Order of convergence of splitting schemes for both deterministic and stochastic nonlinear Schrödinger equations

Jie Liu
National University of Singapore, Singapore

Outline

- The second order convergence of Strang-type splitting scheme for nonlinear Schrödinger equation.

- Mass preserving splitting scheme for stochastic nonlinear Schrödinger equation with multiplicative noise
  - Explicit formula for the nonlinear step
  - First order strong convergence
Strang-type splitting scheme for nonlinear Schrödinger equation

Let $i = \sqrt{-1}$ and $V$ be a real-valued function. Consider the following nonlinear Schrödinger equation

$$idu = \Delta u dt + V(x, |u|) u dt \quad \text{in } \mathbb{R}^d. \quad (1)$$

Note three things:

• If $a \in \mathbb{R}$, $i \frac{du}{dt} = au \Rightarrow u(t) = e^{-iat} u(0)$. Hence $|u(t)| = |u(0)|$.

• The exact solution of $idu = V(x, |u|) u dt$, $u(0) = u_0$ satisfies $|u(t)| = |u(0)|$ and therefore $u(x, t) = \exp \{-itV(x, |u_0(x)|)\} u_0(x)$.

• Recall the Strang splitting for $du = (Au + Bu) dt$:

$$u(\delta t) = e^{\delta t(A+B)} u(0) \approx e^{\frac{\delta t}{2} B} e^{\delta t A} e^{\frac{\delta t}{2} B} u(0).$$
We will study the following scheme
\[ u^{n-1} \rightarrow \tilde{u}^{n-\frac{1}{2}} \rightarrow \breve{u}^{n-\frac{1}{2}} \rightarrow u^n \rightarrow \tilde{u}^{n+\frac{1}{2}} \rightarrow \breve{u}^{n+\frac{1}{2}} \rightarrow u^{n+1} \rightarrow \cdots : \]
\[
\tilde{u}^{n-\frac{1}{2}} = \exp \left\{ -i \frac{\delta t}{2} V(x, |u^{n-1}|) \right\} u^{n-1}, \tag{2}
\]
\[
\breve{u}^{n-\frac{1}{2}} = \exp \left\{ -i \delta t \Delta \right\} \tilde{u}^{n-\frac{1}{2}}, \tag{3}
\]
\[
u^n = \exp \left\{ -i \frac{\delta t}{2} V(x, |\breve{u}^{n-\frac{1}{2}}|) \right\} \breve{u}^{n-\frac{1}{2}}. \tag{4}
\]
It can be implemented as \( \breve{u}^{n-\frac{1}{2}} \rightarrow \tilde{u}^{n+\frac{1}{2}} \rightarrow \breve{u}^{n+\frac{1}{2}} \) (skip \( u^n \)) because
\[
\tilde{u}^{n+\frac{1}{2}} = \exp \left\{ -i \frac{\delta t}{2} V(x, |u^n|) \right\} u^n = \exp \left\{ -i \delta t V(x, |\breve{u}^{n-\frac{1}{2}}|) \right\} \breve{u}^{n-\frac{1}{2}}.
\]
The second order convergence in \( L^2 \) norm has been proved by Lubich\(^2\).

Second order convergence

Theorem 1. Consider (1) and its numerical scheme (2)–(4) in $\mathbb{R}^3$. Assume $V(x,|u|) = |u|^2$, and take any $\gamma > 3/2$ and any $\sigma \in \{0, 1, 2, 3, \ldots\}$. If $\|u^0\|_\gamma = M_\gamma < \infty$, then there are constants $T$ and $C_1$ depending on $M_\gamma$ such that for any $\delta t > 0$,

$$\max_{n=0,1,\ldots,[T/\delta t]} \|u^n\|_\gamma \leq C_1. \quad (5)$$

If $\|u^0\|_{\sigma+4} = M_{\sigma+4} < \infty$, then there are constants $T$ and $C_2$ depending on $M_{\sigma+4}$ such that for any $\delta t > 0$,

$$\max_{n=0,1,\ldots,[T/\delta t]} \|u(t^n) - u^n\|_\sigma \leq C_2 \delta t^2. \quad (6)$$

Assume in addition that there are constants $\bar{T}$ and $\bar{M}_{\sigma+4}$ so that $\sup_{0 \leq t \leq \bar{T}} \|u(s)\|_{\sigma+4} = \bar{M}_{\sigma+4} < \infty$, then there is a constant $\delta t_0 > 0$ so that when $\delta t \leq \delta t_0$, the $T$ in (6) can be taken as $\bar{T}$. 
Integral formulation of the scheme

Fix $T, \delta t$, and $N = [T/\delta t]$. Define a right continuous function $\phi_N(x, t)$ with

- $\phi_N(x, 0) = u^0(x)$.  

