Order of convergence of splitting schemes for both deterministic and stochastic nonlinear Schrödinger equations

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Outline

- The second order convergence of Strang-type splitting scheme for nonlinear Schrödinger equation.
- Mass preserving splitting scheme for stochastic nonlinear Schrödinger equation with multiplicative noise
  - ⋆ Explicit formula for the nonlinear step
  - ⋆ first order strong convergence

Strang-type splitting scheme for nonlinear Schrödinger equation

Let $i=\sqrt{-1}$ and V be a real-valued function. Consider the following nonlinear Schrödinger equation

$$idu = \Delta u dt + V(x,|u|)udt \qquad \text{in } \mathbb{R}^d. \tag{1}$$

Note three things:

- If $a \in \mathbb{R}$, $i\frac{du}{dt} = au \Rightarrow u(t) = e^{-iat}u(0)$. Hence $|u(t)| = |u(0)|$.
- The exact solution of $idu = V(x,|u|)udt$, $u(0) = u_0$ satisfies $|u(t)| = |u(0)|$ and therefore $u(x,t) = \exp\{-itV(x,|u_0(x)|)\}u_0(x)$.
- Recall the Strang splitting for $du = (Au + Bu)dt$:

$$u(\delta t) = e^{\delta t(A+B)}u(0) \approx e^{\frac{\delta t}{2}B}e^{\delta t A}e^{\frac{\delta t}{2}B}u(0).$$

We will study the following scheme [1]

$u^{n-1} \to \tilde{u}^{n-\frac{1}{2}} \to \mathring{u}^{n-\frac{1}{2}} \to u^n \to \tilde{u}^{n+\frac{1}{2}} \to \mathring{u}^{n+\frac{1}{2}} \to u^{n+1} \to \cdots$:

$$\tilde{u}^{n-\frac{1}{2}} = \exp\left\{-i\frac{\delta t}{2}V(x,|u^{n-1}|)\right\}u^{n-1}, \tag{2}$$

$$\mathring{u}^{n-\frac{1}{2}} = \exp\{-i\delta t\Delta\}\,\tilde{u}^{n-\frac{1}{2}}, \tag{3}$$

$$u^n = \exp\left\{-i\frac{\delta t}{2}V(x,|\mathring{u}^{n-\frac{1}{2}}|)\right\}\mathring{u}^{n-\frac{1}{2}}. \tag{4}$$

It can be implemented as $\mathring{u}^{n-\frac{1}{2}} \to \tilde{u}^{n+\frac{1}{2}} \to \mathring{u}^{n+\frac{1}{2}}$ (skip u^n) because

$$\tilde{u}^{n+\frac{1}{2}} = \exp\left\{-i\frac{\delta t}{2}V(x,|u^n|)\right\}u^n = \exp\left\{-i\delta t V(x,|\mathring{u}^{n-\frac{1}{2}}|)\right\}\mathring{u}^{n-\frac{1}{2}}.$$

The second order convergence in L^2 norm has been proved by Lubich[2].

[1] Hardin & Tappert 1973, Taha & Ablowitz 1984.

[2] C. Lubich, On splitting methods for Schrödinger-Poisson and cubic nonlinear Schrödinger equations. Math. Comp., 77 (2008) 2141–2153.

Second order convergence

Theorem 1. *Consider (1) and its numerical scheme (2)–(4) in* $\mathbb{R}^3$*. Assume* $V(x,|u|) = |u|^2$*, and take any* $\gamma > 3/2$ *and any* $\sigma \in \{0,1,2,3,\ldots\}$*. If* $\|u^0\|_\gamma = M_\gamma < \infty$*, then there are constants* T *and* C_1 *depending on* M_γ *such that for any* $\delta t > 0$*,*

$$\max_{n=0,1,\ldots,[T/\delta t]} \|u^n\|_\gamma \leq C_1. \tag{5}$$

If $\|u^0\|_{\sigma+4} = M_{\sigma+4} < \infty$*, then there are constants* T *and* C_2 *depending on* $M_{\sigma+4}$ *such that for any* $\delta t > 0$*,*

$$\max_{n=0,1,\ldots,[T/\delta t]} \|u(t^n) - u^n\|_\sigma \leq C_2 \delta t^2. \tag{6}$$

