function, log-log scale

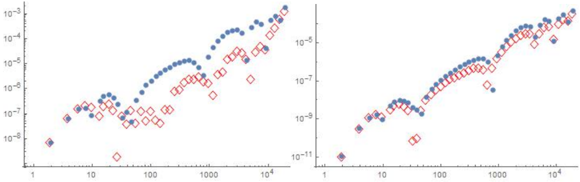

Figure: The Fourth and Sixth Structure function error, log-log scale

Outline

Sobolev Function Spaces

Shahab Karimi in his Ph.D. thesis

- Let $\bar{u} = \int_{\mathbb{T}^n} u\, dx$ and $n = 2$, or 3.
- We work with spaces of periodic functions

$$\dot{H}_{per} = \{u(x,\cdot) \mid u \in L^2_{per}(\mathbb{T}^n),\ \bar{u} = 0,\ \nabla \cdot u = 0\}$$
$$\dot{V}_{per} = \{u(x,\cdot) \mid u \in H^1_{per}(\mathbb{T}^n),\ \bar{u} = 0,\ \nabla \cdot u = 0\}$$

- Define

$$V_s = D(A^{s/2}) = \{u = \sum_{k \in \mathbb{Z}_0^n} c_k e_k \mid \sum |k|^{2s}|c_k|^2 < \infty\},$$

where $A = -P\Delta$ is the Stokes operator.

- The H^s-norm $|\cdot|_s$ on V_s is equivalent to $|\cdot|_{V_s}$. We have $V_0 = \dot{H}_{per}$ and $V_1 = \dot{V}_{per}$.

The Stochastic Navier-Stokes Equation

- The small scale stochastic Navier-Stokes equation (SNS) (1) for incompressible fluid on $\mathbb{T}^n$, $n = 2, 3$, is

$$du = (\nu\Delta u - (u.\nabla)u + \nabla p)dt + Ldt + dW(t)$$

$$\nabla \cdot u = 0, \qquad \bar{u} = 0$$

$$u(x,0,\omega) = u_0(x,\omega)$$

- u_0 is a random variable in $L^p(\Omega; V_\alpha)$, $1 \leq p < \infty$.
- The deterministic term L is the large deviation from mean noise $W(t)$ above

$$L = \sum_{k\in\mathbb{Z}_0^n} \eta_k d_k e^{ik\cdot x}, \qquad dW(t) = \sum_{k\in\mathbb{Z}_0^n} c_k{}^{1/2} db_t{}^k e^{ik\cdot x}.$$

The Integral Equation

- A mild solution of the SNS equation (1), in the space V_α, is a pair (u,τ), where τ is a *strictly positive* stopping time and $u(.\wedge\tau) \in L^p(\Omega; C([0,\tau]; V_\alpha))$ is an $\mathcal{F}_t^{u_0}$-adapted process such that:

$$u(t) = S(t)u_0 - \int_0^t S(t-s)B(u_s)ds + f(t) + W_A(t),$$

a.s. for all $t \in [0,\tau]$, $Bu = Pu \cdot \nabla u$, $S(t) = e^{-At}$,

- where

$$f(t) = \int_0^t S(t-s)Lds, \quad W_A(t) = \int_0^t S(t-s)dW(s).$$

- A mild solution (u,τ) in V_α is unique if for any other mild solution (u',τ'), $u(t\wedge\tau\wedge\tau') = u'(t\wedge\tau\wedge\tau')$ almost surely.

Maximal Mild Solutions

Existence of local solutions

Definition

A local mild solution (u,τ) in V_α is maximal provided that:

i) if (u',τ') is a local mild solution in V_α then $\tau' \le \tau$ a.s.,

ii) There exists a sequence $\{\tau_n\}_n$ of stopping times such that $\tau_n \uparrow \tau$ and for all $n \in \mathbb{N}$, (u,τ_n) is a local mild solution in V_α.

If $\tau = \infty$ almost surely, then the solution is global.

Theorem

Suppose $1 \le \alpha < 3$, $1 \le p < \infty$, $f \in C([0,T_0]; V_\alpha)$ and $W_A(.) \in L^p(\Omega\ ;\ C([0,T_0]; V_\alpha))$ for some fixed $T_0 > 0$, and $u_0 \in L^p(\Omega; V_\alpha)$. Then SNS has a unique mild solution in V_α.

Maximal and Global Solutions

Theorem

(Existence of Maximal Mild Solution) Given the assumptions of above Theorem, there exists a unique (up to null sets) maximal mild solution of SNS (1) in V_α.

$$\sup_{0\le t<\bar{\tau}} |u(t)|_{\alpha-1} + \int_0^{\bar{\tau}} |u(t)|_\alpha^2 dt < \infty$$

Theorem

(Global Existence) Let $\alpha \neq 2$, $u_0 \in L^p(\Omega; V_\alpha)$, $\sum d_k e_k \in V_{\alpha-2}$, $W_A \in \bigcap_{T>0} L^p(\Omega; C([0,T]; V_\alpha))$, $1 < \alpha < 3$, $\mathbf{n} = \mathbf{2}$. Then SNS (1) has a unique global mild solution in V_α.

Outline

Conclusions

- The stochastic closure theory for Navier-Stokes reproduces the statistical theory of K-O with the intermittency corrections of She-Leveque.
- We computed the dependence of the structure functions of turbulence on the Taylor-Reynolds number.
- Comparisons with data from the VDTT tunnel are excellent. The classical Prandtl windtunnel experiment is finally explained. *After 100 years!*
- Very surprisingly the data also determines the smoothness of the fluid velocity u.
- Error analysis favors the Taylor-Reynolds number corrections and confirms the roughness of solutions: $\alpha = 4/3$ and $\alpha = 2$, $n = 2$, $\alpha = 29/18$, $n = 3$.
- Existence of unique global (rough) solutions in V_α is proven, $n = 2$, and unique local (rough) solutions, $n = 3$.

Computation of the Eddy Viscosity
(LES is similar)

- With the stochastic closure, we can now compute the eddy viscosity $\mathcal{R}_{ij} = \overline{u_i u_j}$, using the same method we used to compute the structure functions

$$
\begin{aligned}
\frac{\partial \overline{u u_j}}{\partial x_j} &= \frac{2}{C} e^{-\int_0^t (\nabla u + \nabla u^T) ds} \\
&\times \sum_{k>0} \frac{2\pi[(k \cdot c_k^{1/2}) c_k^{1/2} + (2/C)(k \cdot d_k) d_k]}{|k|^{\zeta_2}} e_k^2 \\
&\approx K |\nabla u|^{(1-\zeta_2)/2} e^{-\int_0^t (\nabla u + \nabla u^T) ds} \Delta^{(1-\zeta_2)/4} u
\end{aligned}
$$

$S = \frac{1}{2}(\nabla u + \nabla u^T)$ is the rate of strain tensor

- The first (multiplicative) term is an exponential (dynamic) Smagorinsky term

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