- $\phi_N(t) = e^{-i\left[tV(x, |\phi_N(0)|)\right]}\phi_N(0)$ when $t \in [0, \frac{\delta t}{2})$  

- On any interval $[t^{n-\frac{1}{2}}, t^{n+\frac{1}{2}})$ ($n = 1, 2, ...$),

$$
\phi_N(t) = \begin{cases} 
    e^{-i\delta t\Delta} \lim_{t^\uparrow t^{n-\frac{1}{2}}} \phi_N(t) & \text{when } t = t^{n-\frac{1}{2}}, \\
    e^{-i\left[(t-t^{n-\frac{1}{2}})V(x, |\phi_N(t^{n-\frac{1}{2}})|)\right]} \phi_N(t^{n-\frac{1}{2}}) & \text{when } t \in [t^{n-\frac{1}{2}}, t^{n+\frac{1}{2}}].
\end{cases}
$$

Then

$$
\phi_N(t^{n-\frac{1}{2}}) = \phi^{n-\frac{1}{2}} \quad \text{and} \quad \phi_N(t^n) = u^n.
$$
Integral formulation of the scheme

Recall $\phi_N(t) = \begin{cases} 
    e^{-i \delta t \Delta} \lim_{t \uparrow t^{n-\frac{1}{2}}} \phi_N(t) & \text{when } t = t^{n-\frac{1}{2}}, \\
    -i \left[(t-t^{n-\frac{1}{2}})V(x, |\phi_N(t^{n-\frac{1}{2}})|)\right] & \\
    e^{-i \left[(t-t^{n-\frac{1}{2}})V(x, |\phi_N(t^{n-\frac{1}{2}})|)\right]} & \phi_N(t^{n-\frac{1}{2}}) \text{ when } t \in [t^{n-\frac{1}{2}}, t^{n+\frac{1}{2}}].
\end{cases}$

- When $t \in [t^{n-\frac{1}{2}}, t^{n+\frac{1}{2}})$, $d\phi_N(t) = -iV(x, |\phi_N(t)|)\phi_N(t)dt$ and therefore

\[
\phi_N(t) = \phi_N(t^{n-\frac{1}{2}}) - i \int_{t^{n-\frac{1}{2}}}^{t} V(x, |\phi_N(s)|)\phi_N(s)ds. \tag{7}
\]

- When $t = t^{n+\frac{1}{2}}$,

\[
\phi_N(t^{n+\frac{1}{2}}) = e^{-i \delta t \Delta} \left[ \phi_N(t^{n-\frac{1}{2}}) - i \int_{t^{n-\frac{1}{2}}}^{t^{n+\frac{1}{2}}} V(x, |\phi_N(s)|)\phi_N(s)ds \right]. \tag{8}
\]
Integral formulation of the scheme

\( \phi_N \) satisfies

\[
\phi_N(t) = e^{-it^n \Delta} \phi_N(0) - i \int_0^t S_{N,n,t}(s) \left( V(x, |\phi_N(s)|) \phi_N(s) \right) ds
\]

when \( t \in [t^{n-\frac{1}{2}}, t^{n+\frac{1}{2}}] \), where \( \phi_N(0) = u(0) \),

\[
S_{N,n,t}(s) = e^{-in\delta t \Delta} I_{[0,\frac{1}{2}]}(s) + \sum_{j=1}^{n-1} I_{[t^{n-\frac{1}{2}} - j, t^{n+\frac{1}{2}} - j]}(s) e^{-i(j\delta t) \Delta} + I_{[t^{n-\frac{1}{2}}, t]}(s).
\]

Let \( S(t - s) = e^{-i(t-s) \Delta} \). The exact solution \( u(t) \) of NLS satisfies

\[
u(t) = e^{-it \Delta} u(0) - i \int_0^t S(t - s) \left( V(x, |u(s)|) u(s) \right) ds.
\]