Assume in addition that there are constants $\bar{T}$ *and* $\bar{M}_{\sigma+4}$ *so that* $\sup_{0\leq t\leq \bar{T}} \|u(s)\|_{\sigma+4} = \bar{M}_{\sigma+4} < \infty$*, then there is a constant* $\delta t_0 > 0$ *so that when* $\delta t \leq \delta t_0$*, the* T *in (6) can be taken as* $\bar{T}$*.*

Integral formulation of the scheme

Fix T, δt, and $N = [T/\delta t]$. Define a right continuous function $\phi_N(x,t)$ with

- $\phi_N(x,0) = u^0(x)$.
- $\phi_N(t) = e^{\{-i[tV(x,|\phi_N(0)|)]\}}\phi_N(0)$ when $t \in [0, \frac{\delta t}{2})$
- On any interval $[t^{n-\frac{1}{2}}, t^{n+\frac{1}{2}})$ $(n = 1, 2, ...)$,

$$\phi_N(t) = \begin{cases} e^{\{-i\delta t\Delta\}} \lim_{t\uparrow t^{n-\frac{1}{2}}} \phi_N(t) & \text{when } t = t^{n-\frac{1}{2}}, \\ e^{\left\{-i\left[(t-t^{n-\frac{1}{2}})V(x,|\phi_N(t^{n-\frac{1}{2}})|)\right]\right\}}\phi_N(t^{n-\frac{1}{2}}) & \text{when } t \in [t^{n-\frac{1}{2}}, t^{n+\frac{1}{2}}). \end{cases}$$

Then

$$\phi_N(t^{n-\frac{1}{2}}) = \mathring{u}^{n-\frac{1}{2}} \qquad \text{and} \qquad \phi_N(t^n) = u^n.$$

Integral formulation of the scheme

Recall $\phi_N(t) = \begin{cases} e^{\{-i\delta t\Delta\}} \lim_{t\uparrow t^{n-\frac{1}{2}}} \phi_N(t) & \text{when } t = t^{n-\frac{1}{2}}, \\ e^{\{-i\left[(t-t^{n-\frac{1}{2}})V(x,|\phi_N(t^{n-\frac{1}{2}})|)\right]\}} \phi_N(t^{n-\frac{1}{2}}) & \text{when } t \in [t^{n-\frac{1}{2}}, t^{n+\frac{1}{2}}). \end{cases}$

- When $t \in [t^{n-\frac{1}{2}}, t^{n+\frac{1}{2}})$, $d\phi_N(t) = -iV(x, |\phi_N(t)|)\phi_N(t)dt$ and therefore

$$\phi_N(t) = \phi_N(t^{n-\frac{1}{2}}) - i \int_{t^{n-\frac{1}{2}}}^{t} V(x, |\phi_N(s)|)\phi_N(s)ds. \tag{7}$$

- When $t = t^{n+\frac{1}{2}}$,

$$\phi_N(t^{n+\frac{1}{2}}) = e^{\{-i\delta t\Delta\}} \left[\phi_N(t^{n-\frac{1}{2}}) - i \int_{t^{n-\frac{1}{2}}}^{t^{n+\frac{1}{2}}} V(x, |\phi_N(s)|)\phi_N(s)ds\right]. \tag{8}$$

Integral formulation of the scheme

ϕ_N satisfies

$$\phi_N(t) = e^{-it^n\Delta}\phi_N(0) - i\int_0^t S_{N,n,t}(s)\Big(V(x,|\phi_N(s)|)\phi_N(s)\Big)ds$$

when $t \in [t^{n-\frac{1}{2}}, t^{n+\frac{1}{2}})$, where $\phi_N(0) = u(0)$,
$S_{N,n,t}(s) = e^{-in\delta t\Delta}I_{[0,t^{\frac{1}{2}}]}(s) + \sum_{j=1}^{n-1} I_{[t^{n-\frac{1}{2}-j},t^{n+\frac{1}{2}-j}]}(s)e^{-i(j\delta t)\Delta} + I_{[t^{n-\frac{1}{2}},t]}(s)$.