Note that \( \int_0^{t^n} S_{N,n,t^n}(s) ds \) is the midpoint rule approximation of

\[
\int_0^{t^n} S(t^n - s) ds \text{ on the partition } \int_0^{\frac{1}{2}} + \sum_{j=1}^{n-1} \int_{t^{n-\frac{1}{2}} - j}^{t^{n+\frac{1}{2}} - j} + \int_{t^{n-\frac{1}{2}}}^t
\]
Error equation

Let \( r(t) = u(t) - \phi_N(t) \). We have

\[
\begin{align*}
    r(t) &= (e^{-it\Delta} - e^{-it^n\Delta})u^0 \\
        &\quad - i \int_0^t (S(t - s) - S_{N,n,t}(s)) V(x, |\phi_N(s)|) \phi_N(s) \, ds \\
        &\quad - i \int_0^t S(t - s) [V(x, |u(s)|)u(s) - V(x, |\phi_N(s)|)\phi_N(s)] \, ds \\
        &=: J_1(t) + J_2(t) + J_3(t)
\end{align*}
\]

for \( t \in [t^n - \frac{1}{2}, t^n + \frac{1}{2}] \) (but \( n \) is arbitrary). Note that only at \( t = t^n \), \( J_1(t^n) \) vanishes instead of being of size \( O(\delta t) \).
Error estimate

- Step 1: (First order error estimate over a short time interval)
  Since $\|fg\|_\gamma \leq c\|f\|_\gamma\|g\|_\gamma$ when $\gamma > d/2$ and $\|e^{-it\Delta}w\|_\sigma = \|w\|_\sigma$, $\|(e^{-it\Delta} - I)w\|_\alpha = t\|\Delta w\|_\alpha$, one immediately get $\sup_{t \leq T} \|\phi_N(t)\|_4 < C$ and
  $$\sup_{t \leq T} \|r(t)\|_2 \leq C\delta t$$
  for some $T > 0$. Here $r(t) = u(t) - \phi_N(t)$, $\|r\|_\alpha = \|r\|_{H^\alpha}$

- Step 2: (Error estimate over a short time interval) There is a constant $T$ which depends on the initial data so that
  $$\|r(t^n)\|_0 \leq C_3\delta t \sum_{j=1}^{n-1} \|r(t^j)\|_0 + C_5\delta t^2$$
  for all $n \leq \lceil T/\delta t \rceil$. So, second order convergence follows from the standard discrete Gronwall inequality.
Step 3: (Error estimate up to the blowing up time of the exact solution) 
Note that to prove the 2nd order convergence in $L^2$, one has to prove the stability in $H^4$.

By Calculus inequality in the Sobolev space,

$$
\|\| \phi_N \|^2 \phi_N \|_4 \leq C \|\phi_N\|^2_2 \|\phi_N\|_4. \tag{10}
$$

Hence

$$
\|\phi_N(t)\|_4 \leq \|\phi_N(0)\|_4
+ C \int_0^t (\|u(s)\|_2 + \|\phi_N(s) - u(s)\|_2)^2 \|\phi_N(s)\|_4 ds.
$$
Consider the stochastic Schrödinger equation with multiplicative noise:

\[
    idv = \Delta v dt + V(x, |v|)v dt + v \circ dW \quad \text{in } \mathbb{R}^d, \tag{11}
\]

where \( v \) is a complex valued function, \( i = \sqrt{-1} \), \( W \) is a real valued Wiener process, \( \circ \) means Stratonovich product, \( V \) is also real valued. Let \( \{\beta_k : k \in \mathbb{N}\} \) be a sequence of independent Brownian motions that are associated with \( \{\mathcal{F}_t : t \geq 0\} \). Let \( \{e_k : k \in \mathbb{N}\} \) be an orthonormal basis of \( L^2(\mathbb{R}^d, \mathbb{R}) \). Then

\[
    \hat{W} = \sum_k \beta_k(t, \omega)e_k(x)
\]

and

\[
    W = \Phi \hat{W} = \sum_k \beta_k(t, \omega)(\Phi e_k(x))
\]

\( \Phi \) is a Hilbert-Schmidt operator from \( L^2 \) to \( H^\alpha \). \(^3\)

\(^3\)To present our scheme, we need \( \alpha = 0 \). To prove stability and convergence, we will need larger \( \alpha \).
Applications of Stochastic NLS

• It can be used to model the wave propagation in nonhomogeneous or random media\textsuperscript{4}. (Prof. Solna’s talk on Tuesday)

• It has been introduced by Bang etc\textsuperscript{5} as a model for molecular monolayers arranged in Scheibe aggregates with thermal fluctuations of the phonons.

• It has also been widely used in quantum trajectory theory\textsuperscript{6}, which determines the evolution of the state of a continuously measured quantum system (e.g. the continuous monitoring of an atom by the detection of its fluorescence light).


\[
\begin{aligned}
    \left\{ \begin{array}{l}
    d\psi(t) = \left( -iH(t) - \frac{1}{2} \sum_{j=1}^{d} R_j(t)^* R_j(t) \right) \psi(t) \, dt + \sum_{j=1}^{d} R_j(t) \psi(t) \, dW_j(t), \\
    \psi(0) = \psi_0, \quad \psi_0 \in \mathcal{H}.
    \end{array} \right.
\end{aligned}
\]

(2.28)
Known results

Exact formula for the nonlinear part

Let us for the moment drop the $\Delta v$ term in stochastic NLS:

$$idv = V(x, |v|)vdt + v \circ dW(t).$$  \hspace{1cm} (12)

For the above equation, we have the following observation:

**Theorem 2.** The following $v(x, t)$ is a solution of (12)

$$v(x, t) = v(x, 0) \exp \{ -i [t V(x, |v(x, 0)|) + W(x, t)] \}.$$ \hspace{1cm} (13)

*In particular, $|v(x, t)| = |v(x, 0)|$ and $V(x, |v(x, t)|) = V(x, |v(x, 0)|).$*
Mass preserving splitting scheme

It is now clear that we can define the following splitting scheme initialized with $v^0 = v(x, 0)$:

\begin{align*}
\tilde{v}^n &= \exp \left\{ -i \left[ \delta t V(x, |v^{n-1}|) + (W(x, t^n) - W(x, t^{n-1})) \right] \right\} v^{n-1}, \\
v^n &= \exp \left\{ -i \delta t \Delta \right\} \tilde{v}^n. 
\end{align*}

(14) \quad (15)

It is clear $\|v^n\|_{L^2}^2 = \|	ilde{v}^n\|_{L^2}^2 = \|v^{n-1}\|_{L^2}^2$. We summarize this property into the following Theorem:

**Theorem 3.** The scheme (14)–(15) is unconditionally stable and is mass preserving, i.e, for any $n$

$$\|v^n\|_{L^2}^2 = \|v^0\|_{L^2}^2.$$
First order strong convergence

**Theorem 4.** Consider (11) and the scheme (14)–(15) in $\mathbb{R}^d$. Assume $V(x,|v|) = |v|^2$. Take any integer $\gamma > d/2$ and assume $\Phi \in \mathcal{L}_2(L^2, H^{\gamma+2})$. Assume the $\mathcal{F}_0$-measurable initial data $v^0$ satisfies $(\mathbb{E}\|v^0\|_{H^{\gamma+2}}^p)^{1/p} < M_{\gamma+2} < \infty$ uniformly for $p \geq 1$. Then, for any $L > 0$ with associated stopping time $\tau_L = \inf\{t, \|v(t)\|_{\gamma} \geq L\}$, we have

$$\lim_{K \to \infty} \mathbb{P}\left(\delta t^{-1} \max_{n=0,1,\ldots,[\tau_L/\delta t]} \|v(t^n) - v^n\|_{\gamma} \geq K \right) = 0. \quad (16)$$

Moreover, for any $\varepsilon > 0$, there exists a random variable $K_{L,\varepsilon}$ with finite $\mathbb{E}|K_{L,\varepsilon}|^q$ for any $q \geq 1$, such that

$$\max_{n=0,1,\ldots,[\tau_L/\delta t]} \|v(t^n) - v^n\|_{\gamma} \leq K_{L,\varepsilon} \delta t^{1-\varepsilon}. \quad (17)$$
Integral representation of the scheme