Let $S(t-s) = e^{-i(t-s)\Delta}$. The exact solution $u(t)$ of NLS satisfies

$$u(t) = e^{-it\Delta}u(0) - i\int_0^t S(t-s)\left(V(x,|u(s)|)u(s)\right)ds.$$

Note that $\int_0^{t^n} S_{N,n,t^n}(s)ds$ is the midpoint rule approximation of $\int_0^{t^n} S(t^n-s)ds$ on the partition $\int_0^{t^{\frac{1}{2}}} + \sum_{j=1}^{n-1}\int_{t^{n-\frac{1}{2}-j}}^{t^{n+\frac{1}{2}-j}} + \int_{t^{n-\frac{1}{2}}}^{t}$

Error equation

Let $r(t) = u(t) - \phi_N(t)$. We have

$$
\begin{aligned}
r(t) =& (e^{-it\Delta} - e^{-it^n\Delta})u^0 \\
&- i\int_0^t \left(S(t-s) - S_{N,n,t}(s)\right) V(x, |\phi_N(s)|)\phi_N(s)ds \\
&- i\int_0^t S(t-s)\left[V(x,|u(s)|)u(s) - V(x,|\phi_N(s)|)\phi_N(s)\right] ds \\
=:& J_1(t) + J_2(t) + J_3(t) \qquad (9)
\end{aligned}
$$

for $t \in [t^{n-\frac{1}{2}}, t^{n+\frac{1}{2}})$ (but n is arbitrary). Note that only at $t = t^n$, $J_1(t^n)$ vanishes instead of being of size $O(\delta t)$.

Error estimate

- Step 1: (First order error estimate over a short time interval) Since $\|fg\|_\gamma \leq c\|f\|_\gamma\|g\|_\gamma$ when $\gamma > d/2$ and $\|e^{-it\Delta}w\|_\sigma = \|w\|_\sigma$, $\|(e^{-it\Delta} - I)w\|_\alpha = t\|\Delta w\|_\alpha$, one immediately get $\sup_{t\leq T}\|\phi_N(t)\|_4 < C$ and
$$\sup_{t\leq T}\|r(t)\|_2 \leq C\delta t$$
for some $T > 0$. Here $r(t) = u(t) - \phi_N(t)$, $\|r\|_\alpha = \|r\|_{H^\alpha}$

- Step 2: (Error estimate over a short time interval) There is a constant T which depends on the initial data so that
$$\|r(t^n)\|_0 \leq C_3\delta t\sum_{j=1}^{n-1}\|r(t^j)\|_0 + C_5\delta t^2$$
for all $n \leq [T/\delta t]$. So, second order convergence follows from the standard discrete Gronwall inequality.

Step 3: (Error estimate up to the blowing up time of the exact solution) Note that to prove the 2nd order convergence in L^2, one has to prove the stability in H^4:

By Calculus inequality in the Sobolev space,

$$\||\phi_N|^2\phi_N\|_4 \leq C\|\phi_N\|_2^2\|\phi_N\|_4. \tag{10}$$

Hence

$$\begin{aligned}\|\phi_N(t)\|_4 \leq &\|\phi_N(0)\|_4 \\ &+ C\int_0^t \left(\|u(s)\|_2 + \|\phi_N(s) - u(s)\|_2\right)^2 \|\phi_N(s)\|_4 ds.\end{aligned}$$

Stochastic Schrödinger equation

Consider the stochastic Schrödinger equation with multiplicative noise:

$$idv = \Delta v dt + V(x, |v|)v dt + v \circ dW \qquad \text{in } \mathbb{R}^d, \tag{11}$$

where v is a complex valued function, $i = \sqrt{-1}$, W is a real valued Wiener process, $\circ$ means Stratonovich product, V is also real valued. Let $\{\beta_k : k \in \mathbb{N}\}$ be a sequence of independent Brownian motions that are associated with $\{\mathcal{F}_t : t \geq 0\}$. Let $\{e_k : k \in \mathbb{N}\}$ be an orthonormal basis of $L^2(\mathbb{R}^d, \mathbb{R})$. Then

$$\hat{W} = \sum_k \beta_k(t, \omega) e_k(x)$$

and

$$W = \Phi \hat{W} = \sum_k \beta_k(t, \omega)(\Phi e_k(x)).$$

Φ is a Hilbert-Schmidt operator from L^2 to H^α [3].

[3] To present our scheme, we need $\alpha = 0$. To prove stability and convergence, we will need larger α.

Applications of Stochastic NLS

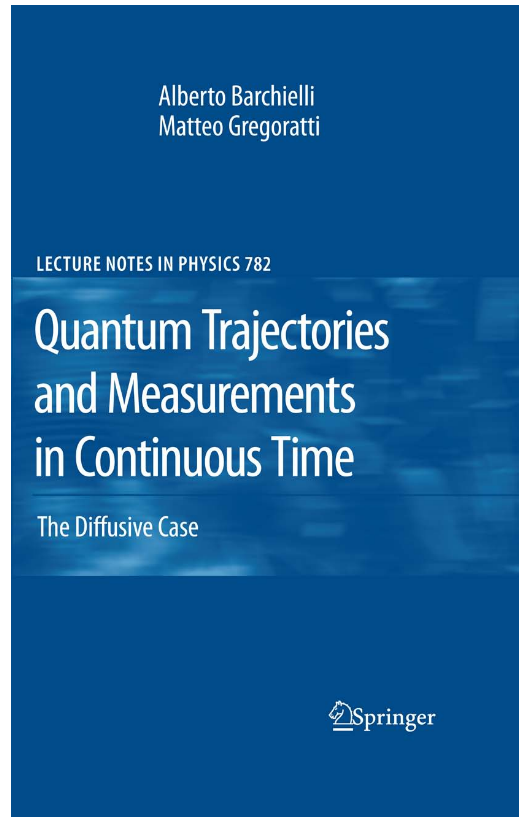
- It can be used to model the wave propagation in nonhomogeneous or random media[4]. (Prof. Solna's talk on Tuesday)
- It has been introduced by Bang etc[5] as a model for molecular monolayers arranged in Scheibe aggregates with thermal fluctuations of the phonons.
- It has also been widely used in quantum trajectory theory[6], which determines the evolution of the state of a *continuously measured* quantum system (e.g. the continuous monitoring of an atom by the detection of its fluorescence light).

[4] V. Konotop, and L. Vázquez, Nonlinear random waves, World Scientific, NJ, 1994.

[5] O. Bang, P.L. Christiansen, F. If, K. Ø. Rasmussen and Y. B. Gaididei, Temperature effects in a nonlinear model of monolayer Scheibe aggregates, Phys. Rev. E, 49 (1994) 4627–4636.

[6] A. Barchielli and M. Gregoratti, Quantum trajectories and measurements in continuous time: the diffusive case, Lecture Notes in Physics 782, Springer, Berlin, 2009.

$$
\begin{cases}
\mathrm{d}\psi(t) = \left(-\mathrm{i}H(t) - \frac{1}{2}\sum_{j=1}^{d} R_j(t)^* R_j(t) \right) \psi(t)\,\mathrm{d}t + \sum_{j=1}^{d} R_j(t)\psi(t)\,\mathrm{d}W_j(t)\,, \\
\psi(0) = \psi_0\,, \qquad\qquad \psi_0 \in \mathcal{H}.
\end{cases}
\tag{2.28}
$$



Known results

- A. de Bouard and A. Debussche, A stochastic nonlinear Schrödinger equation with multiplicative noise, Commun. Math. Phys. 205 (1999) 161-181.
- A. de Bouard and A. Debussche, A semi-discrete scheme for the stochastic nonlinear Schrödinger equation, Numerische Math. 96 (2004) 733–770.
- A. de Bouard and A. Debussche, Weak and strong order of convergence of a semi discrete scheme for the stochastic nonlinear Schrödinger equation, Applied Mathematics and Optimization, 54 (2006) 369–399.
- R. Marty, On a splitting scheme for the nonlinear Schrödinger equation in a random medium. Commun. Math. Sci., 4 (2006) 679–705.
- Z. Brzezniak and A. Millet, On the splitting method for Schrodinger-like evolution equations. Stochastic Analysis and Related Topics, 2012 57–90.