Fix $T, \delta t$ and $N = [T/\delta t]$. Introduce a right continuous function $\psi_N(x, t)$ with $\psi_N(x, 0) = \psi(x, 0) = \psi_0$ and on any interval $[t^{n-1}, t^n],

$$\psi_N(t) = \begin{cases} 
  e^{\{-i[(t-t^{n-1})V(x,|\psi_N(t^{n-1})|)+W(t)-W(t^{n-1})]\}} \times \psi_N(t^{n-1}) & t \in [t^{n-1}, t^n), \\
  e^{\{-i\delta t \Delta\}} \lim_{t \uparrow t^n} \psi_N(t) & t = t^n.
\end{cases}$$

We have $\psi_N(t^n) = \psi^n$.
Integral representation of the scheme

\[ \psi_N(t) = e^{-it^{n-1}\Delta} \psi_N(0) - i \int_0^t S_{N,n,t}(s) \times \]

\[ \left( \tilde{V}(x, |\psi_N(s)|)\psi_N(s)ds + \psi_N(s)dW(s) \right), \]

where

\[ S_{N,n,t}(s) = \sum_{j=1}^{n-1} I_{[tn-1-j,tn-j]}(s)e^{-i(j\delta t)\Delta} + I_{[tn-1,t]}(s). \]
Truncated stochastic Schrödinger equation

Because we do not have $\mathbb{E}|X|^3 \leq C(\mathbb{E}|X|)^3$, the previous argument cannot be applied here. To prove error estimate, we need to study the following truncated equation (motivated the works of Profs. de Bouard & Debussche and Prof. Gyöngy and his co-workers)

$$ idv_R = \Delta v_R dt + \bar{V}_R(x, v_R) v_R dt + v_R dW $$

in $\mathbb{R}^d$, \hspace{1cm} (19)

with initial condition $v_R(x, 0) = v^0(x)$. Here

$$ \theta_R(w) = \theta \left( \frac{\|w\|_\gamma}{R} \right), $$

$$ V_R(x, w) = \theta_R(w) V(x, |w|), $$

$$ \bar{V}_R(x, w) = V_R(x, w) - \frac{i}{2} F_{\Phi}. $$

$\theta \in C^\infty(\mathbb{R})$ is a cut-off function satisfying $\theta(x) = 1$ for $x \in [0, 1]$ and $\theta(x) = 0$ for $x \geq 2$. $\gamma$ is any integer $> d/2$. 
Step 1: Stability of the truncated stochastic NLS

One can prove that even though we only truncate in the $H^\gamma$ norm in $v_R$, the solution automatically becomes $H^{\gamma+2}$-regular:

**Lemma 1.** *(de Bouard and Debussche)* Take any integer $\gamma > d/2$ and consider (19) in $\mathbb{R}^d$ with $\hat{V}_R(x,v) = \theta (\|v\|_\gamma/R) |v|^2 - \frac{i}{2} F_{\Phi}$. Assume $\Phi \in L_2(L^2, H^{\gamma+2})$. Take any $T > 0$ and any $p \geq 1$, for any $\mathcal{F}_0$-measurable initial data $v^0$ satisfying $\mathbb{E}\|v^0\|_{\gamma+2}^p = M_{\gamma+2} < \infty$, there is a constant $M_R$ depending on $R$, such that

$$\mathbb{E}\sup_{t \leq T} \|v_R(t)\|_{\gamma+2}^p \leq M_R. \quad (20)$$

Once again. the proof uses $\|\|v_R\|^2 v_R\|_{\gamma+2} \leq C \|v_R\|^2 \|v_R\|_{\gamma+2}$. 
Step 2: splitting scheme for the truncated stochastic NLS

Proposition 1. Take any integer $\gamma > d/2$ and consider the splitting scheme in $\mathbb{R}^d$ with $V_R(x,v) = \theta_R(v)|v|^2$. Assume $\Phi \in L_2(L^2, H^{\gamma+2})$. For any $R, T > 0$ and any $p \geq 1$, for any $\mathcal{F}_0$-measurable initial data $v^0$ satisfying $\mathbb{E}\|v^0\|_{\gamma+2}^p \leq M_{p,\gamma+2} < \infty$, there is a constant $C_R$ which depends on $R, T, p, M_{p,\gamma+2}$ and $\|\Phi\|_{L_2(L^2, H^{\gamma+2})}$ so that

$$\mathbb{E}\max_{n=0,1,...,[T/\delta t]} \|v_R(t^n) - v_R^n\|_{\gamma}^p \leq C_R \delta t^p. \quad (21)$$