Exact formula for the nonlinear part

Let us for the moment drop the Δv term in stochastic NLS:

$$idv = V(x, |v|)vdt + v \circ dW(t). \tag{12}$$

For the above equation, we have the following observation:

Theorem 2. *The following $v(x,t)$ is a solution of (12)*

$$v(x,t) = v(x,0)\exp\left\{-i[tV(x,|v(x,0)|) + W(x,t)]\right\}. \tag{13}$$

In particular, $|v(x,t)| = |v(x,0)|$ and $V(x,|v(x,t)|) = V(x,|v(x,0)|)$.

Mass preserving splitting scheme

It is now clear that we can define the following splitting scheme initialized with $v^0 = v(x, 0)$:

$$\tilde{v}^n = \exp\left\{-i\left[\delta t V(x, |v^{n-1}|) + \big(W(x, t^n) - W(x, t^{n-1})\big)\right]\right\} v^{n-1}, \quad (14)$$

$$v^n = \exp\{-i\delta t \Delta\} \tilde{v}^n. \quad (15)$$

It is clear $\|v^n\|_{L^2}^2 = \|\tilde{v}^n\|_{L^2}^2 = \|v^{n-1}\|_{L^2}^2$. We summarize this property into the following Theorem:

Theorem 3. *The scheme (14)–(15) is unconditionally stable and is mass preserving, i.e, for any* n

$$\|v^n\|_{L^2}^2 = \|v^0\|_{L^2}^2.$$

First order strong convergence

Theorem 4. *Consider (11) and the scheme (14)–(15) in* $\mathbb{R}^d$*. Assume* $V(x,|v|) = |v|^2$*. Take any integer* $\gamma > d/2$ *and assume* $\Phi \in \mathcal{L}_2(L^2, H^{\gamma+2})$*. Assume the* $\mathcal{F}_0$*-measurable initial data* v^0 *satisfies* $\left(\mathbb{E}\|v^0\|^p_{H^{\gamma+2}}\right)^{1/p} < M_{\gamma+2} < \infty$ *uniformly for* $p \geq 1$*. Then, for any* $L > 0$ *with associated stopping time* $\tau_L = \inf\{t, \|v(t)\|_\gamma \geq L\}$*, we have*

$$\lim_{K\to\infty} \mathcal{P}\left(\delta t^{-1} \max_{n=0,1,\ldots,[\tau_L/\delta t]} \|v(t^n) - v^n\|_\gamma \geq K\right) = 0. \qquad (16)$$

Moreover, for any $\varepsilon > 0$*, there exists a random variable* $K_{L,\varepsilon}$ *with finite* $\mathbb{E}|K_{L,\varepsilon}|^q$ *for any* $q \geq 1$*, such that*

$$\max_{n=0,1,\ldots,[\tau_L/\delta t]} \|v(t^n) - v^n\|_\gamma \leq K_{L,\varepsilon}\delta t^{1-\varepsilon}. \qquad (17)$$

Integral representation of the scheme

Fix T, δt and $N = [T/\delta t]$. Introduce a right continuous function $\psi_N(x,t)$ with $\psi_N(x,0) = v(x,0) = v^0$ and on any interval $[t^{n-1}, t^n]$,

$$\psi_N(t) = \begin{cases} e^{\{-i\left[(t-t^{n-1})V(x,|\psi_N(t^{n-1})|)+\left(W(t)-W(t^{n-1})\right)\right]\}} \\ \qquad \times\psi_N(t^{n-1}) & t \in [t^{n-1}, t^n), \\ e^{\{-i\delta t\Delta\}} \lim_{t\uparrow t^n} \psi_N(t) & t = t^n. \end{cases}$$

We have $\psi_N(t^n) = v^n$

Integral representation of the scheme

$$\psi_N(t) = e^{\{-it^{n-1}\Delta\}}\psi_N(0) - i\int_0^t S_{N,n,t}(s)\times \Big(\bar{V}(x,|\psi_N(s)|)\psi_N(s)ds + \psi_N(s)dW(s)\Big),$$

where

$$S_{N,n,t}(s) = \sum_{j=1}^{n-1} I_{[t^{n-1-j},t^{n-j}]}(s)e^{-i(j\delta t)\Delta} + I_{[t^{n-1},t]}(s). \qquad (18)$$

Truncated stochastic Schrodinger equation

Because we do not have $\mathbb{E}|X|^3 \leq C(\mathbb{E}|X|)^3$, the previous argument cannot be applied here. To prove error estimate, we need to study the following truncated equation (motivated the works of Profs. de Bouard & Debussche and Prof. Gyöngy and his co-workers)

$$idv_R = \Delta v_R dt + \bar{V}_R(x, v_R)v_R dt + v_R dW \qquad \text{in } \mathbb{R}^d, \tag{19}$$

with initial condition $v_R(x,0) = v^0(x)$. Here

$$\theta_R(w) = \theta\left(\frac{\|w\|_\gamma}{R}\right),$$
$$V_R(x,w) = \theta_R(w)V(x,|w|),$$
$$\bar{V}_R(x,w) = V_R(x,w) - \frac{i}{2}F_\Phi.$$

$\theta \in C^\infty(\mathbb{R})$ is a cut-off function satisfying $\theta(x) = 1$ for $x \in [0,1]$ and $\theta(x) = 0$ for $x \geq 2$. γ is any integer $> d/2$.

Step 1: Stability of the truncated stochastic NLS

One can prove that even though we only truncate in the H^γ norm in v_R, the solution automatically becomes $H^{\gamma+2}$-regular:

Lemma 1. *(de Bouard and Debussche) Take any integer $\gamma > d/2$ and consider (19) in $\mathbb{R}^d$ with $\bar{V}_R(x, v) = \theta\left(\|v\|_\gamma/R\right)|v|^2 - \frac{i}{2}F_\Phi$. Assume $\Phi \in \mathcal{L}_2(L^2, H^{\gamma+2})$. Take any $T > 0$ and any $p \geq 1$, for any $\mathcal{F}_0$-measurable initial data v^0 satisfying $\mathbb{E}\|v^0\|^p_{\gamma+2} = M_{\gamma+2} < \infty$, there is a constant M_R depending on R, such that*

$$\mathbb{E}\sup_{t\leq T}\|v_R(t)\|^p_{\gamma+2} \leq M_R. \tag{20}$$

Once again. the proof uses $\||v_R|^2 v_R\|_{\gamma+2} \leq C\|v_R\|^2_\gamma \|v_R\|_{\gamma+2}$.

Step 2: splitting scheme for the truncated stochastic NLS

Proposition 1. *Take any integer $\gamma > d/2$ and consider the splitting scheme in $\mathbb{R}^d$ with $V_R(x,v) = \theta_R(v)|v|^2$. Assume $\Phi \in \mathcal{L}_2(L^2, H^{\gamma+2})$. For any $R, T > 0$ and any $p \geq 1$, for any $\mathcal{F}_0$-measurable initial data v^0 satisfying $\mathbb{E}\|v^0\|^p_{\gamma+2} \leq M_{p,\gamma+2} < \infty$, there is a constant C_R which depends on $R, T, p, M_{p,\gamma+2}$ and $\|\Phi\|_{\mathcal{L}_2(L^2,H^{\gamma+2})}$ so that*

$$\mathbb{E} \max_{n=0,1,\dots,[T/\delta t]} \|v_R(t^n) - v_R^n\|^p_\gamma \leq C_R \delta t^p. \tag{21}$$