Proof: Let $r(t) = v_R(t) - \psi_N^R(t)$. Then

$$r(t) = \sum_{k=1}^{7} I_k(t), \quad (22)$$
where

\[ I_1(t) = \left( e^{-i(t-t^n-1)\Delta} - 1 \right) e^{-it^n-1\Delta}v^0, \]  
(23)

\[ I_2(t) = -i \int_0^t S_{N,n,t}(s) \left( V_R(x,v_R)v_R(s) - V_R(x,\psi_N^R)\psi_N^R(s) \right) ds, \]  
(24)

\[ I_3(t) = -\frac{1}{2} \int_0^t S_{N,n,t}(s)r(s)F_{\Phi}ds, \]  
(25)

\[ I_4(t) = -i \int_0^t (S(t-s) - S_{N,n,t}(s)) V_R(x,v_R)v_R(s)ds, \]  
(26)

\[ I_5(t) = -\frac{1}{2} \int_0^t (S(t-s) - S_{N,n,t}(s)) v_R(s)F_{\Phi}ds, \]  
(27)

\[ I_6(t) = -i \int_0^t S_{N,n,t}(s)r(s)dW(s), \]  
(28)

\[ I_7(t) = -i \int_0^t (S(t-s) - S_{N,n,t}(s)) v_R(s)dW(s). \]  
(29)
Finally,

\[
\mathbb{E} \sup_{0 \leq \tau \leq t} \|r(\tau)\|_\gamma^p \leq C_1 \int_0^t \left( \mathbb{E} \sup_{0 \leq \tau \leq s} \|r(\tau)\|_\gamma^p \right) ds + C_2 \delta t^p.
\]

The \(C_1\) term comes from the estimates of \(I_2, I_3, I_6\) and the \(C_2\) term comes from the estimates of \(I_1, I_4, I_5, I_7\).
Step 3: from truncated equation to un-truncated equation

(Follows the idea of Gyöngy and Nualart.) Next, for any $\sigma \geq 0$, $L > 0$, we define the stopping time $\tau_L = \inf\{t, \|v(t)\|_\sigma \geq L\}$. We then prove that for any $\varepsilon \in (0, 1)$

$$\left\{ \max_{0 \leq n \leq \lceil \tau_L/\delta t \rceil} \|v^n - v(t^n)\|_\sigma \geq \varepsilon \right\}$$

$$\subset \left\{ \sup_{0 \leq t \leq \tau_L} \|v(t)\|_\sigma \geq R - 1 \right\} \cup \left\{ \max_{0 \leq n \leq \lceil \tau_L/\delta t \rceil} \|v^n_R - v_R(t^n)\|_\sigma \geq \varepsilon \right\}$$

(30)

for any $R > 0$. This follows from $A \subset B \cup \{\overline{B} \cap A\}$.

(30) allows us to prove the convergence in probability before $\tau_L$ for the splitting scheme for the un-truncated stochastic NLS.
Theorem 5. Consider (11) and the scheme (14)–(15) in $\mathbb{R}^d$. Assume $V(x,|v|) = |v|^2$. Take any integer $\gamma > d/2$ and assume $\Phi \in L_2(L^2, H^{\gamma+2})$. Assume the $\mathcal{F}_0$-measurable initial data $v^0$ satisfies $(\mathbb{E}\|v^0\|_{H^{\gamma+2}}^p)^{1/p} < M_{\gamma+2} < \infty$ uniformly for $p \geq 1$. Then, for any $L > 0$ with associated stopping time $\tau_L = \inf\{t, \|v(t)\|_{\gamma} \geq L\}$, we have

$$\lim_{K \to \infty} \mathcal{P}\left( \delta t^{-1} \max_{n=0,1,\ldots,[\tau_L/\delta t]} \|v(t^n) - v^n\|_{\gamma} \geq K \right) = 0. \quad (31)$$

Moreover, for any $\varepsilon > 0$, there exists a random variable $K_{L,\varepsilon}$ with finite $\mathbb{E}|K_{L,\varepsilon}|^q$ for any $q \geq 1$, such that

$$\max_{n=0,1,\ldots,[\tau_L/\delta t]} \|v(t^n) - v^n\|_{\gamma} \leq K_{L,\varepsilon} \delta t^{1-\varepsilon}. \quad (32)$$