Proof: Let $r(t) = v_R(t) - \psi_N^R(t)$. Then

$$r(t) = \sum_{k=1}^{7} I_k(t), \tag{22}$$

where

$$I_1(t) = \left(e^{\{-i(t-t^{n-1})\Delta\}} - 1\right) e^{\{-it^{n-1}\Delta\}} v^0, \tag{23}$$

$$I_2(t) = -\, i \int_0^t S_{N,n,t}(s)\big(V_R(x, v_R)v_R(s) - V_R(x, \psi_N^R)\psi_N^R(s)\big)ds, \tag{24}$$

$$I_3(t) = -\, \frac{1}{2}\int_0^t S_{N,n,t}(s)r(s)F_\Phi ds, \tag{25}$$

$$I_4(t) = -\, i \int_0^t \left(S(t-s) - S_{N,n,t}(s)\right) V_R(x, v_R)v_R(s)ds, \tag{26}$$

$$I_5(t) = -\, \frac{1}{2}\int_0^t \left(S(t-s) - S_{N,n,t}(s)\right) v_R(s)F_\Phi ds, \tag{27}$$

$$I_6(t) = -\, i \int_0^t S_{N,n,t}(s)r(s)dW(s), \tag{28}$$

$$I_7(t) = -\, i \int_0^t \left(S(t-s) - S_{N,n,t}(s)\right) v_R(s)dW(s). \tag{29}$$

Finally,

$$\mathbb{E} \sup_{0\le\tau\le t} \|r(\tau)\|_\gamma^p \le C_1 \int_0^t \left(\mathbb{E} \sup_{0\le\tau\le s} \|r(\tau)\|_\gamma^p \right) ds + C_2 \delta t^p.$$

The C_1 term comes from the estimates of I_2, I_3, I_6 and the C_2 term comes from the estimates of I_1, I_4, I_5, I_7.

Step 3: from truncated equation to un-truncated equation

(Follows the idea of Gyöngy and Nualart.) Next, for any $\sigma \geq 0$, $L > 0$, we define the stopping time $\tau_L = \inf\{t, \|v(t)\|_\sigma \geq L\}$. We then prove that for any $\varepsilon \in (0,1)$

$$\left\{\max_{0 \leq n \leq [\tau_L/\delta t]} \|v^n - v(t^n)\|_\sigma \geq \varepsilon\right\}$$
$$\subset \left\{\sup_{0 \leq t \leq \tau_L} \|v(t)\|_\sigma \geq R - 1\right\} \cup \left\{\max_{0 \leq n \leq [\tau_L/\delta t]} \|v_R^n - v_R(t^n)\|_\sigma \geq \varepsilon\right\} \tag{30}$$

for any $R > 0$. This follows from $A \subset B \cup \{\bar{B} \cap A\}$.

(30) allows us to prove the convergence in probability before τ_L for the splitting scheme for the un-truncated stochastic NLS.

First order strong convergence

Theorem 5. *Consider (11) and the scheme (14)–(15) in $\mathbb{R}^d$. Assume $V(x,|v|) = |v|^2$. Take any integer $\gamma > d/2$ and assume $\Phi \in \mathcal{L}_2(L^2, H^{\gamma+2})$. Assume the $\mathcal{F}_0$-measurable initial data v^0 satisfies $\left(\mathbb{E}\|v^0\|^p_{H^{\gamma+2}}\right)^{1/p} < M_{\gamma+2} < \infty$ uniformly for $p \geq 1$. Then, for any $L > 0$ with associated stopping time $\tau_L = \inf\{t, \|v(t)\|_\gamma \geq L\}$, we have*

$$\lim_{K\to\infty} \mathcal{P}\left(\delta t^{-1} \max_{n=0,1,\ldots,[\tau_L/\delta t]} \|v(t^n) - v^n\|_\gamma \geq K\right) = 0. \qquad (31)$$

Moreover, for any $\varepsilon > 0$, there exists a random variable $K_{L,\varepsilon}$ with finite $\mathbb{E}|K_{L,\varepsilon}|^q$ for any $q \geq 1$, such that

$$\max_{n=0,1,\ldots,[\tau_L/\delta t]} \|v(t^n) - v^n\|_\gamma \leq K_{L,\varepsilon} \delta t^{1-\varepsilon}. \qquad (32)$$